

Basic Sample rate alteration devices, Multirate structures for rate converters, Multistage design of decimator and interpolator, polyphase decomposition, Nyquist criteria

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The DSP structures discussed earlier belong to the class of single-rate systems since the sampling rates at i/p and at o/p and at all internal nodes are same. There are certain applications where the signal at a given sampling rate needs to be converted into another signal with a different sampling rate. E.g. 32 kHz in broadcasting, 44.1 kHz in CD, 48 kHz in digital audio tape.

So, conversion of sampling rates of audio signals among these three different rates is necessary in many situations. To change the sampling rates of sampling of a digital signal, multirate digital signal processing systems use a down sampler and an up-sampler, the two basic sample rate alteration devices in addition to conventional elements such as adder, multiplier and delay. Discrete time systems with unequal sampling rates at various parts of the system are called Multirate systems.

Basic Sampling rate alteration Devices

The two basic components in sampling rate alteration are up sampler and down sampler.

* Time domain characterization

— i — — —

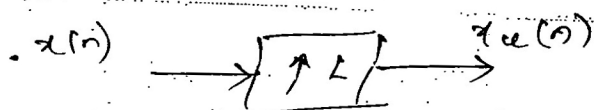
An up sampler with an up-sampling factor L ,

where L is a +ve integer develops an o/p sequence

$x_u(n)$ with a sampling rate that is L times larger than that of the sequence $x(n)$. This operation is implemented by inserting $L-1$ zero-valued samples b/w two consecutive samples of the sequence $x(n)$ according to relation.

$$x_u(n) = \begin{cases} x(n/L) & n = 0, \pm L, \pm 2L, \dots \\ 0 & \text{otherwise} \end{cases}$$

UP Sampler is also called Sample rate expander



In practice zero-valued samples inserted by up-sampler are interpolated using some filtering process in order that new higher rate sequence has no unnecessary spectral components. This process is called Interpolation

⋮

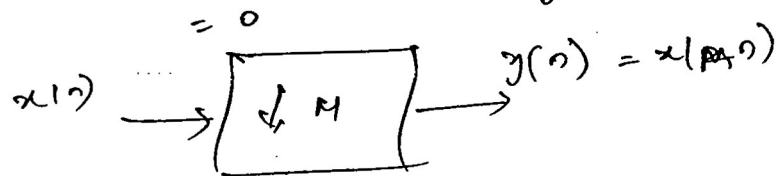
Eg: $x(n) = \{1, 4, 6, 8, 10\}$

$y(n) = (\uparrow 2) x(n)$ or $x(n/2)$ i.e. Insert 2-1 zeros.

$= \{1, 0, 4, 0, 6, 0, 8, 0, 10\}$

* A Down Sampler with a down sampling factor M , where M is the integer creates an op sequence $y(n)$ with a sampling rate that is $(1/M)^{th}$ of the seq $x(n)$. This operation is implemented by keeping every M^{th} sample of the sequence and removing $M-1$ in between

① Samples to generate o/p sequence according to relation
 $y(n) = x(nM)$ $n=0, \pm M, \pm 2M, \dots$
 otherwise

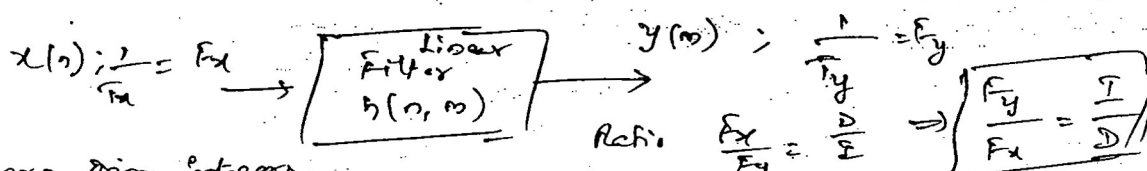


Eg: $x(n) = \{1, 4, 6, 8, 10, 12, 13, 2, 3, 15, 5\}$

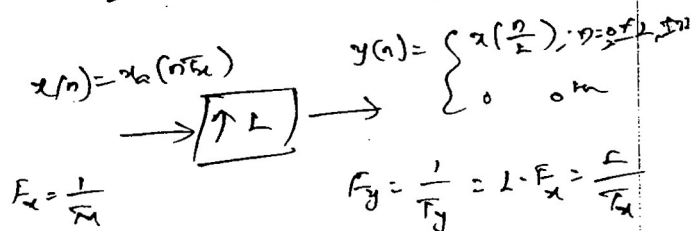
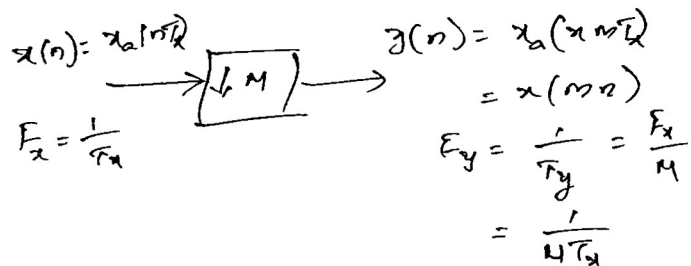
$y(n) = x(2n) = \{1, 6, 10, 13, 3, 5\}$
 (i.e. $x(n)$)

Down sampler is also called sample rate compressor
 or decimation.

The process of sample rate conversion in digital domain can be viewed as linear filtering operation. If $x(n)$ characterized by sampling rate $F_x = 1/T_x$ and o/p signal $y(m)$ characterized by sampling rate $F_y = 1/T_y$



I & D are prime integers.
 Simplified sample rate building blocks are



We know $\frac{F_y}{F_x} = \frac{I}{D}$

①

I & D are prime integers and ratio is to be rational

D - decimation process reducing sample rate factor by D
(down-sampling)

I - Increases sample rate, factor by I (up-sampling)

⇒ Frequency domain characterization

* Decimation (down-sampler)

Let $x(n)$ be ip and $y(n)$ be op. The $I=0$ relation
of down sampler and by applying z^{-I}

$$y(n) = x(nM)$$

$$Y(z) = \sum_{n=-\infty}^{\infty} x(nM) \cdot z^{-n} \quad \text{--- (1)}$$

z -t.s. cannot be directly expressed in $X(z)$ so define intermediate sequence $x_{int}(n)$ whose sample

$x_{int}(n) = x(n)$: $n = 0, \pm M, \pm 2M, \dots$ values are same as $x(n)$ at 0 otherwise. Values of n that are multiples of M and zero at other values.
--- (2)

$$\Rightarrow Y(z) = \sum_{n=-\infty}^{\infty} x(nM) z^{-n} = \sum_{n=-\infty}^{\infty} x_{int}(nM) \cdot z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_{int}(k) \cdot z^{-k/M} = X_{int}(z^{1/M}) \quad \text{--- (3)}$$

$x_{int}(n)$ can be related to $x(n)$ through $x_{int}(n) = C(n)x(n)$

$$C(n) = \begin{cases} 1 & n = 0, \pm M, \pm 2M, \dots \\ 0 & \text{otherwise.} \end{cases} \quad \text{--- (4)}$$

DFS expression

$$C(n) = \frac{1}{M} \sum_{k=0}^{M-1} W_M^{kn} \quad ; \quad W_M = e^{+j \frac{2\pi}{M}}$$

$$X_{int}(z) = \sum_{n=-\infty}^{\infty} x_{int}(n) z^{-n} = \sum_{n=-\infty}^{\infty} c(n) x(n) \cdot z^{-n}$$

$$= \frac{1}{M} \sum_{n=-\infty}^{\infty} \left(\sum_{k=0}^{M-1} W_M^{kn} \right) \cdot x(n) \cdot z^{-n}$$

$$= \frac{1}{M} \sum_{k=0}^{M-1} \left(\sum_{n=-\infty}^{\infty} x(n) W_M^{kn} \cdot z^{-n} \right)$$

$$= \frac{1}{M} \sum_{k=0}^{M-1} X(z W_M^{-k}) \quad \text{--- (6)}$$

Derived HP-SP relation in transform domain for factor M down sampler is obtained by substituting

(6) in (3)

$$\Rightarrow Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} X(z W_M^{-k}) \quad \text{--- (7)}$$

put $z = e^{j\omega}$ in above eqn we get freq domain

representation

$$Y(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(e^{j(\omega - \frac{2\pi k}{M})}\right)$$

The above eqn implies that $Y(e^{j\omega})$ is a sum of M -uniformly shifted and stretched versions of $X(e^{j\omega})$ scaled by a factor $1/M$. Aliasing due to a factor of M -down sampling is absent if and only if the signal $x(n)$ is band limited to $\pm \pi/M$.

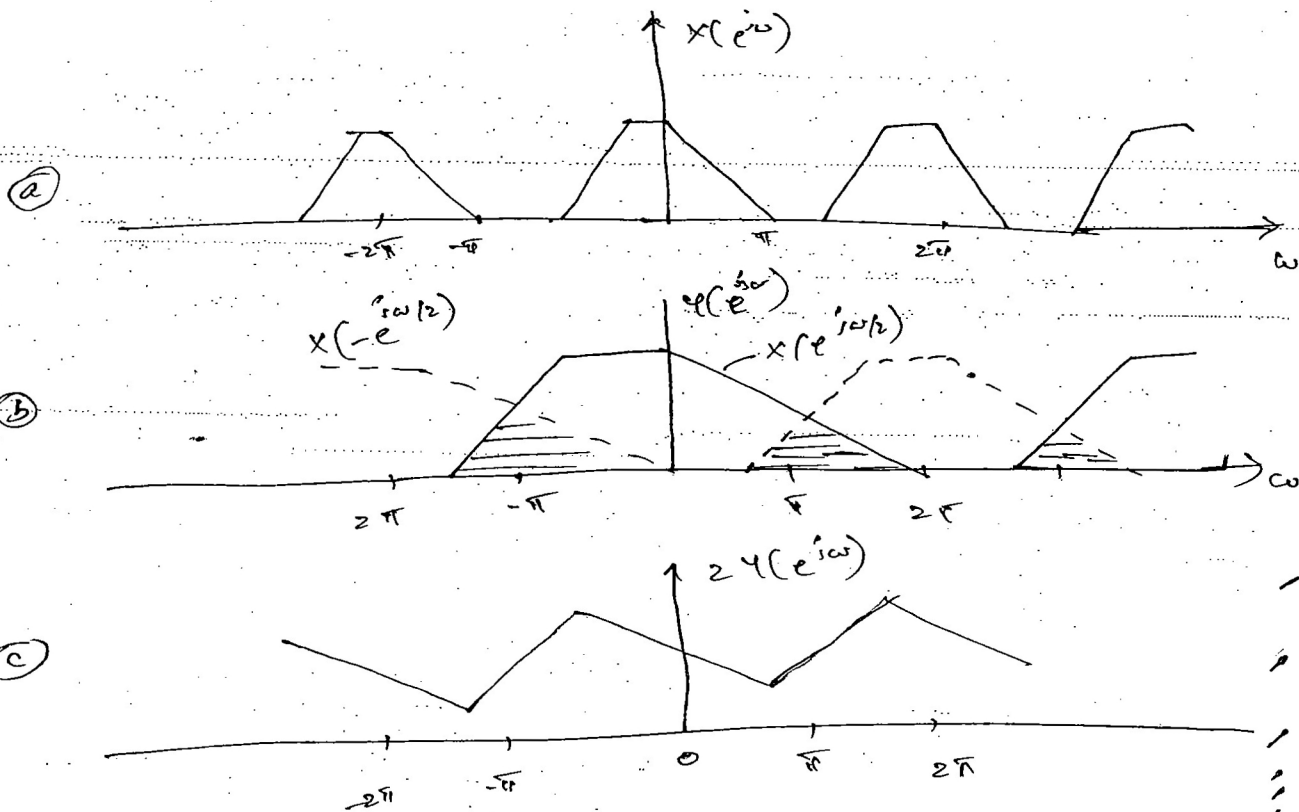
Eg to understand aliasing effect. due to down sampling.

R/p- stp relation is $y(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X(e^{j(\omega - 2\pi k/M)})$

of factor of M- down samples

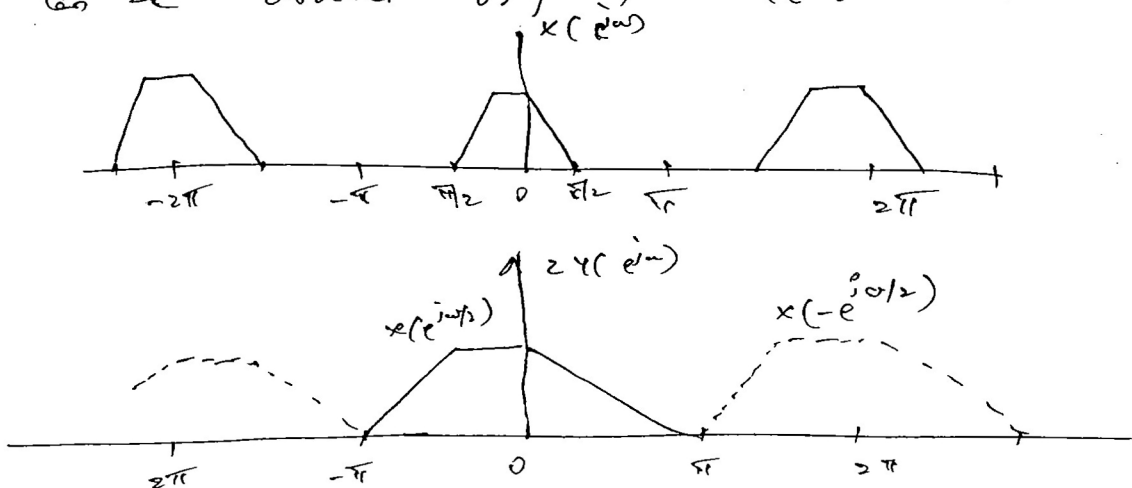
for $M=2$;

$$y(e^{j\omega}) = \frac{1}{2} \left(X(e^{j\omega/2}) + X(-e^{j\omega/2}) \right)$$



Original shape of $X(e^{j\omega})$ is lost when $X(e^{j\omega})$ is down sampled. Resulting in overlap causing aliasing.

Aliasing can be avoided only if $X(e^{j\omega})$ is 0 for $|\omega| \geq \pi/2$



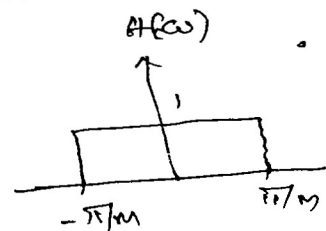
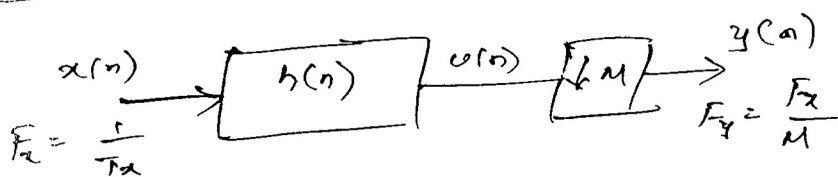
Conclusions :-

* If the spectrum of original signal $x(n)$ i.e. $X(e^{j\omega})$ is band limited to $\omega = \frac{\pi}{M}$, the spectrum being periodic with 2π , the spectrum of downsampled signal $y(n)$ is sum of all uniformly shifted and stretched versions of $X(e^{j\omega})$ scaled by factor $\frac{1}{M}$. In every interval of 2π , in addition to original spectrum there are $M-1$ equally spaced replica. Due to added replica, spectrum period is $\frac{2\pi}{M}$.

The relation b/w original and reduced sampling rate is $\boxed{F_y = \frac{F_x}{M}}$ (or) $\boxed{\omega_y = M \cdot \omega_x}$

* Aliasing effect - Spectrum $Y(e^{j\omega})$ obtained after down sampling will overlap if original spectrum is not band limited to $\omega = \frac{\pi}{M}$. Resulting in aliasing. This effect can be avoided if $x(n)$ is band limited to $\pm \frac{\pi}{M}$. If signal $x(n)$ is not band limited to $\pm \frac{\pi}{M}$, then a LPF with cut off freq $\frac{\pi}{M}$ is used prior to down sampling as shown below. The filter is called antialiasing filter. Its ideal response is shown below.

The complete process of filtering and down sampling is called decimation



* Interpolator (up-sampling)

This is a process of increasing the sampling rate by an integer factor of L . The interpolation adds $L-1$ new samples b/w successive value of the signal.

Let $x(n)$ and $y(n)$ be N -dp. The N -dp relation of up-sampler by applying IT

$$y(n) = x\left(\frac{n}{L}\right) \quad ; \quad n = 0, \pm L, \pm 2L, \dots$$

otherwise.

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{L}\right) \cdot z^{-n} = \sum_{m=-\infty}^{\infty} x(m) \cdot z^{-Lm}$$

$$= X(z^L) \quad \text{--- (1)} \quad z = e^{j\omega}$$

$$\therefore Y(e^{j\omega}) = X(e^{j\omega L}) \quad \text{--- (2)}$$

freq variable ω_x and ω_y are related by

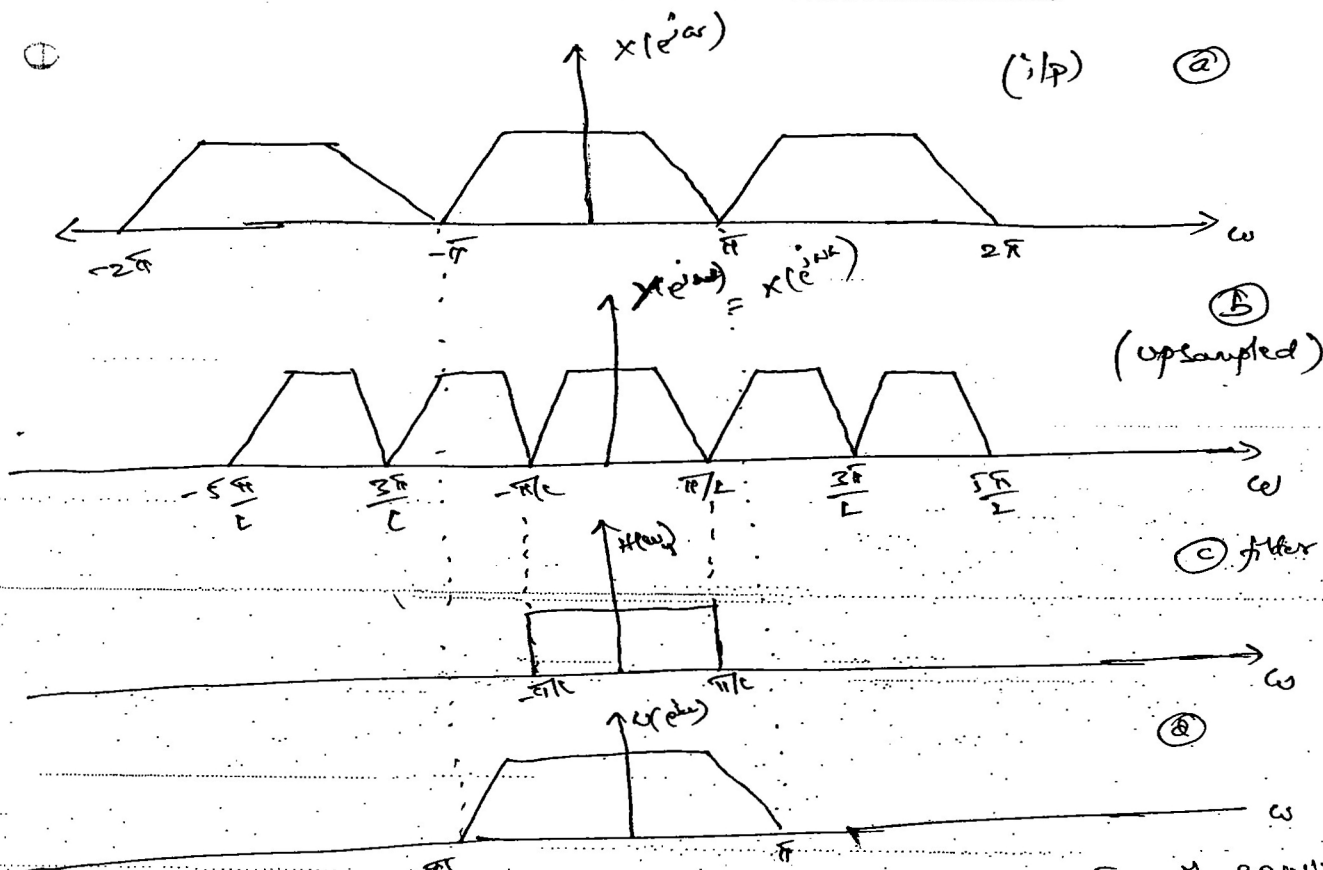
$$\boxed{\omega_y = \frac{\omega_x}{L}}$$

$$\text{(or)} \quad \boxed{F_y = L \cdot F_x}$$

Inserting $L-1$ zeros b/w successive value of $x(n)$ results in a signal whose spectrum $X(e^{j\omega L})$ is an L -fold periodic repetition of N -dp signal spectrum $X(e^{j\omega})$.

Fig (a) and (b)

UNIT - VII: MULTIRATE DIGITAL SIGNAL PROCESSING FUNDAMENTALS



The original spectral range is $-\pi \leq \omega_x \leq \pi$, it results in repeated of spectral range $-\frac{\pi}{L} \leq \omega_y \leq \frac{\pi}{L}$. The periodic repetition of original spectra $x(e^{j\omega})$ are the images of original spectra $x(e^{j\omega})$. So these images can be rejected by passing the sequence $y(n)$ through a LPF whose freq response $H(\omega)$ is given by fig (c)

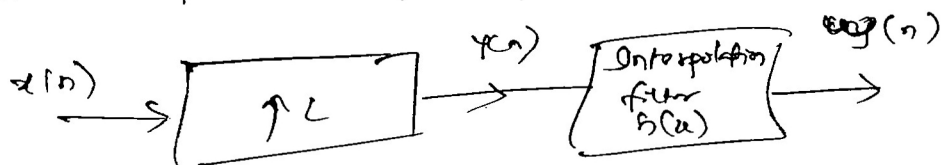
$$H(\omega) = k \quad 0 \leq |\omega| \leq \frac{\pi}{L}$$

$$= 0 \quad \text{otherwise}$$

k is a scaling factor for normalization purpose. The original sequence after passing through LPF is given by fig (d)

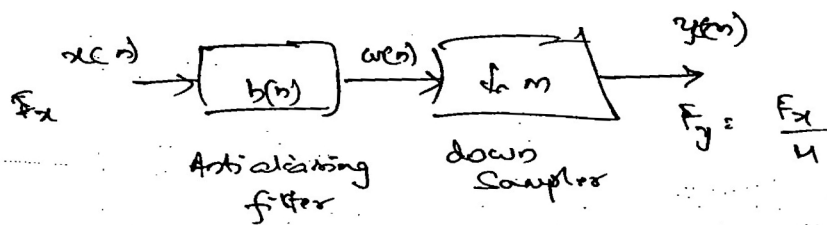
$$y(e^{j\omega}) = \begin{cases} k \cdot x(e^{j\omega L}) & 0 \leq |\omega| \leq \frac{\pi}{L} \\ 0 & \text{otherwise} \end{cases}$$

Required interpolation for perfect reconstruction



⇒ O/P is of

① Decimator



filter op is convolution of $x(n)$ & $b(n)$

$$\Rightarrow w(n) = \sum_{k=0}^{\infty} b(k) x(n-k)$$

O/P of filter is down sampled by factor M resulting in $y(n)$

$$\Rightarrow y(m) = w(Mm)$$

$$\Rightarrow y(m) = \sum_{k=0}^{\infty} b(k) x(Mm-k)$$

Even though filtering operation is linear and time invariant,
the down sampling operation in combination with filtering
results in time-variant system but only linear

$$y(n) = x(nM)$$

$$y(m, k) = x(mM - k)$$

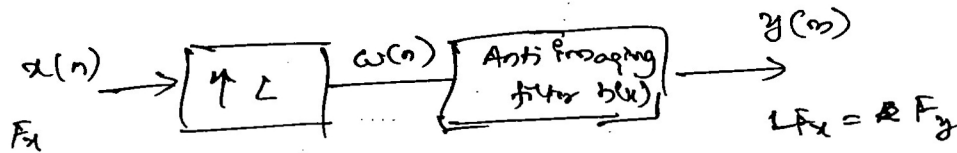
$$y(m-k) = x(m(M-k))$$

m - dummy variable

$$y(m, k) \neq y(m-k)$$

∴ time-variant.

① Interpolator



$$y(n) = \sum_{k=-\infty}^{\infty} h(n-k) w(k)$$

Since $w(k) = 0$ except at multiples of L , where

$$w(Lk) = x(k)$$

$$\Rightarrow y(n) = \sum_{k=-\infty}^{\infty} h(n-kL) x(k)$$

Up Sampling in combination with filtering operation is time-variant even though filtering LTI.

$$y(n) = x\left(\frac{n}{L}\right)$$

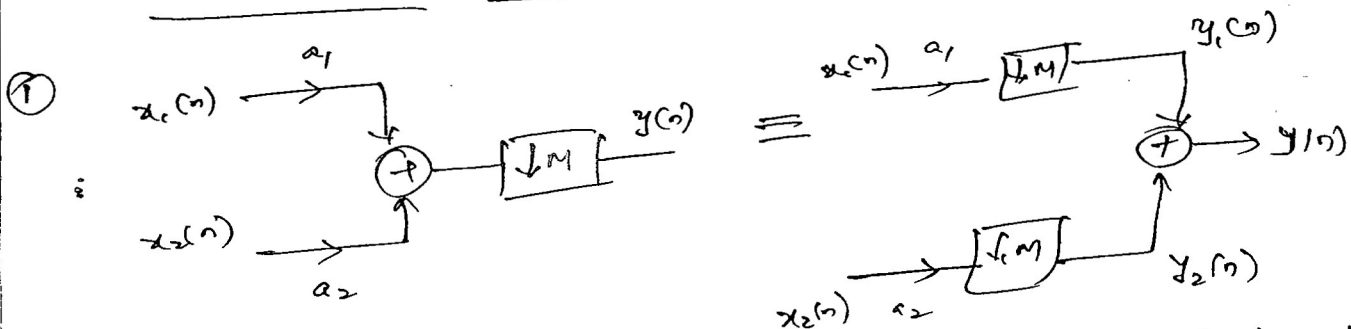
$$y(n, k) = x\left(\frac{n}{L} - k\right)$$

$$y(n-k) = x\left(\frac{n-k}{L}\right)$$

$$y(n, k) \neq y(n-k)$$

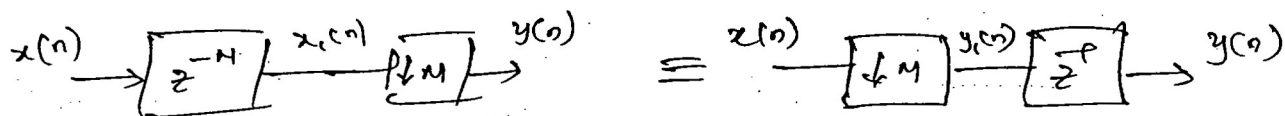
Hence time-variant.

$\Rightarrow$ Important Identities of basic sample rate devices



Scaling of signals and their addition at nodes are independent of sampling rates.

② A delay of M sample periods before a downsampler is the same as delay of one sample period after downsampler.



$$x_1(z) = z^{-M} X(z)$$

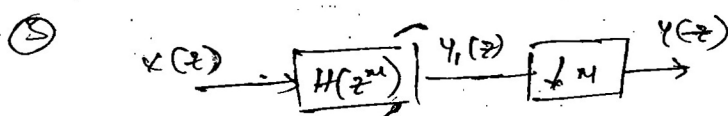
$$y_1(z) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(e^{j\frac{2\pi k}{M}} \cdot \frac{1}{z}\right)$$

$$y(z) = \frac{1}{M} \sum_{k=0}^{M-1} x_1\left(e^{j\frac{2\pi k}{M}} \cdot \frac{1}{z}\right)$$

$$y(z) = \frac{1}{M} \cdot z \sum_{k=0}^{M-1} X\left(e^{j\frac{2\pi k}{M}} \cdot \frac{1}{z}\right)$$

$$= \frac{1}{M} \sum_{k=0}^{M-1} z^{-M} X\left(e^{j\frac{2\pi k}{M}} \cdot \frac{1}{z}\right)$$

$$= \frac{1}{M} \sum_{k=0}^{M-1} z^{-1} X\left(e^{j\frac{2\pi k}{M}} \cdot \frac{1}{z}\right)$$



$$y_1(z) = x(z) \cdot H(z^M)$$

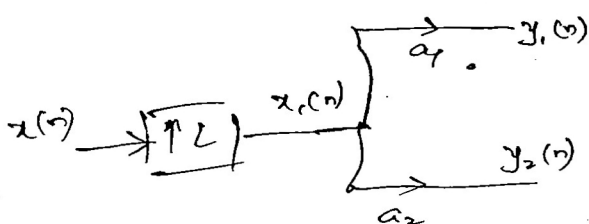
$$y(z) = \frac{1}{M} \sum_{k=0}^{M-1} y_1\left(e^{j\frac{2\pi k}{M}} \cdot \frac{1}{z}\right)$$

$$x_1(z) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(e^{j\frac{2\pi k}{M}} \cdot \frac{1}{z}\right)$$

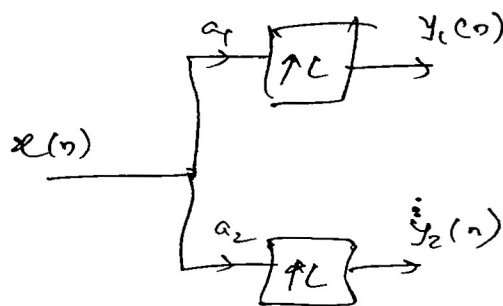
$$y(z) = H(z) \cdot \frac{1}{M} \sum_{k=0}^{M-1} X\left(e^{j\frac{2\pi k}{M}} \cdot \frac{1}{z}\right)$$

$$= H(z) \cdot \frac{1}{M} \sum_{k=0}^{M-1} X\left(e^{j\frac{2\pi k}{M}} \cdot \frac{1}{z}\right)$$

③ Scaling of D.T. Signals and their addition at nodes are independent of sampling rate.



$\equiv$



$$x_1(z) = X(z^2)$$

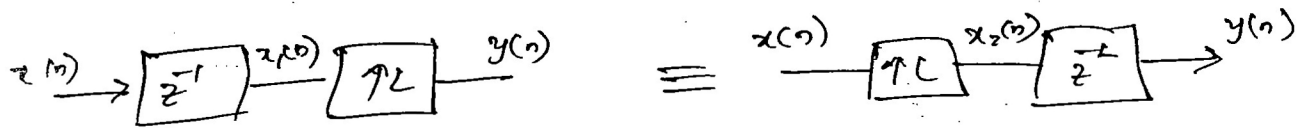
$$y_1(z) = a_1 X(z^2)$$

$$y_1(z) = a_1 X(z^2)$$

$$y_2(z) = a_2 X(z^2)$$

$$y_2(z) = a_2 X(z^2)$$

③ A delay of one sample before upsampling leads to a delay of L periods after upsampling

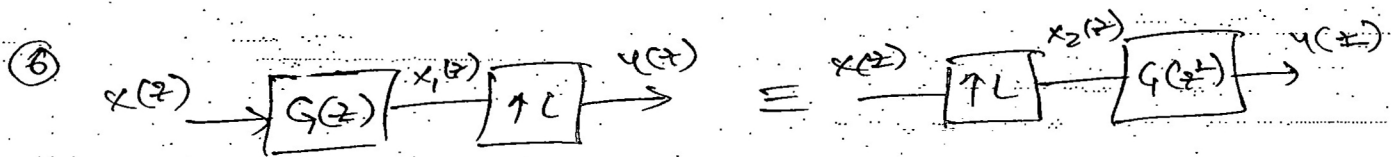


$$x_1(z) = z^{-1} x(z)$$

$$y(z) = x_1(z^L) = z^{-L} x(z^L)$$

$$x_2(z) = x(z^L)$$

$$y(z) = z^{-L} x(z^L)$$



$$x_1(z) = x(z) \cdot G(z)$$

$$y(z) = x_1(z^L) = x(z^L) G(z^L)$$

$$x_2(z) = x(z^L)$$

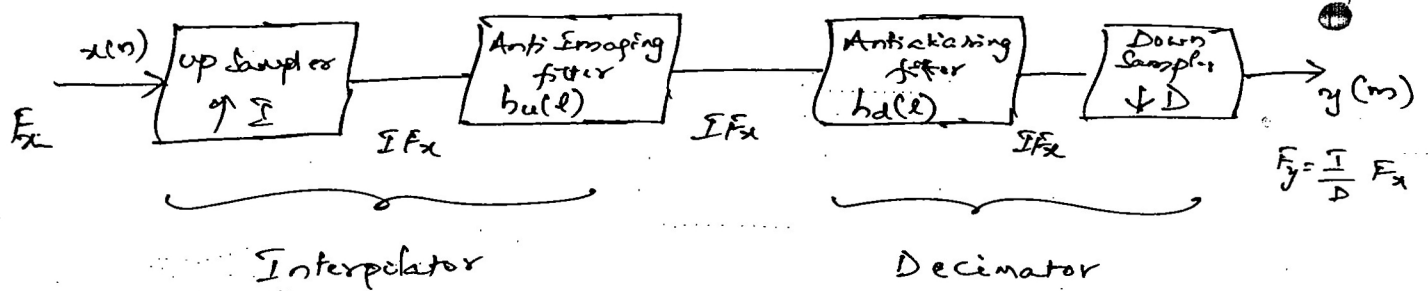
$$y(z) = x_2(z) \cdot G(z^L) = x(z^L) \cdot G(z^L)$$

⇒ Multirate structures for Sampling rate Conversion :-

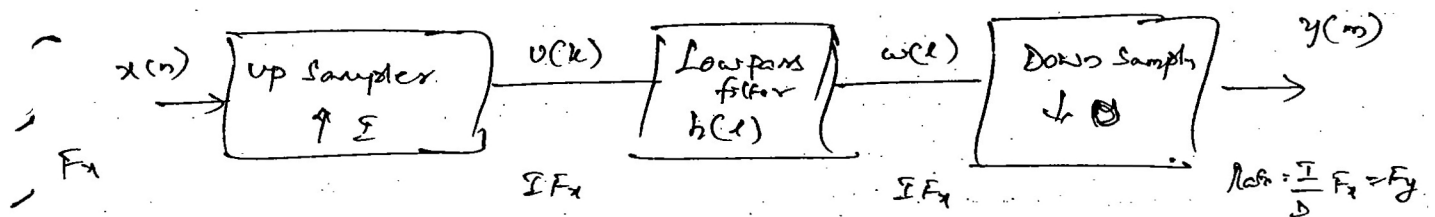
(Sampling rate conversion by a rational factor I/D) or (H_m)

A sampling rate conversion by a rational factor I/D or (H_m) can be obtained by first interpolating $x[n]$ by factor I and then decimating $x_1[n]$ of interpolator by factor D .

In other words a sampling rate conversion by a rational factor I/D is accomplished by cascading an interpolator with a decimator as shown below.



The two filters are operated at same sampling rate i.e. IF_x and hence can be combined into a single low pass filter with impulse response $h(l)$. The frequency response of combined filter must incorporate the filtering operations for both interpolation and decimation. Hence its cut off freq $\omega_c = \min\left(\frac{\pi}{I}, \frac{\pi}{D}\right)$ i.e. it has to eliminate spectral images and avoid aliasing.



Interpolation is first performed and decimation second ~~so as~~ to preserve the desired spectral characteristics of $x(n)$

In time domain o/p of up sampler is

$$U(l) = \begin{cases} x(l/I) & l = 0, \pm I, \pm 2I, \dots \\ 0 & \text{otherwise} \end{cases}$$

o/p of linear time invariant filter is

$$\begin{aligned} w(l) &= h(l) * U(k) \\ &= \sum_{k=-\infty}^{\infty} h(l-k) U(k) = \sum_{k=-\infty}^{\infty} h(l-kI) x(k) \end{aligned}$$

① Finally, the op of sampling rate converter is sequence $y(n)$ which is obtained by downsampling the sequence $u(n)$ by factor D .

$$y(n) = u(nD)$$

$$= \sum_{k=-\infty}^{\infty} h(nD - kI) x(k)$$

Frequency domain :- Relationships can be obtained by combining results of interpolation and decimation processes.

Spectrum of op of linear filter with response $h(n)$ is

$$Y(e^{j\omega_y}) = H(e^{j\omega_y}) \cdot X(e^{j\omega_y I})$$

$$= \begin{cases} I X(e^{j\omega_y I}) & 0 \leq |\omega_y| \leq \min\left(\frac{\pi}{D}, \frac{\pi}{I}\right) \\ 0 & \text{otherwise} \end{cases}$$

Spectrum of op sequence $y(n)$ is obtained by decimating sequence $v(n)$ by factor D .

$$Y(e^{j\omega_y}) = \frac{1}{D} \sum_{k=0}^{D-1} V\left(e^{j(\omega_y - 2\pi k/D)}\right)$$

$$\omega_y = D \omega_x$$

the spectrum of op

Linear filter prevents aliasing, the sequence is given by

$$Y(e^{j\omega_y}) = \begin{cases} \frac{I}{D} X\left(e^{j\frac{\omega_y}{D}}\right) & 0 \leq |\omega_y| \leq \min\left(\pi, \frac{\pi D}{I}\right) \\ 0 & \text{otherwise} \end{cases}$$

Prob:- The spectrum of a signal $x(n)$ is symmetrical triangular pulse with amplitude of 2 and frequency boundaries are -0.2 to 0.2 . Sketch the spectrum and spectrum of

- (i) Zero interpolated signal $y(n) = x(n/2)$
- (ii) Decimated signal $d(n) = x(2n)$
- (iii) Signal $g(n)$ that equals to $x(n)$ for even n , and zero for odd n .

Solⁿ: Give symmetrical triangular pulse with amplitude of 2 and freq boundaries are -0.2 to 0.2 .

- (i) Interpolated signal $L=2$

$$\text{or } Y(e^{j\omega}) = X(e^{j\omega L}) = X(e^{j2\omega})$$

- (ii) Decimated signal $d(n) = x(2n)$; $M=2$

$$\text{or } Y(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X(e^{j(\omega - 2\pi k/M)})$$

$$= \frac{1}{2} \sum_{k=0}^1 X(e^{j(\omega - \frac{2\pi k}{2})})$$

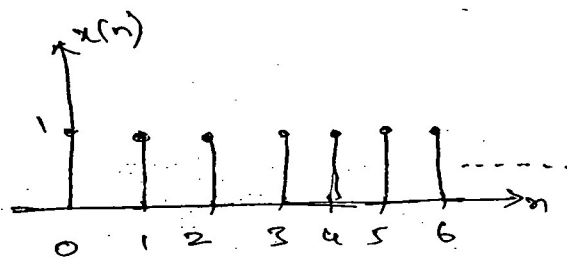
$$= \frac{1}{2} \left[X(e^{j\omega/2}) + X(e^{-j\omega/2}) \right]$$

Prob: Consider a signal $x(n) = u(n)$

- i) obtain a signal with decimation factor 3
- ii) obtain a signal with interpolation factor 3

Sol: $x(n) = u(n)$

$$x(n) = \{1, 0, 1, 1, \dots\}$$



(i)

$$y(n) = x\left(\frac{n}{3}\right)$$

$$y(0) = x(0) = 1$$

$$y(1) = x(1/3) = 0$$

$$y(2) = x(2/3) = 0$$

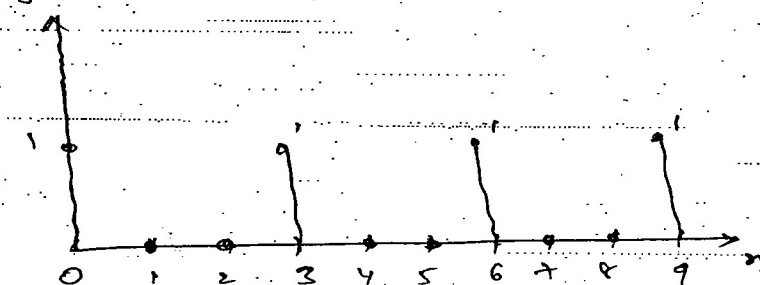
$$y(3) = x(1) = 1$$

$$y(4) = x(4/3) = 0$$

$$y(5) = x(5/3) = 0$$

$$y(6) = x(2) = 1 \text{ and so on}$$

$$y(n) = x\left(\frac{n}{3}\right)$$



(ii)

$$y(n) = x(3n)$$

$$y(0) = x(0) = 1$$

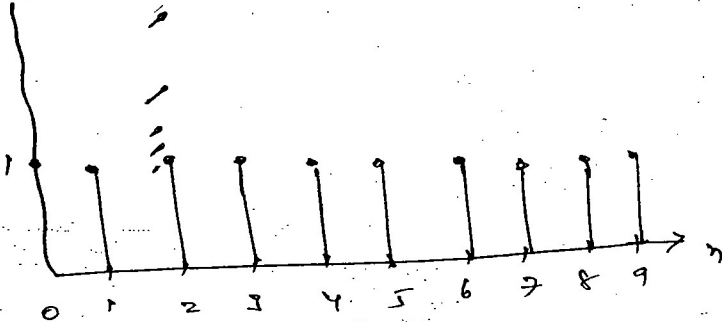
$$y(1) = x(3) = 1$$

$$y(2) = x(6) = 1$$

$$y(3) = x(9) = 1$$

and so on

$$y(n) = x(3n)$$



Prob: Consider a signal $x(n) = \sin \pi n \cdot u(n)$

- (i) Obtain a signal with decimation factor 2
- (ii) Obtain a signal with interpolation factor 2

Sol: $x(n) = \sin \pi n \cdot u(n)$

$$u(n) = 1 \quad n \geq 0$$

$$= 0 \quad n < 0$$

So $x(n) = \sin \pi n \quad n \geq 0$

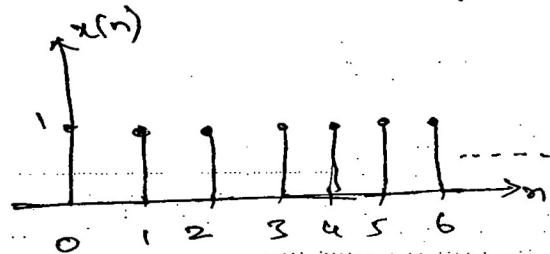


Prob: Consider a signal $x(n) = u(n)$

- i) obtain a signal with decimation factor '3'
- ii) obtain a signal with interpolation factor 3

Sol: $x(n) = u(n)$

$$x(n) = \{1, 0, 1, 1, \dots\}$$



(i)

$$y(n) = x\left(\frac{n}{3}\right)$$

$$y(0) = x(0) = 1$$

$$y(1) = x(1/3) = 0$$

$$y(2) = x(2/3) = 0$$

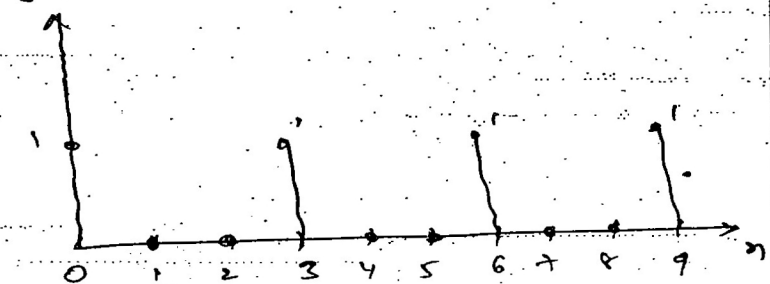
$$y(3) = x(1) = 1$$

$$y(4) = x(4/3) = 0$$

$$y(5) = x(5/3) = 0$$

$$y(6) = x(2) = 1 \text{ and so on}$$

$$y(n) = x\left(\frac{n}{3}\right)$$



(ii)

$$y(n) = x(3n)$$

$$y(0) = x(0) = 1$$

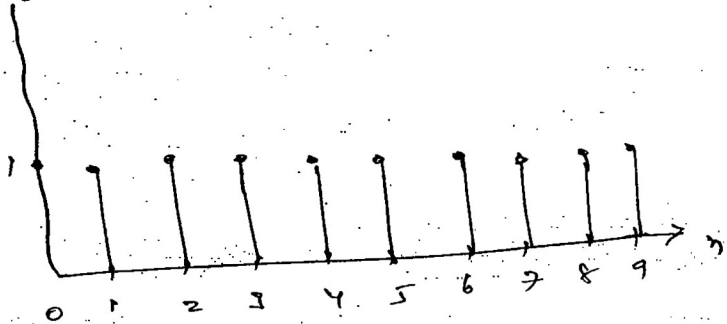
$$y(1) = x(3) = 1$$

$$y(2) = x(6) = 1$$

$$y(3) = x(9) = 1$$

and so on

$$y(n) = x(3n)$$



Prob: Consider a signal $x(n) = \sin \pi n \cdot u(n)$

- (i) Obtain a signal with decimation factor 2
- (ii) Obtain a signal with interpolation factor 2

Sol: $x(n) = \sin \pi n \cdot u(n)$

$$u(n) = 1 \quad n \geq 0$$

$$= 0 \quad n < 0$$

So $x(n) = \sin \pi n \quad n \geq 0$



Prob: Sketch the following signals

(i) $x_1(n) = n \quad n > 0$
 $= 0 \quad \text{otherwise}$

(ii) $x_2(n) = n^2 \quad n > 0$
 $= 0 \quad \text{otherwise}$

Also sketch decimated and interpolated version of above signals with factor of 2.

Sol: $x_1(n) = n \quad n > 0$
 $= 0 \quad \text{otherwise}$

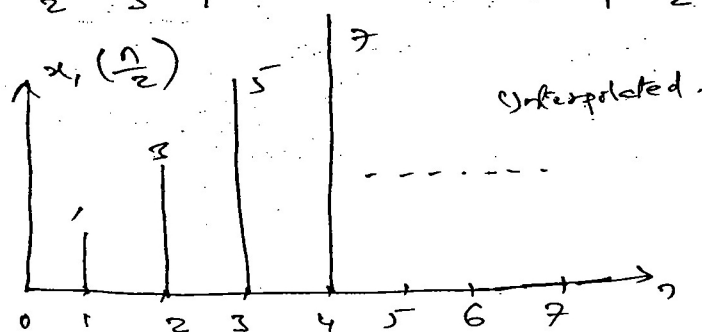
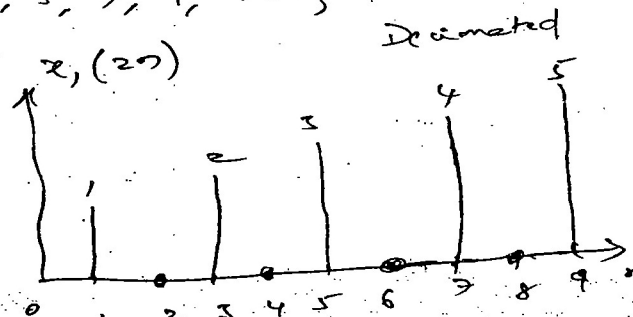
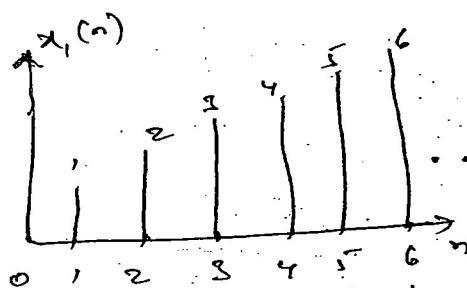
$x_1(n) = \{ 1, 2, 3, 4, 5, 6, 7, \dots \}$

Interpolated $x_1(n)$ with factor 2

$x_1\left(\frac{n}{2}\right) \text{ (or) } (\uparrow 2) x_1(n) = \{ 1, 0, 2, 0, 3, 0, 4, 0, 5, 0, 6, 0, 7, 0, \dots \}$

Decimated version of $x_1(n)$ with factor 2

$x_1(2n) \text{ (or) } (\downarrow 2) x_1(n) = \{ 1, 3, 5, 7, 9, \dots \}$



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(ii) $x_2(n) = n^2 \quad n > 0$
 $= 0 \quad \text{otherwise}$

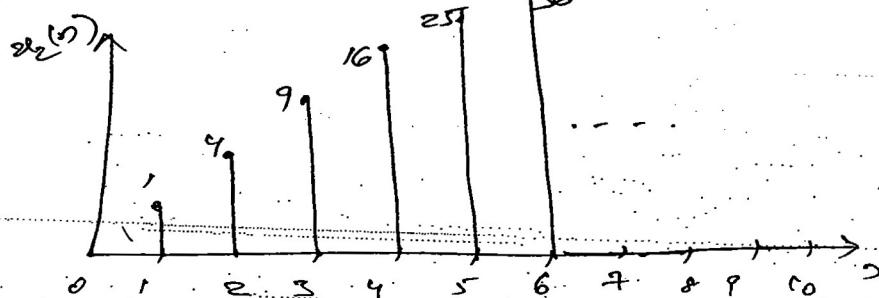
$x_2(n) = \{ 1, 4, 9, 16, 25, 36, \dots \}$

Interpolated

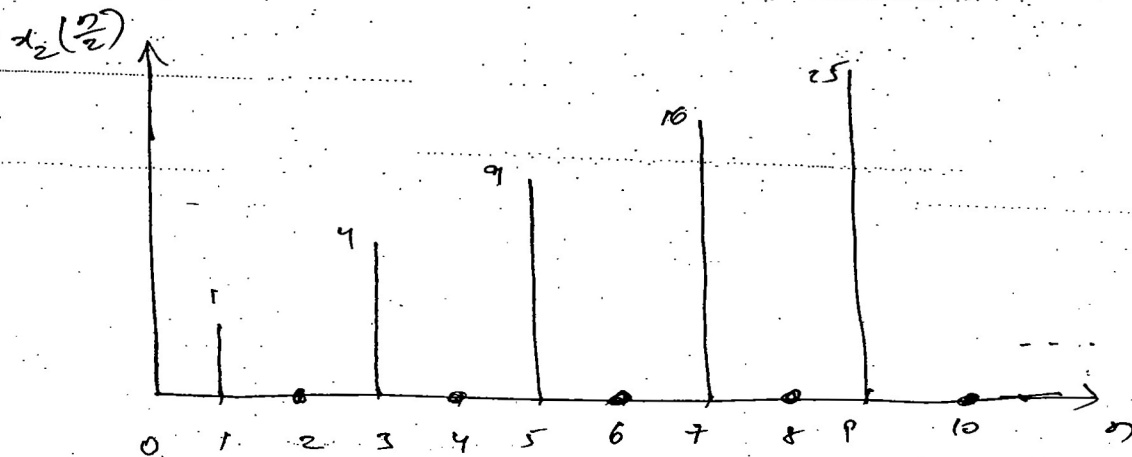
$$x_2\left(\frac{n}{2}\right) = \{1, 0, 4, 0, 9, 0, 16, 0, 25, 0, 36, \dots\}$$

Decimated

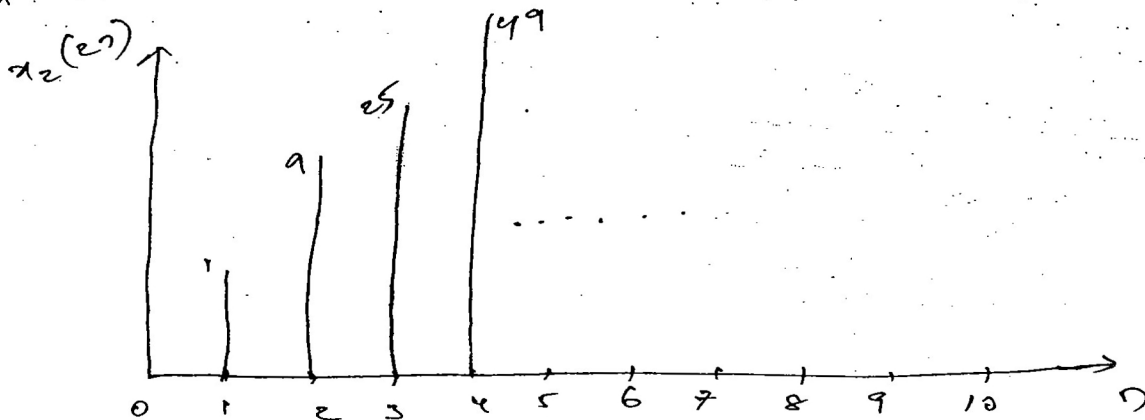
$$x_2(2n) = \{1, 9, 25, 49, \dots\}$$



Interpolated



Decimated



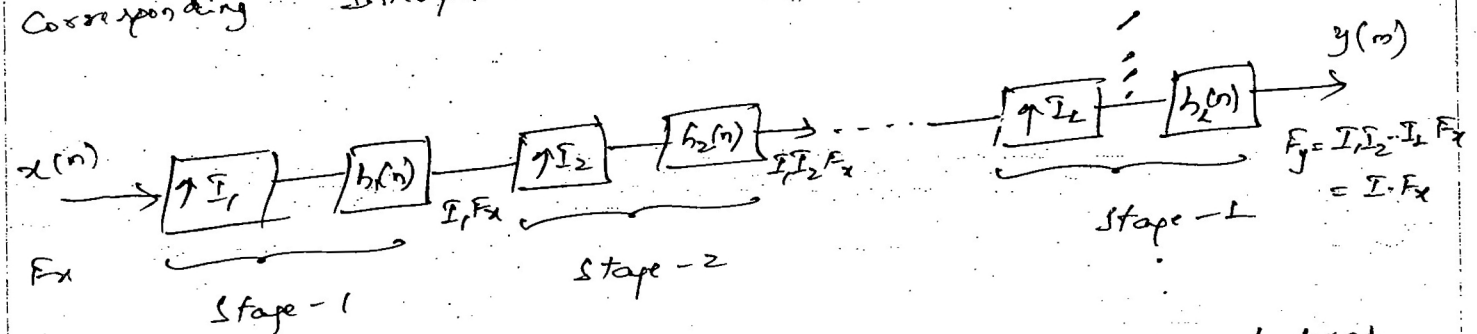
• Multistage Implementation of Sample Rate Conversion (Multistage design of decimator and interpolator)

In practical applications of sampling rate conversion, decimation factors and interpolation factors are much larger than unity (i.e. $D \gg 1$ & $I \gg 1$). So, implementing sample rate conversion in a single stage is computationally inefficient. $\therefore$ performing sampling rate conversion for $I \gg 1$ & $D \gg 1$ in multistages is efficient.

* Interpolation by factor $I \gg 1$. Assume that I can be factored into a product of 'r' integers as

$$I = \prod_{i=1}^L I_i$$

Then interpolation can be achieved by cascading L stages of interpolation and filtering as shown. Filter in each stage eliminates images introduced by up-sampling process in corresponding interpolator.



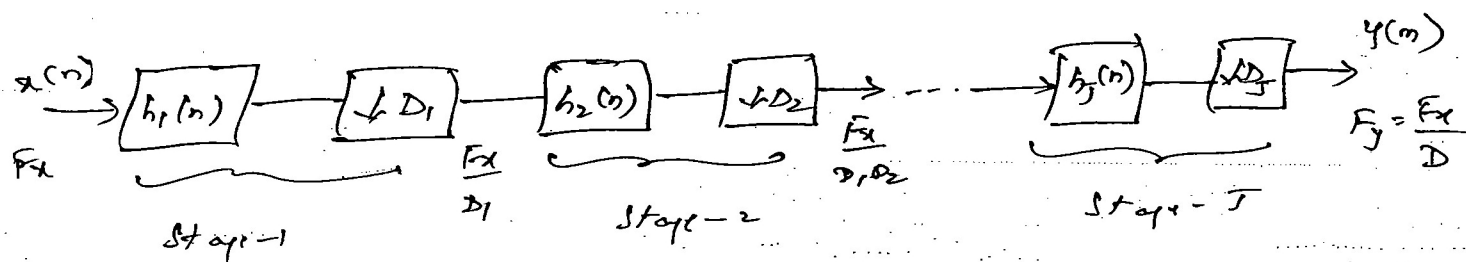
* Decimation by factor $D \gg 1$. If D can be factored into a product of 'r' integers as

$$D = \prod_{i=1}^J D_i$$

Decimation can be achieved by Cascade of J stages as shown. The sampling rate at the i -th stage is

$$F_i = \frac{F_{i-1}}{D_i} \quad i = 1, 2, \dots, J$$

To ensure no aliasing occurs in overall decimation process, filters are designed to avoid aliasing within the frequency band of interest.



⇒ Specifications :-

Let us define desired PB and TB in overall decimator as

$$PB : 0 \leq F \leq F_{pc}$$

$$TB : F_{pc} \leq F \leq F_{sc}$$

Where $F_{sc} \leq \frac{F_x}{2D}$. Then aliasing in band $0 \leq F \leq F_{sc}$ is avoided by selecting freq bands of each filter stage as

$$PB : 0 \leq F \leq F_{pc}$$

$$TB : F_{pc} \leq F \leq F_c - F_{sc}$$

$$SB : F_c - F_{sc} \leq F \leq \frac{F_c - 1}{2}$$

By designing a filter with in subsequent stages to satisfy above specifications, we can ensure that no aliasing occurs in primary freq band $0 \leq F \leq F_{sc}$.

Q. 2011
Prob

Implement a two stage decimator for the following specifications. Sampling rate of $M = 20,000 \text{ Hz}$. $M = 100$

$$PB = 0 \text{ to } 50 \text{ Hz}$$

$$SB \text{ ripple} = 0.002$$

$$TB = 50 \text{ to } 70 \text{ Hz}$$

$$PB \text{ ripple} = 0.01$$

For the case of multistage interpolator, the sampling rate at the i^{th} of the i^{th} stage is

$$F_{i+1} = \sum_i F_i \quad i = 1, 2, \dots, I$$

and the i^{th} rate is $F_0 = \sum F_i$ when the i^{th} sampling rate is F_i . The corresponding frequency

band specifications are

Pass band: $0 \leq F \leq F_{pc}$

Transition band: $F_{pc} < F \leq F_{sc} - F_{pc}$

Problem: Implement a two stage decimator for the following specifications. Sampling rate of $x(n)$ signal = 8 kHz.

Pass band: $0 \leq F \leq 75$

PB ripple = $\delta_1 = 0.01$ (10^{-2})

Transition band: $75 \leq F \leq 80$

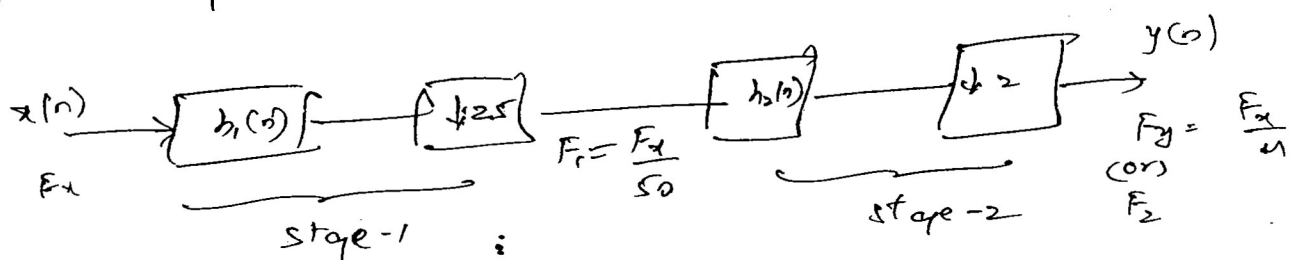
SB ripple = $\delta_2 = 0.0001$ (10^{-4})

soln. Give data $F_{pc} = 75 \text{ Hz}$; $F_{sc} = 80 \text{ Hz}$
 $\delta_1 = 0.01$; $\delta_2 = 0.002$
 $F_s = 8000 \text{ Hz}$

We know $\boxed{F_{sc} \leq \frac{F_x}{2D}} \Rightarrow D = \frac{F_x}{2F_{sc}} = \frac{8000}{2 \times 80} = 50$

Let $D = D_1 D_2 = 25 \times 2 \Rightarrow D_1 = 25$
 $D_2 = 2$

First stage decimator is



1st stage : $D_1 = 25$

$$F_1 = \frac{F_s}{25} = 320 \text{ Hz}$$

Specifications of 1st stage filter $\delta_{11} = \frac{\delta_1}{2}$; $\delta_{21} = \delta_2$

$$\text{PB: } 0 \leq F \leq \frac{F_s}{PCL} \Rightarrow 0 \leq F \leq 75 \text{ Hz}$$

$$\text{TB: } F_{PC} \leq F \leq F_1 - F_{SC} \Rightarrow 75 \leq F \leq 240 \text{ Hz}$$

$$\text{SB: } F_1 - F_{SC} \leq F \leq \frac{F_D}{2} \Rightarrow 240 \leq F \leq 4000 \text{ Hz}$$

2nd stage : $D_2 = 2$; $F_2 = \frac{F_1}{D_2} = \frac{320}{2} = 160 \text{ Hz}$

Specifications of 2nd stage filter $\delta_{12} = \frac{\delta_1}{2}$; $\delta_{22} = \delta_2$

$$\text{PB: } 0 \leq F \leq F_{PC} \Rightarrow 0 \leq F \leq 75 \text{ Hz}$$

$$\text{SB: } F_P \leq F \leq F_2 - F_{SC} \Rightarrow 75 \text{ Hz} \leq F \leq 80 \text{ Hz}$$

i.e. There is no aliasing in freq band $0 \leq F \leq F_{SC} = 800 \text{ Hz}$
 but aliasing occurs above freqs $F_{ac} = 80 \text{ Hz}$.

The above are freq band specifications of $h_1(n)$ and $h_2(n)$
 Now we have to find the length of linear phase FIR filter that satisfy the above specifications.

$$ND = \frac{-10 \log_{10} \delta_1 \delta_2 - 13}{14.6 \Delta f} + 1$$

$$\Delta f = \frac{(F_{SC} - F_P)}{F_{band}}$$

1st stage : $\delta_{11} = \frac{\delta_1}{2}$ (so that PB ripple is equal to 2-stage)

$\delta_{21} = \delta_2$ (SB ripple is same for two stages)

$$\text{1st stage filter length } ND_1 = \frac{-10 \log_{10} \left(\frac{\delta_1}{2}\right) (\delta_2) - 13}{14.6 \left(\frac{165}{8000}\right)} \approx 167$$

UNIT - VII: MULTIRATE DIGITAL SIGNAL PROCESSING FUNDAMENTALS

For 2nd stage $N_2 = \frac{-10 \log_{10} \left(\frac{\delta_1}{2} \right) (\delta_2) - 13}{14.6 \times \Delta f} + 1$

$$\Delta f = \left(\frac{f}{320} \right)$$

$$N \approx 220.$$

Total length of AP FIR = $167 + 220 = \underline{387}$

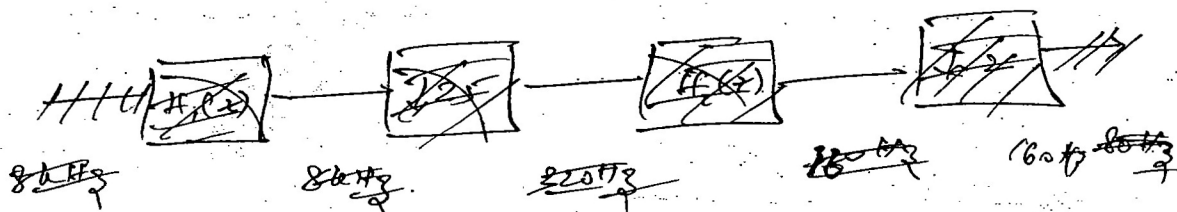
If for 1st stage $N = \frac{-10 \log_{10} \delta_1 \delta_2 - 13}{14.6 \cdot \Delta f} + 1$

$$\Delta f = \frac{f}{8000}$$

$$N \approx 5152 \quad \underline{3728}$$

Conclusion: Efficient implementation is possible in multistage than in single stage

Overall multistage decimator is shown below.



prob: design a multistage decimator structure to reduce the sampling rate from 10 kHz to 500 Hz. The specifications of decimator filter $H(z)$ are

PB freq: $f_p = 1500 \text{ Hz}$; PB ripple $\delta_p = 0.022$

SB freq: $f_s = 1750 \text{ Hz}$; SB ripple $\delta_s = 0.001$

Soln: Note: For decimator and Interpolator order is

length of filter $N = \frac{-10 \log_{10} \delta_p \delta_s - 13}{14.6 \Delta f} + 1$ is same

$$\Delta f = \frac{f_s - f_p}{N}$$

Order of decimator filter $H(z)$ is (single stage)

$$N = \frac{-10 \log_{10} \delta_p \delta_s - 13}{14.6 \Delta f} + 1 \quad (\text{or}) \quad \frac{-20 \log_{10} \sqrt{\delta_p \delta_s} - 13}{14.6 \Delta f} + 1$$

$$\Delta f = \frac{f_s - f_p}{f_{\text{dec}}} = \frac{175 - 150}{10 \text{ kHz}} = 2.5 \times 10^{-3}$$

$$N = \frac{-10 \log_{10} (0.002 \times 0.001) - 13}{14.6 \times 2.5 \times 10^{-3}} + 1$$

$$= 1205.2 \approx 1206$$

Decimation factor $D = \frac{10 \times 10^3}{500} = 20$

So, for $D = 20$, order of filter is 1206.

Two stage : $D = 10 \times 2$; $D_1 = 10$; $D_2 = 2$

Filter specifications are of 1st stage $F_0 = 10 \times 10^3 \text{ Hz}$

PB: $0 \leq F \leq 150$

TB: $150 \leq F \leq F_1 - F_{sc}$
 $= 825 \text{ Hz}$

SB: $825 \leq F \leq \frac{F_0}{2} = 5000 \text{ Hz}$

$\delta_{p1} = \frac{\delta_p}{2}$; $\delta_{s1} = \delta_s$

order of filter in 1st stage

$$N_1 = \frac{-10 \log_{10} \left(\frac{\delta_p}{2}\right) (\delta_s) - 13}{14.6 \Delta f}$$

$$= 47.6 \approx 48$$

$$\Delta f = \frac{825 - 150}{10000} = 0.0675$$

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Second stage filter specifications

$$PB: 0 \leq F \leq 150 \text{ Hz}$$

$$TB: 150 \text{ Hz} \leq F \leq F_2 - F_c = 325 \text{ Hz}$$

$$F_2 = \frac{F_1}{D_2} = 500 \text{ Hz}$$

$$\delta_{p2} = \frac{\delta_p}{2} = 0.001; \quad \delta_{s2} = \delta_s = 0.001$$

Order of filter $N_2 = \frac{-10 \log_{10} \left(\frac{\delta_p}{\delta_s} \right) \delta_p - 13}{14.6 \Delta f} + 1$

$$\Delta f = \frac{f_s - f_p}{1000} = \frac{325 - 150}{1000} = 17.5 \times 10^{-3}$$

$$N_2 = 183.8 + 1 \approx 185$$

Total length of two filters = $185 + 48 = 233$

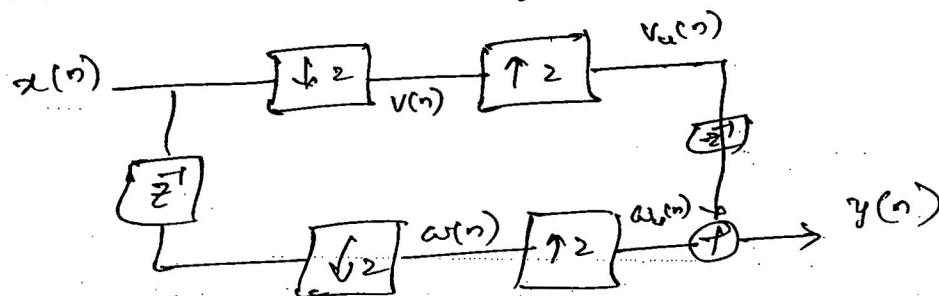
Reduction achieved = $\frac{1206}{233} = 5.17 \text{ (factor)}$

Number of multiplications for a filter is = $\frac{N \times F_{\text{sample}}}{D}$



UNIT - VII: MULTIRATE DIGITAL SIGNAL PROCESSING FUNDAMENTALS

Prob: A multirate system is shown below. Find the relation b/w $x(n)$ and $y(n)$



Soln: After down sampling $v(n) = x(2n)$
 up-sampling $u_u(n) = x(n/2)$
 $= x(n)$ for $n = 0, \pm 2, \pm 4, \dots$
 $= 0$ otherwise.

Delay $x(n)$ we get $x(n-1)$
 after down sample $w(n) = x(2n-1)$
 up sample $u_d(n) = w(n/2)$ for $n = 0, \pm 2, \pm 4, \dots$
 $= 0$ otherwise
 $u_d(n) = x(n-1)$ for $n = 1, \pm 3, \pm 5, \dots$
 $= 0$ otherwise.

overall o/p $y(n) = u_u(n-1) + u_d(n)$

$$u_u(n) = \{x(0), 0, x(2), 0, x(4), 0, \dots\}$$

$$u_u(n-1) = \{0, x(0), 0, x(2), 0, x(4), 0, \dots\}$$

$$u_d(n) = \{x(-1), 0, x(1), 0, x(3), \dots\}$$

$$u_u(n-1) + u_d(n) = \{x(-1), x(0), x(1), x(2), x(3), \dots\}$$

$$= x(n-1)$$

In z-domain

L-factor up sampler $Y(z) = X(z^L)$

M-factor down sampler $Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(\frac{z}{M} e^{j\frac{2\pi k}{M}}\right)$

$$V(z) = \frac{1}{2} \sum_{k=0}^1 X\left(z^k e^{-j\frac{2\pi k}{M}}\right)$$

$$= \frac{1}{2} X\left(\frac{z}{2}\right) + \frac{1}{2} X\left(-\frac{z}{2}\right)$$

$$V_u(z) = \frac{1}{2} X(z) + \frac{1}{2} X(-z)$$

$W(z) =$ Z.T of $x(n-1)$ decimated by factor 2

$$= \frac{1}{2} \left[\frac{1}{2} X\left(\frac{z}{2}\right) + \frac{1}{2} X\left(-\frac{z}{2}\right) \right]$$

$$W_u(z) = \frac{1}{2} \left[\frac{1}{2} X(z) + \frac{1}{2} X(-z) \right]$$

or $Y(z) = \frac{1}{2} V_u(z) + W_u(z)$

$$= \frac{1}{2} \left[\frac{1}{2} X(z) + \frac{1}{2} X(-z) \right] + \frac{1}{2} X(z) - \frac{1}{2} X(-z)$$

$$= \frac{1}{2} X(z)$$

taking Inverse.

$$\Rightarrow \boxed{y(n) = x(n-1)}$$

⇒ Filter Design and Implementation for Sampling Rate Conversion

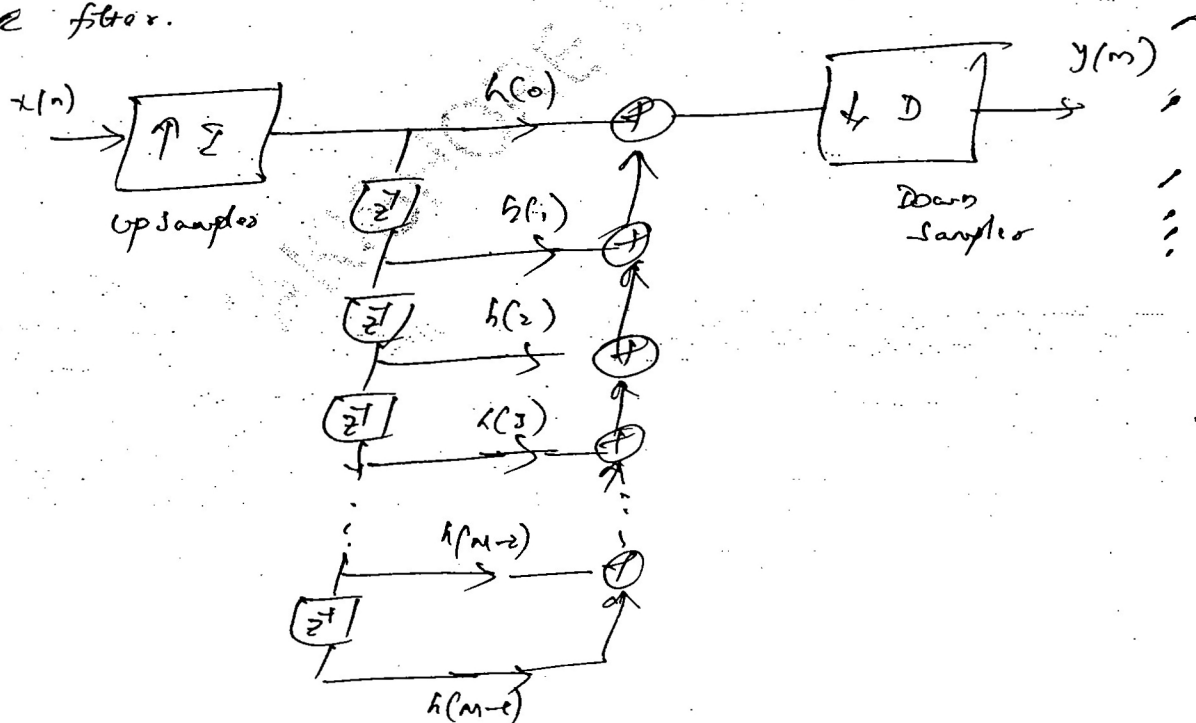
Sampling rate conversion by factor I/D can be achieved by first increasing the sampling rate by I , followed by linear filtering to eliminate unwanted images of $x(e^{j\omega})$ and finally down sampling by factor D .

① Direct form filter - FIR

Simplest realization of filter is direct form FIR structure with system function.

$$H(z) = \sum_{k=0}^{M-1} h(k) \cdot z^{-k}$$

$\{h(k)\}$ is unit sample response of FIR filter. The LPF should have linear phase, specified pass ripple and stop attenuation. Methods like window, frequency sampling etc can be used to design FIR filter.

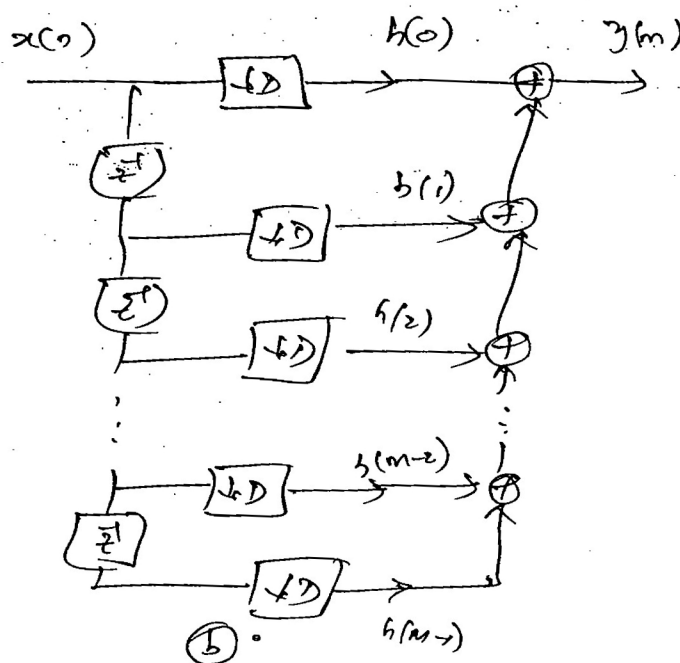
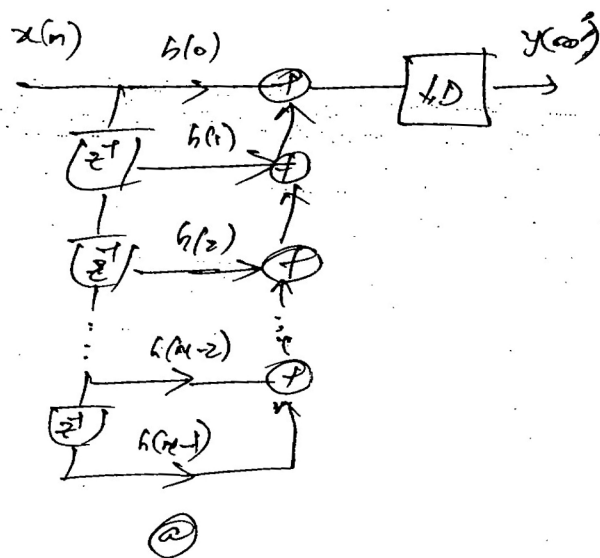


Above direct form FIR realization is simple but inefficient. This is because up sampling process introduces $I-1$ zeros b/w successive points of $x[n]$

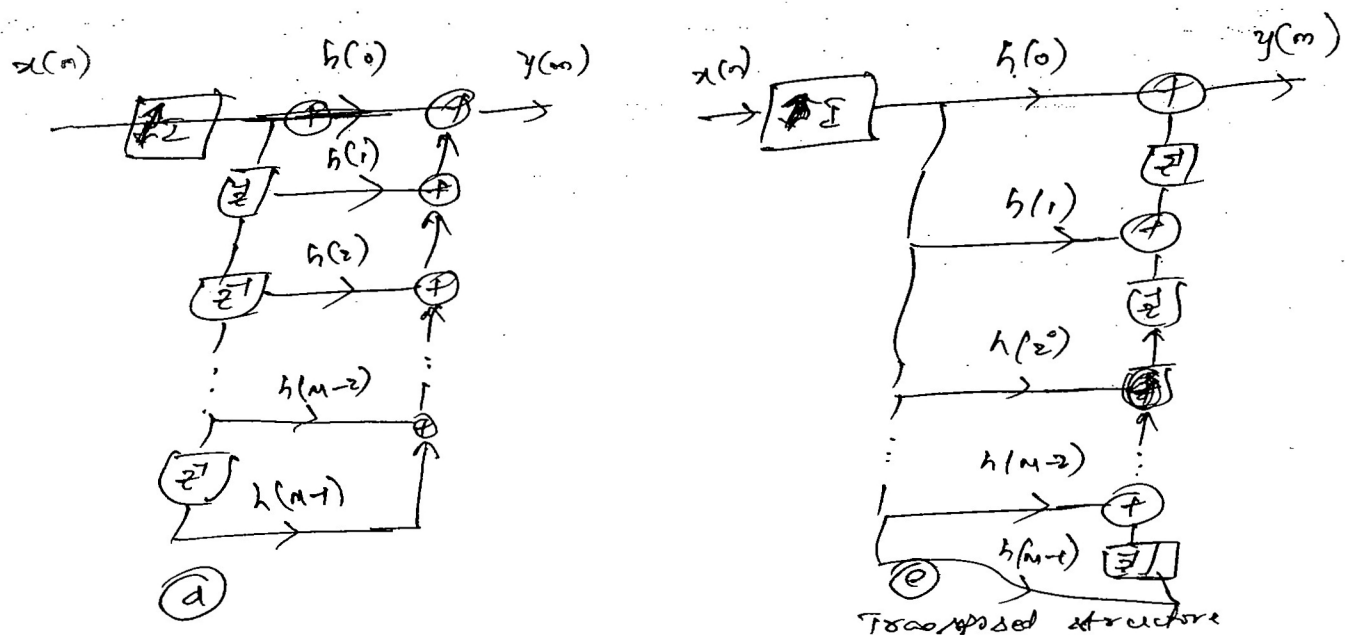
Signal. If I is large, most of signal components in FIR filter are zero. Consequently most of multiplications and additions results are zeros. Further down sampling process at op of filter implies that only one out of every D of samples is required at op of filter. Consequently only one out of every D possible values at op of filter should be computed.

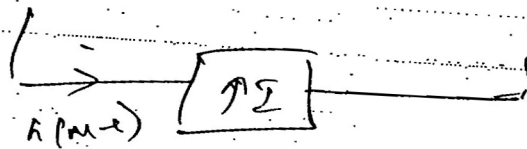
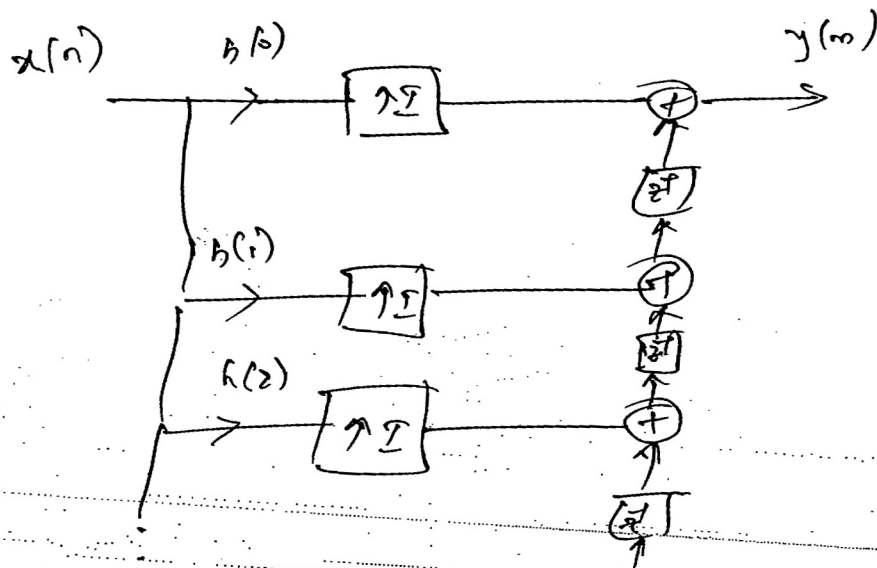
For more efficient filter structure

(a) **Decimator** o/p of filter is down sampled by factor D as shown below (a). The filter is operating at high rate F_s and only one out of D of samples is actually needed. So, we can embed down sampling operation within the filter as shown in (b). In this all operations are performed (multiplications and additions) are performed at lower sampling rate F_s/D achieving desired efficiency. Additional reduction in computation can be achieved by using symmetric characteristics of $\{h(n)\}$ shown in fig (c) below



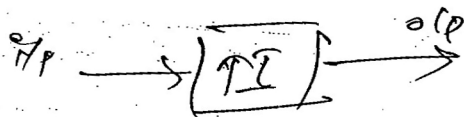
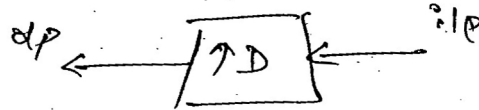
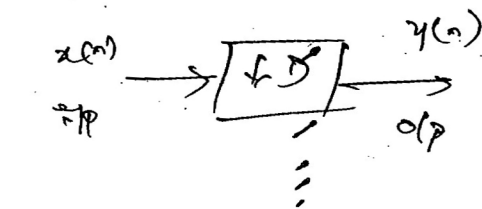
(ii) Interpolator : first inserting I-1 zeros into samples of $x(n)$ and then filtering resulting sequence. Direct form realization is shown in fig (a). Here all filtering computations are performed at high sampling rate IF_x . Derived simplification is achieved by using the transposed form of FIR structure (b) and embedding upsampler within filter. Thus filter multiplications are performed at low rate F_x and upsampling process introduces I-1 zeros in each filter branches of structure (c).





Efficient realization of parallel

⇒ Duality relation ship is obtained through Transposition



② Polyphase Decomposition

① Decimator. : If of decimator is



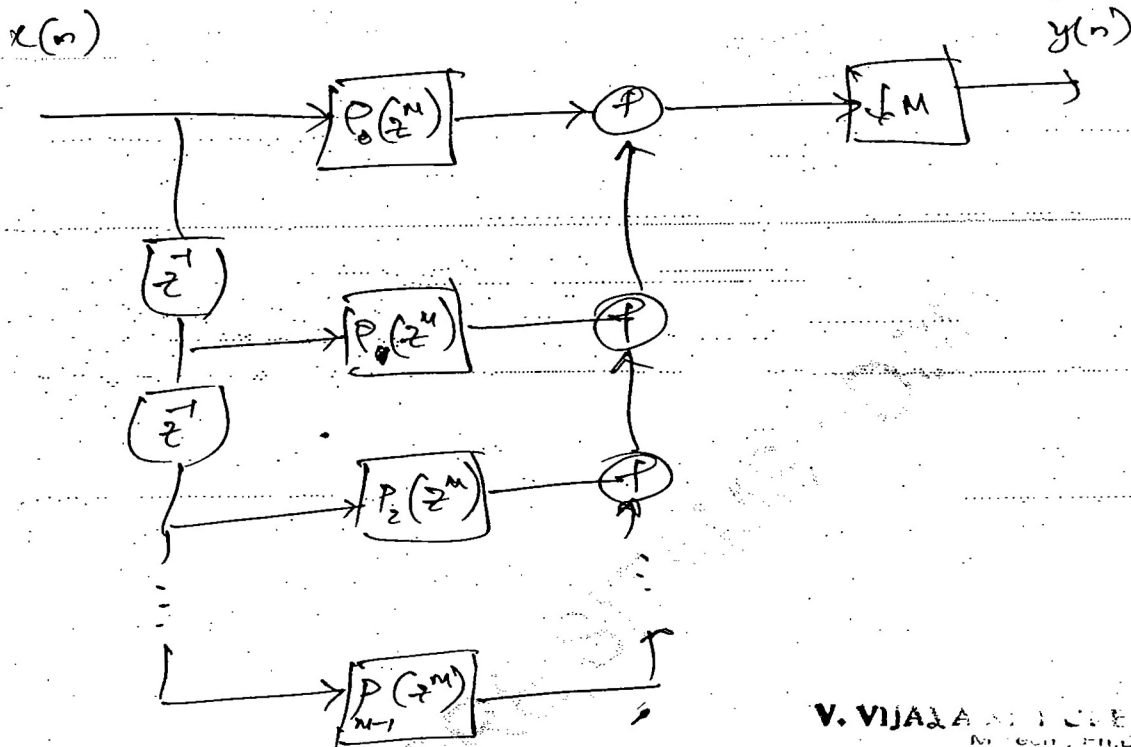
M-branch polyphase decomposition of $H(z)$ is given by

$$H(z) = \sum_{m=0}^{M-1} z^{-m} P_m(z^M) ; P_m(z) = \sum_{k=0}^{M-1} h_k z^k = \sum_{k=0}^{M-1} h_{k+Mm} z^{k+Mm}$$

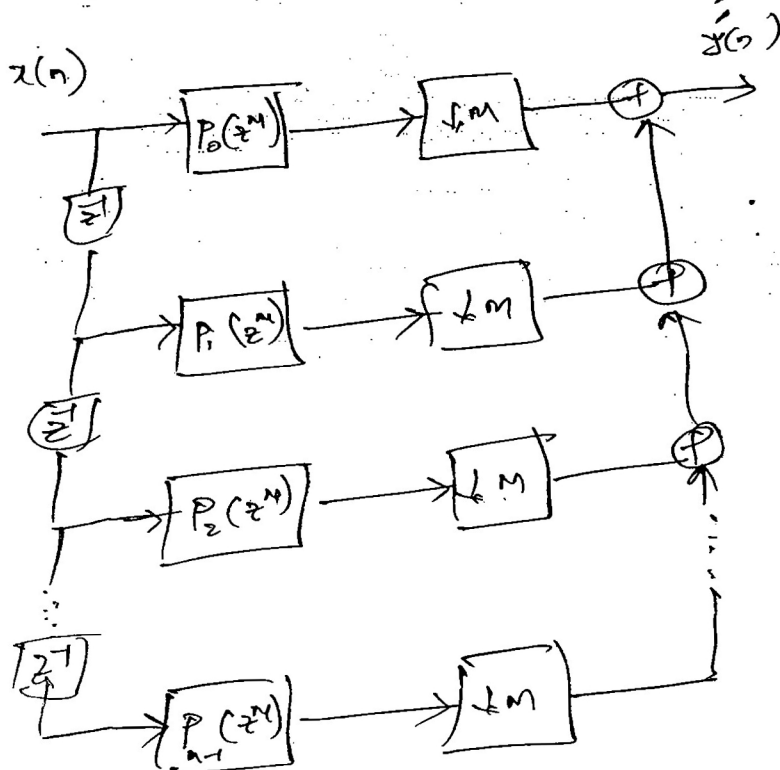
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The subfilters $P_0(z)$, $P_1(z)$, ..., $P_{M-1}(z)$ are FIR filters and when combined in right phase sequence produce original filter $H(z)$.

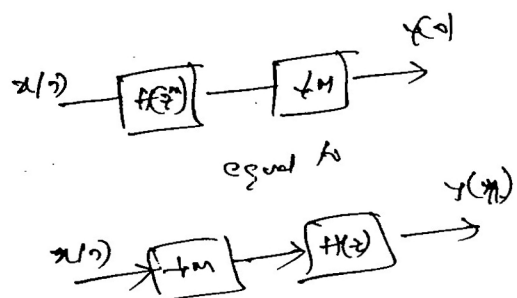
The polyphase structure is shown below

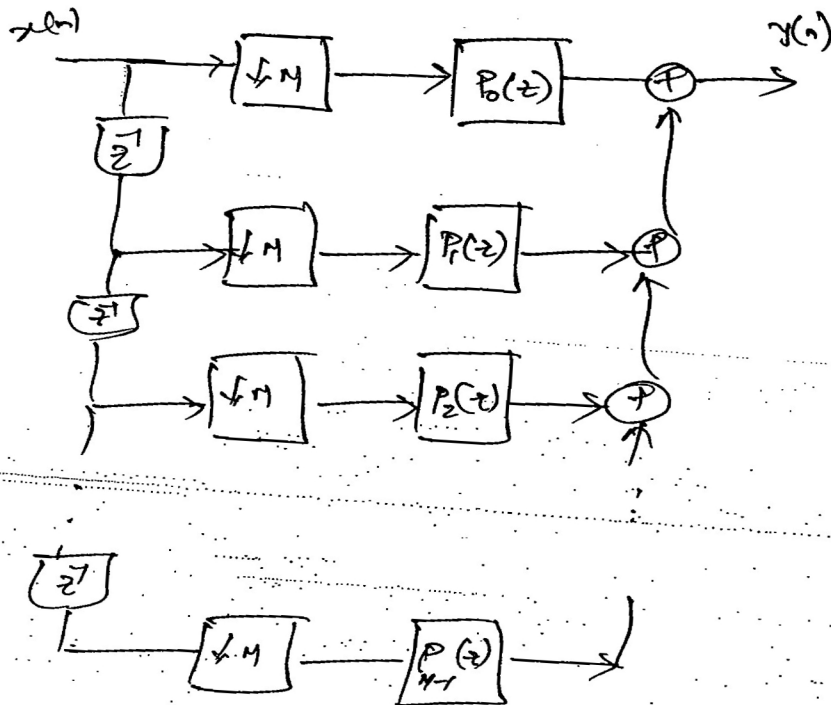


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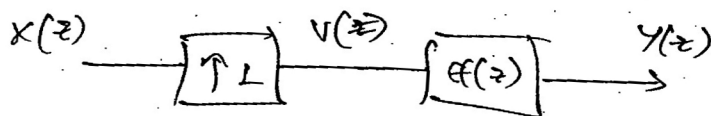


using identity



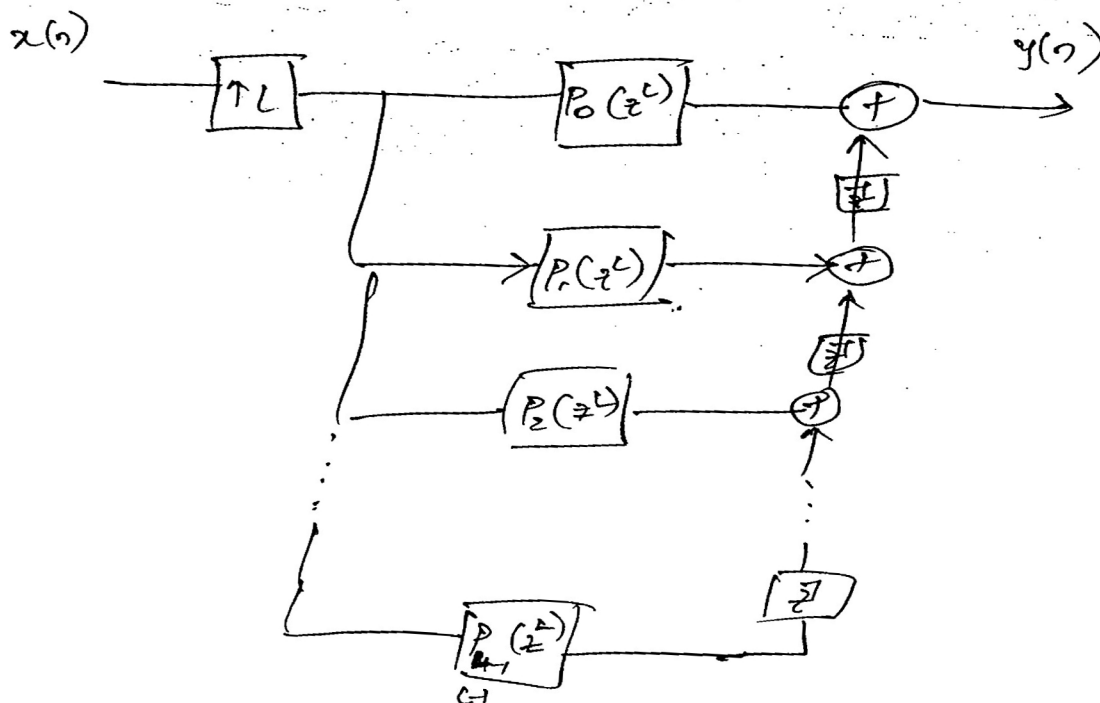


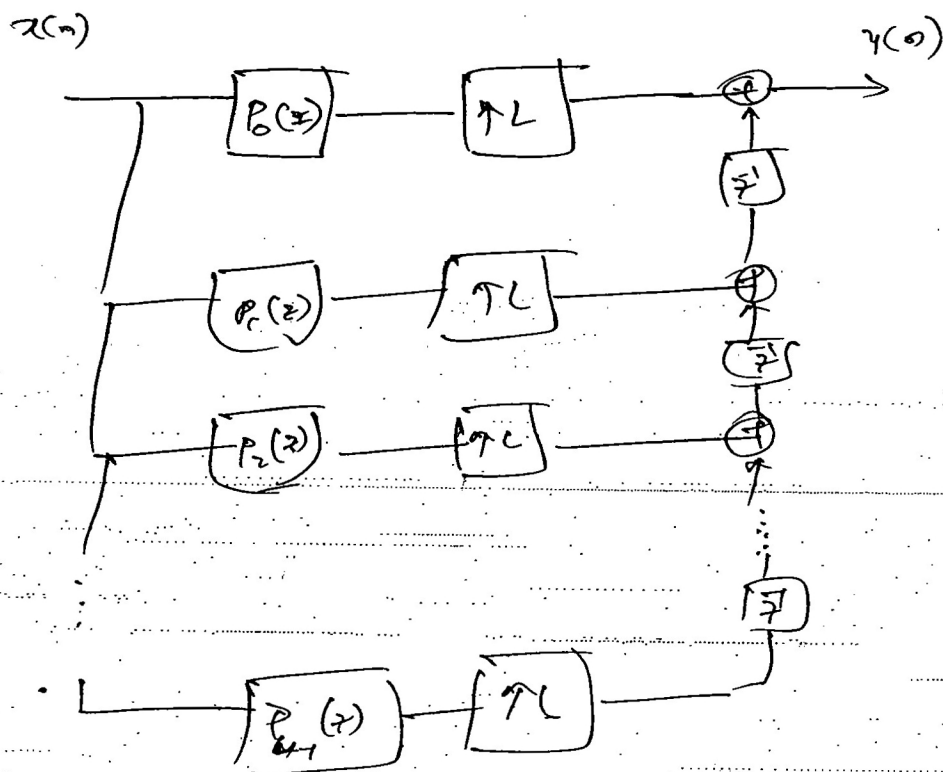
⑧ Interpolator :-



Transfer function $H(z)$ of interpolator is given by

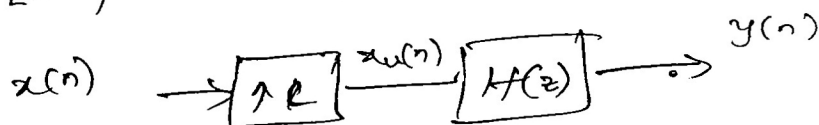
$$H(z) = \sum_{m=0}^{L-1} z^{-m} P_m(z^L)$$





⇒ Nyquist Filters :- These are special type of LPF's with a transfer function that by design has certain zero-valued co-efficients. Due to presence of these zero-valued co-efficients, these filters are computationally more efficient than other LPFs of same order. In addition when used as interpolator filters, they preserve the non-zero samples of up-sampler o/p at interpolator o/p. These filters are called L^{th} Band filters (a) Nyquist filters. They can be used in both (i) single rate and multi-rate signal processing. (ii) decimator and interpolator design (iii) quadrature mirror filter bank (iv) Hilbert Transformer.

Consider L -factor interpolator



The relation b/w op and ip of Interpolator given by

$$Y(z) = H(z) X(z^L) \quad \text{--- (1)}$$

If the interpolation filter $H(z)$ is realized in 1-band Polyphase form, we have

$$H(z) = P_0(z^L) + z^{-1} P_1(z^L) + z^{-2} P_2(z^L) + \dots + z^{-(L-1)} P_{L-1}(z^L) \quad \text{--- (2)}$$

Assume that k^{th} polyphase component of $H(z)$ is a constant that is $P_k(z) = \alpha$

$$H(z) = P_0(z^L) + z^{-1} P_1(z^L) + \dots + z^{-(L-1)} P_{L-1}(z^L) + \alpha z^{-k} + z^{-(L+k)} P_{L+1}(z^L) + \dots + z^{-(L-1+k)} P_{L-1+k}(z^L) \quad \text{--- (3)}$$

Then we can express

$$Y(z) = \alpha z^{-k} X(z^L) + \sum_{\substack{l=0 \\ l \neq k}}^{L-1} z^{-l} P_l(z^L) X(z^L) \quad \text{--- (4)}$$

as a result for a given k , $y(Ln+k) = \alpha x(n)$;

i.e. α samples appear at o/p without any distortion at time instants $Ln+k$, $-\infty < n < \infty$ where as $(L-1)$ o/p samples are determined by interpolation.

A filter with above property is called Nyquist filter and its impulse response has many zero valued samples making it computationally attractive.

i.e. for eg: impulse response of zero phase L^{th} band filter for $k=0$ i.e. $P_0(z) = \alpha$ in eqn (3)

$$\sum_{k=0}^{L-1} H(z \omega_L^k) = L\alpha = 1$$

L^{th} band filters can be FIR & IIR filters.

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⇒ Spectral Analysis of non-stationary signals

DFT can be used for spectral analysis of finite length signal composed of sinusoidal components as long as freq, amplitude and phase of each sinusoidal component are time-invariant and independent of signal length.

But in practical situations the signal to be analyzed is non-stationary, for which these signal parameters are time-varying. Eg (Chirp, Speech, Radar, Sonar signals)

Such signals in frequency domain cannot be described using a DFT. An alternate approach is to segment the sequence into a set of subsequences of short length with each subsequence centred at uniform intervals of time and its DFT computed separately. If subsequence length is reasonably small, it can be assumed as stationary for practical purpose. As a result the freq-domain decomposition of a long sequence is given by a set of short length DFTs that is a time dependent DFT.

To represent a non-stationary signal $x(n)$ in terms of a set of short-length sequences, we can multiply it with a window $w(n)$ that is stationary w.r.to time and move the signal through the window. A DFT of short sequence obtained by windowing is called short-term Fourier Transform (STFT), which is thus a function of location of the window relative to original long sequence and frequency.

The short-term Fourier Transform (STFT) also known as the time-dependent Fourier Transform of a sequence $x(n)$

$$\text{defined as } X_{\text{STFT}}(e^{j\omega}, n) = \sum_{m=-\infty}^{\infty} x(n-m) w(m) e^{-j\omega m}$$

$w(n)$ is suitably chosen window. The function of window is to extract a finite-length portion of the sequence $x(n)$ such that the spectral characteristics of the section extracted are approximately stationary over the duration of the window for practical purpose.

If $w(n)=1$, STFT eqn reduces to conventional DFT of $x(n)$.

The window w is finite in length, ensures the existence of STFT for any sequence $x(n)$.

STFT is a f^n of two variables: Integer Variable in time n and Continuous freq Variable ω .

STFT is a periodic f^n of ω with period 2π .

Display of magnitude of STFT is called Spectrogram.

Since STFT is a function of two variables, the display of its magnitude require three dimensions. Often, it is plotted in 2-D, with magnitude represented by darkness of plot. White areas represent zero-valued magnitudes, gray areas represent non zero valued magnitudes, largest magnitudes are shown in black. In STFT magnitude display vertical axis represent freq variable ω , horizontal axis represent time index n .

In general STFT is computed at finite set of discrete values of ω . Due to finite length of windowed sequence STFT is accurately represented by its freq samples as long as no of freq samples is greater than window length. The portion of a sequence $x(n)$ inside the window can be fully recovered from freq samples of STFT!

Let window be of length R , defined in the range $0 \leq m \leq R-1$. Let sample $x_{STFT}(e^{j\omega}, n)$ at N equally spaced freqs $\omega_k = \frac{2\pi k}{N}$, with $n \geq 0$ in

$$X_{STFT}(k, n) = X_{STFT}\left(e^{j\omega_k}, n\right) = X_{STFT}\left(e^{j\frac{2\pi k}{N}}, n\right)$$

$$= \sum_{m=0}^{R-1} x(n-m) w(m) e^{j\frac{2\pi k m}{N}}, \quad 0 \leq k \leq N-1 \quad \text{--- (2)}$$

In (2) assuming $w(m) \neq 0$, $X_{STFT}(k, n)$ is simply the DFT of $x(n-m)w(m)$.

$X_{STFT}(k, n)$ is 2π sequence which is periodic in k with N

Applying IDFT

$$x(n-m)w(m) = \frac{1}{N} \sum_{k=0}^{N-1} X_{STFT}(k, n) e^{j\frac{2\pi k m}{N}}, \quad 0 \leq m \leq R-1$$

$$\Rightarrow x(n-m) = \frac{1}{N \cdot w(m)} \sum_{k=0}^{N-1} X_{STFT}(k, n) e^{j\frac{2\pi k m}{N}}, \quad 0 \leq m \leq R-1$$

Window selection $w(n)$ is to extract a portion of signal for analysis and ensure that the extracted section of $x(n)$ is approximately stationary. Window length R should be small for signals with widely varying spectral parameters.

Decrease in window length increases time-resolution property of STFT. Increase in window length increases frequency-resolution property of STFT.

Shorter window provides wide band spectrogram and a longer window results in narrow-band spectrogram.

Analysis of speech is performed using STFT.

Musical sound processing :-

All musical programs are produced basically in two stages.

- * Sound from each individual instrument is recorded in an acoustically inert studio on a single track of multitrack tape recorder.

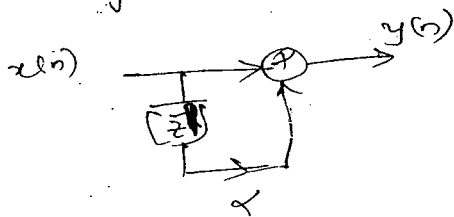
- * Signals from each track are manipulated by the sound engineer to add special effects and are combined in a mix down system to finally generate the stereo recording on two-track tape recorder.

The audio effects are performed using digital signal processing. Commonly used time domain operations carried on musical sound signals are

① Echo generation - Echoes are generated by delay units.

- * Single echo filter

Direct sound and single echo, appearing R sampling periods later, can be generated by FIR filter. $R \rightarrow$ delay in samples



Characterized as

$$y(n) = x(n) + \alpha x(n-R)$$

$$\Rightarrow H(z) = 1 + \alpha z^{-R}$$

R - denotes time the sound wave takes to travel from sound source to listener after bouncing back from reflecting wall. α - with $|\alpha| < 1$ represents the signal loss caused by propagation and reflection.

- * Multiple echo filter To generate a fixed no. of multiple echoes spaced R sampling periods apart with exponentially decaying amplitudes, FIR filter with T.F of the form

$$H(z) = 1 + \alpha z^{-R} + \alpha^2 z^{-2R} + \dots + \alpha^{N-1} z^{-(N-1)R} = \frac{1 - \alpha^N z^{-NR}}{1 - \alpha z^{-R}}$$

Applications

Infinite no of echoes spaced T sampling periods apart with exponentially decay amplitudes can be created by IIR filter with T.F

$$H(z) = 1 + \alpha z^{-T} + \alpha^2 z^{-2T} + \alpha^3 z^{-3T} + \dots$$

$$= \frac{1}{1 - \alpha z^{-T}} ; |\alpha| < 1$$

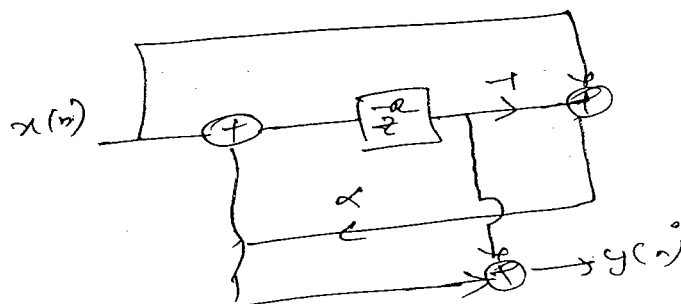
ii) Reverberation

Sound reaching a listener in a closed space, such as concert hall, consists of several components: direct sound, early reflections, and reverberation.

Early reflections are composed of several closely spaced echoes that are basically delayed and attenuated copies of direct sound, whereas reverberation is composed of densely packed echoes.

Sound recorded in a live studio does not sound natural to listener. But digital filtering can be employed to convert sound recorded in a live studio into a natural sounding one by artificially creating the echoes and adding them to original signal. More realistic reverberation, a reverbulator with an all pass structure whose T.F is.

$$H(z) = \frac{\alpha + z^{-T}}{1 + \alpha z^{-T}} ; |\alpha| < 1$$



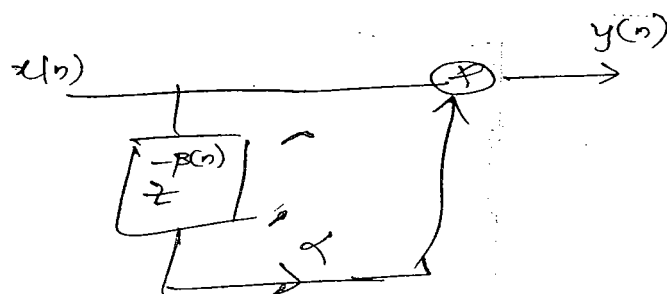
③ Flanging - special sound effect used in mix down

process. Originally, it was created by feeding same musical piece in to two tape recorders and then combining their delayed o/p's while varying the difference Δt b/w their delay times.

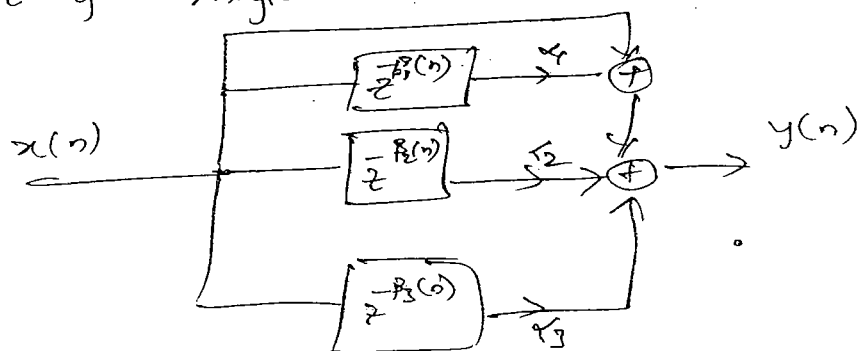
FIR Comb filter can be modified to create flanging effect. The I/P-o/p relation is

$$y(n) = x(n) + \alpha x(n - \beta(n))$$

The delay of β samples is made a time-varying delay $\beta(n)$ b/w 0 and R with a low periodically varying delay $\beta(n)$ b/w 0 and R with a low freq ω_0 such as $\beta(n) = \frac{R}{2} (1 - \cos \omega_0 n)$ generator flanging effect on sound



④ Chorus Generator : Chorus effect is achieved when several musicians are playing same musical piece at same time but with small changes in amplitudes and small timing differences b/w their sounds. Such an effect can be created synthetically by chorus generator from the music of single musician. The FIR structure is



$\beta_i(n)$ are randomly varied with very slow variations.

② Signal Compression

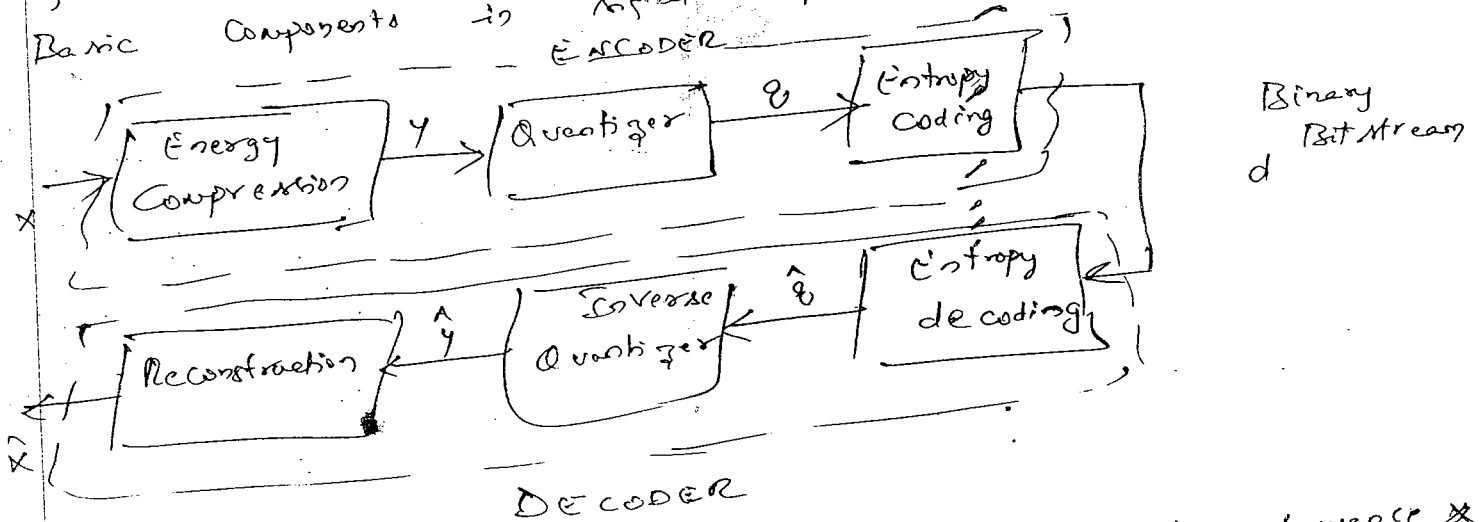
Signals carry information, the objective of signal processing is to preserve the information carried by the signal and extract and manipulate it when required.

Signal compression is needed for reducing the amount of data, while preserving the information content of signal with some acceptable fidelity criteria. and also to reduce bandwidth requirement for transmission.

Most signals exhibit data redundancy i.e. some amount of data with no relevant information or either repetition. There are three types of redundancies they are Coding, Inter sample, and Psycho-visual redundancies. Signal

compression can be achieved by removing one or more of these redundancies to achieve data reduction.

Basic Components in Signal Compression System



ENCODER

Energy compression block transforms the x sequence into another sequence y with same total energy, while packing most of energy in very few of samples of y . Quantizer develops an approximate representation of y for a given level of accuracy in the form of an integer-valued sequence q by adjusting quantizer

step size to control trade-off b/w distortion and bit rate. Entropy coding block uses variable length entropy coding to encode the integers in sequence $\hat{q}$ into bitstream d with the aim of minimizing total no. of bits in d by making use of statistics of class of samples in q .

DECODER

Entropy decoding block regenerates the integer valued sequence $\hat{q}$ from binary bit stream d . The inverse quantizer develops $\hat{y}$, a best estimate of y from q . Finally the reconstruction block develops $\hat{x}$, the best approximation of original ip sequence x from $\hat{y}$.

Signal Compression Methods

- ① Lossless Compression - No information is lost due to compression and original signal can be recovered exactly from compressed data by decoder.
- ② Lossy Compression - Some amount of information that is less relevant is lost due to compression and signal reconstructed by decoder is not a perfect replica of original signal but it is only acceptable approximation.

④ Trans Multiplexers :-

To transmit multiple low freq voice signals over a wide band channel multiplexing technique is used.

* Frequency division Multiplexing (FDM)

In this multiplex analog voice signals are first modulated by single sideband (SSB) modulators onto several subcarriers, combined and transmitted simultaneously

UNIT - VII MULTIRATE DIGITAL SIGNAL PROCESSING

Applications.

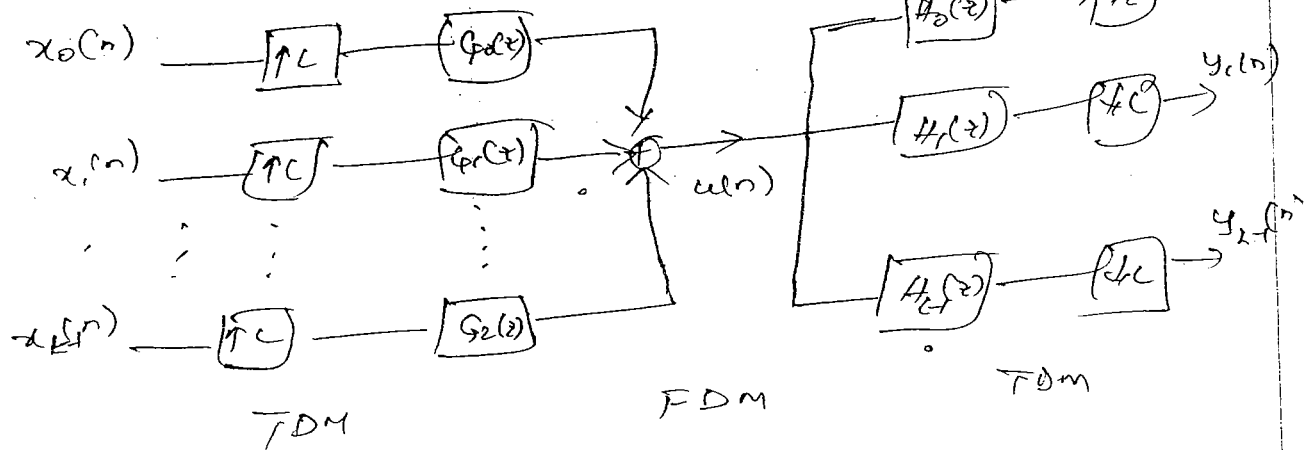
over a common wide band channel. To avoid cross talk, the subcarriers are chosen to ensure that the spectra of modulated signals do not overlap. At the receiving end, the modulated subcarriers signals are separated by band pass filters and demodulated to reconstruct individual voice signals.

* Time Division Multiplexing - In this the voice signals are first converted into digital signals by sampling and A/D conversion. The samples of digital signals are time-interleaved by a digital multiplexer, and combined signal is transmitted. At the receiving end the digital voice signals are separated by a digital demultiplexer and then passed through a D/A converter and an analog reconstruction filter to recover the original analog voice signals.

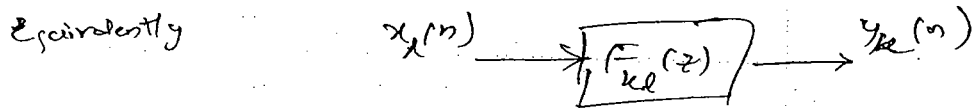
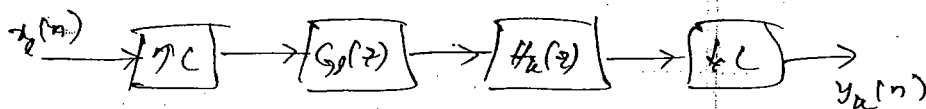
Until all services become digital, it is necessary to translate signals b/w two formats. This is achieved by

Transmultiplexer system

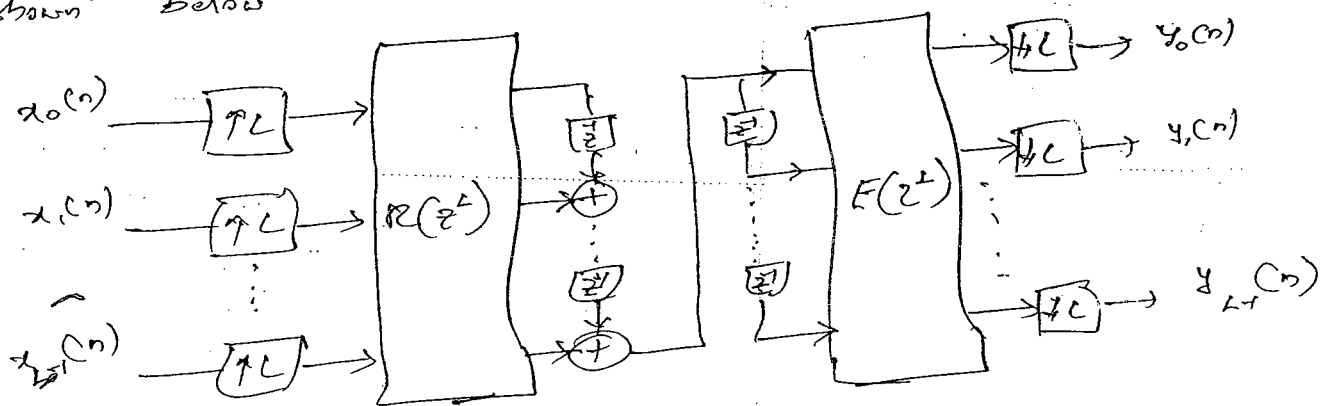
Transmultiplexer is a multiport input, multi output, multirate structure that consists of L -channel synthesis filter bank at input and L -channel analysis filter bank at output.



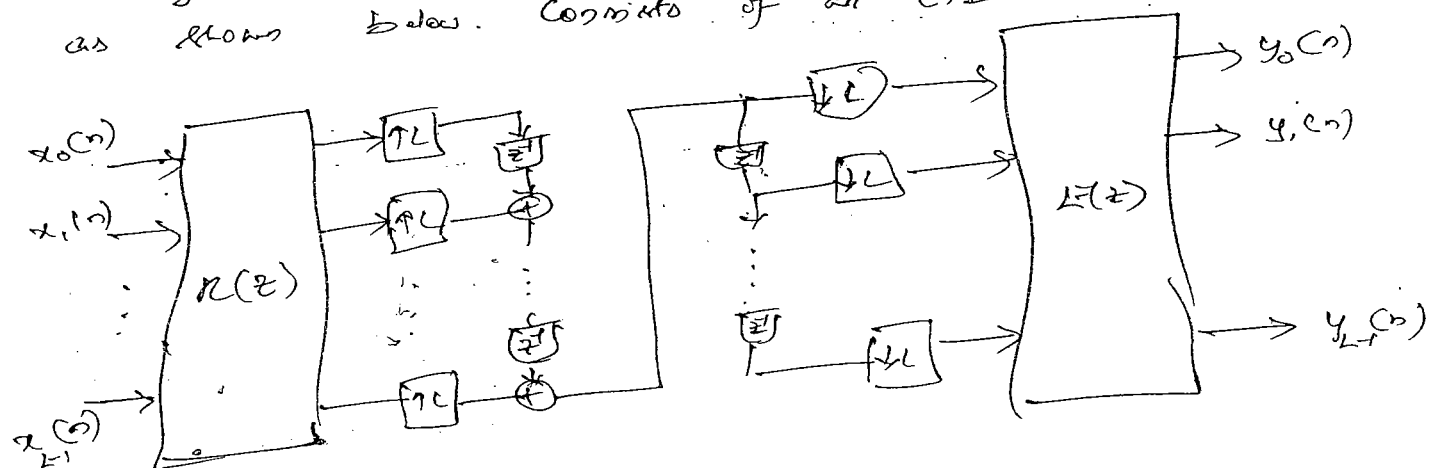
To determine i/p - o/p relation of transmultiplexer
 Consider one path from k^{th} i/p to j^{th} o/p
 as shown



A polyphase representation of the structure is
 shown below



Using identities we can obtain an efficient realization
 as shown below. Consists of an LTI branch with a



transfer function $F_{kl}(z)$ that is zeroth polyphase
 component of $H_k(z)G(z)$. The i/p - o/p relation of
 transmultiplexer is given by

$$Y_k(z) = \sum_{l=0}^{L-1} F_{kl}(z) X_l(z) ; 0 \leq k \leq L-1$$

$$Y(z) = \begin{bmatrix} Y_0(z) & Y_1(z) & \dots & Y_{L-1}(z) \end{bmatrix}^t$$

$$X(z) = \begin{bmatrix} X_0(z) & X_1(z) & \dots & X_{L-1}(z) \end{bmatrix}^t$$

$$Y(z) = F(z) X(z)$$

$F(z)$ is $L \times L$ matrix whose $(k, l)^{th}$ element is given by $F_{kl}(z)$. The objective of transmultiplexed design is to ensure that $y_k(n)$ is a reasonable replica of $x_k(n)$. If $y_k(n)$ contains contributions from $x_l(n)$ with $l \neq k$ then there is cross-talk b/w these two channels. Cross talk is absent totally if $F(z)$ is a diagonal matrix $\Rightarrow Y_k(z) = F_{kk}(z) X_k(z) ; 0 \leq k \leq L-1$

⑤ Discrete Multitone Transmission of Digital Data

Discrete multitone transmission (DMT) is a scheme in which modulation and demodulation processes are implemented via DFT efficiently realized using Fast Fourier Transform (FFT). This approach leads to as all digital systems eliminating the arrays of sinusoidal generators and coherent demodulators.

Let $\{a_k(n)\}$ and $\{b_k(n)\}$, $0 \leq k \leq M-1$ be two $M-1$ real valued data sequences operating at a sampling

UNIT - VII: MULTIRATE DIGITAL SIGNAL PROCESSING FUNDAMENTALS

Rate F_T to be transmitted. Define a new set of complex sequences $\{x_k(n)\}$ of length $N = 2M$ according to

$$x_k(n) = \begin{cases} a_0(n) & k=0 \\ a_k(n) + j b_k(n) & 1 \leq k \leq \frac{N}{2}-1 \\ b_0(n) & \\ a_{N-k}(n) - j b_{N-k}(n) & \frac{N}{2}+1 \leq k \leq N-1 \end{cases}$$

We apply IDFT to above set of N sequences that is transformed into another new set of N

signal $\{y_k(n)\}$ given by

$$y_k(n) = \frac{1}{N} \sum_{l=0}^{N-1} x_l(n) W_N^{-lk}, \quad 0 \leq l \leq N-1$$

$$W_N = e^{-j \frac{2\pi}{N}}$$

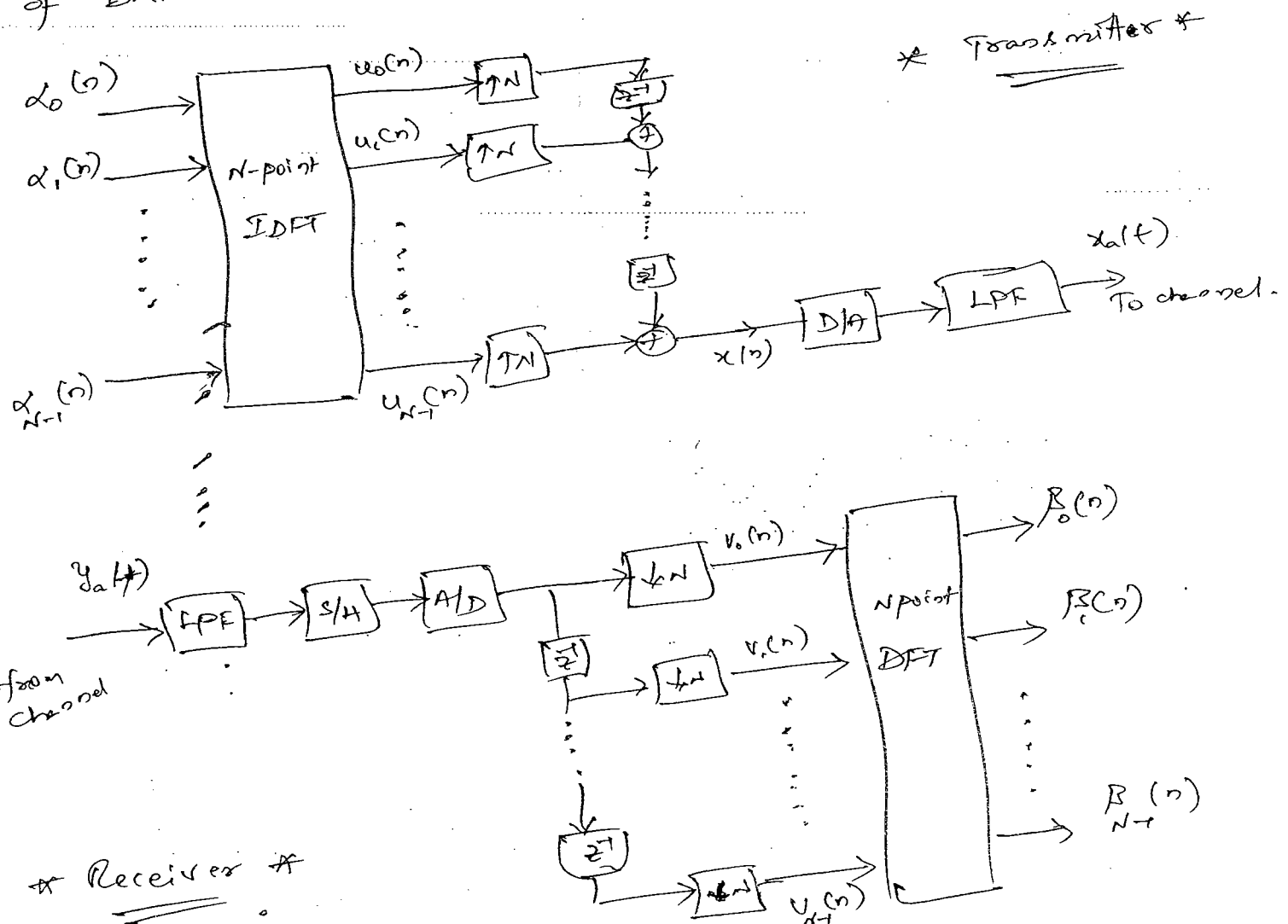
Each of these N signals is then upsampled by a factor N and time interleaved, generating a composite signal $\{x(n)\}$ operating at a rate $N F_T$ that is assumed to be equal to $2 F_c$. The composite signal is converted into an analog signal $x_a(t)$ by passing through D/A Converter. followed by an analog reconstruction filter. The analog signal $x_a(t)$ is transmitted over channel.

At the receiver, the received analog signal $y_a(t)$ is passed through an analog anti-aliasing filter and then converted into a digital signal $\{y(n)\}$ by an A/D circuit, followed by A/D Converter operating at a rate of $N F_T = 2 F_c$. The

received digital signal is then deinterleaved by a delay chain containing $N-1$ unit delays, whose o/p's are next down sampled by a factor N generating the set of signals $\{u_k(n)\}$. Applying DFT to these N signals, final N signals $\{R_k(n)\}$

$$P_k(n) = \sum_{l=0}^{N-1} u_l(n) W_N^{lk}, \quad 0 \leq k \leq N-1$$

Overall Block diagram of Transmitter and Receiver of DMT scheme is shown below.



1. Designing procedure for Analog Butterworth LPF :-

Step 1 :- Find order N using
$$N \geq \frac{\log \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log(\omega_s/\omega_p)}$$

Step 2 :- Round off ' N ' to the next highest integer.

Step 3 :- Find the poles using $S_K = e^{j\phi_K}$

$$\phi_K = \frac{\pi}{2} + \frac{(2K-1)\pi}{2N} \quad ; K=1 \text{ to } N.$$

Step 4 :- Find $H(s)$ for $\omega_c = 1 \text{ rad/sec.}$

Step 5 :- Calculate ω_c i.e., $\omega_c = \frac{\omega_p}{\epsilon^{1/N}} = \frac{\omega_s}{\lambda^{1/N}}$

Step 6 :- Denormalize $H(s)$ by replacing ' s ' according to the analog domain transformations to obtain $H_a(s)$.

2. Analog Domain transformations :- (To obtain $H_a(s)$ from $H(s)$)

(i) low pass to low pass :- ' s ' replaced by $\frac{s}{\omega_c}$

(ii) low pass to High pass :- ' s ' replaced by $\frac{\omega_c}{s}$

(iii) low pass to Band pass :- ' s ' replaced by $\frac{s^2 + \omega_p \omega_u}{s(\omega_u - \omega_l)}$

(iv) low pass to Band Reject :- ' s ' replaced by $\frac{s(\omega_u - \omega_l)}{s^2 + \omega_p \omega_u}$

3. Designing procedure for chebyshev filter Approximation :- (LPF)

Type 1 :-

1. Find the order of filter N' ,
$$N \geq \frac{\cosh^{-1}(\lambda/\epsilon)}{\cosh^{-1}(1/k)}$$

where $\cosh^{-1}x = \ln(x + \sqrt{x^2 - 1})$

2. Round off 'N' to the next highest integer.

3. Find the constants a, b and μ

$$a = \cosh^{-1} \left(\frac{\mu^{1/N} - \mu^{-1/N}}{2} \right) \quad \text{and} \quad \mu = e^{-1} + \sqrt{e^{-2} + 1}$$

$$b = \cosh^{-1} \left(\frac{\mu^{1/N} + \mu^{-1/N}}{2} \right)$$

4. Find the poles $s_k = a \cos \phi_k + jb \sin \phi_k$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} ; k=1 \text{ to } N$$

5. Find denominator polynomial

$$D(s) = (s-s_1)(s-s_2) \dots (s-s_N)$$

6. Find numerator ~~pol~~ value $N(s)$

$$N\text{-odd} \Rightarrow N(s) = D(s)|_{s=0}$$

$$N\text{-even} \Rightarrow N(s) = \frac{D(s)|_{s=0}}{\sqrt{1+e^2}}$$

$$7. H(s) = \frac{N(s)}{D(s)}$$

8. obtain $h_b(s)$ from $H(s)$ by replacing s by analog domain transformations.

Type (2) :-

1. Find the order of the filter 'N'

$$N \geq \frac{\cosh^{-1}(1/\epsilon)}{\cosh^{-1}(1/k)} ; \cosh^{-1}x = \ln(x + \sqrt{x^2 - 1})$$

2. Round off 'N' to the next highest integer

3. Find the constants a, b and μ

$$a = \Omega_p \left(\frac{\mu^{1/N} - \mu^{-1/N}}{2} \right)$$

$$\& \quad \mu = \epsilon^{-1} + \sqrt{\epsilon^{-2} + 1}$$

$$b = \Omega_p \left(\frac{\mu^{1/N} + \mu^{-1/N}}{2} \right)$$

(4) Find zeros using $S_n = \frac{j\Omega_s}{\sin \phi_k}$

$$\phi_k = \frac{\pi}{2} + \frac{\pi(2k-1)}{2N} \quad ; \quad k=1 \text{ to } N.$$

(5) Find poles using

$$x_k = \frac{\Omega_s a_k}{a_k^2 + \Omega_k^2} \quad ; \quad y_k = \frac{\Omega_s \cdot \Omega_k}{a_k^2 + \Omega_k^2}$$

$$\text{where } \left. \begin{array}{l} a_k = a \cos \phi_k \\ \Omega_k = b \sin \phi_k \end{array} \right\} \quad k=1 \text{ to } N.$$

4. Overall Design procedure using Butterworth Approximation:-

(1.) collect the parameters (or) specifications from the given problem. If frequency is given in Hz, obtain normalised angular frequency using $\omega = \frac{2\pi f}{f_s}$ (rad)

(2) Select a transformation technique

(a) If BLT is used then, pre-warp the given frequency

$$\Omega = \frac{2}{T} \tan\left(\frac{\omega}{2}\right) \text{ rad/sec.}$$

$$\Omega_p = \frac{2}{T} \tan\left(\frac{\omega_p}{2}\right) \quad \& \quad \Omega_s = \frac{2}{T} \tan\left(\frac{\omega_s}{2}\right)$$

(b) If IIM (or) IIF method is used, then

$$\Omega = \frac{\omega}{T} \text{ rad/sec}$$

$$\Omega_p = \frac{\omega_p}{T} \quad \& \quad \Omega_s = \frac{\omega_s}{T} \text{ rad/sec}$$

if 'T' is not given, we assume as $T=1\text{sec}$

3) find order of the filter

$$N \geq \frac{\log_{10} \sqrt{\frac{10^{0.1K_S} - 1}{10^{0.1K_P} - 1}}}{\log_{10} \left(\frac{\omega_s}{\omega_p} \right)}$$

; LP & HP

$$N \geq \frac{\log_{10} \sqrt{\frac{10^{0.1K_S} - 1}{10^{0.1K_P} - 1}}}{\log_{10} (\omega_s)}$$

; Bandpass & BandReject

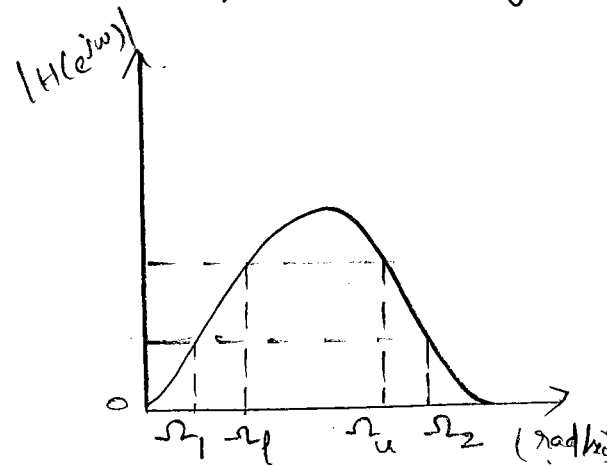
where

$$\omega_{\pi} = \min \{ |A|, |B| \}$$

For Bandpass

$$A = \frac{-\omega_1^2 + \omega_1 \cdot \omega_u}{\omega_1 (\omega_u - \omega_l)} \quad \&$$

$$B = \frac{-\omega_2^2 + (\omega_1 \cdot \omega_u)}{\omega_2 (\omega_u - \omega_l)}$$



For Band Reject

$$A = \frac{\omega_1 (\omega_u - \omega_l)}{-\omega_1^2 + \omega_p \cdot \omega_u}$$

$$; B = \frac{\omega_2 (\omega_u - \omega_l)}{-\omega_2^2 + (\omega_1 \cdot \omega_u)}$$

4) Find the poles & determine the normalised butterworth L.P.F transfer function

$$S_K = e^{j\phi_K}$$

$$\phi_K = \frac{\pi}{2} + \frac{(2K-1)\pi}{2N} \quad ; K=1 \text{ to } N.$$

$$H(s) = \frac{1}{(s-s_1)(s-s_2)\dots(s-s_N)} \quad (\text{Or}) \quad \sum_{i=1}^N \frac{1}{(s-s_i)}$$

$$H_a(s) = H(s) \Big|_{s \rightarrow \text{Analog domain transformations.}}$$

(5) Apply transformations to convert analog filter into equivalent digital filter (to convert $H_a(s)$ to $H(z)$)

(a) If BLT is used, replace $s \rightarrow \frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}} \right)$ in $H_a(s)$

(b) If IIT is used, convert $H_a(s)$ into sequence of partial fractions in the form $H_a(s) = \sum_{i=1}^M \frac{A_i}{s-p_i}$ and apply

transformations $\frac{1}{s-p_i} \rightarrow \frac{1}{1 - e^{p_i T} z^{-1}}$

Note:- If realisation is to be performed after design then use realization of IIR systems (Direct form I, II, cascade & parallel)

5. Overall Design procedure for Chebyshev approximation:-

(1) Collect the parameters (or) specifications from the given problem. If frequency is given in Hz, obtain normalized angular frequency using $\omega = \frac{2\pi f}{f_s}$ (rad)

(2) Select a transformation technique.

(a) If BLT is used then, pre-warp the given frequency

$$\omega = \frac{2}{T} \tan\left(\frac{\omega_s}{2}\right) \text{ rad/sec}$$

$$\omega_p = \frac{2}{T} \tan\left(\frac{\omega_{p_s}}{2}\right) \quad \& \quad \omega_s = \frac{2}{T} \tan\left(\frac{\omega_{s_s}}{2}\right)$$

(b) If IIM (or) IIF method is used, then

$$\Omega = \frac{\omega}{T} \text{ rad/sec}$$

$$\Omega_p = \frac{\omega_p}{T} \text{ and } \Omega_s = \frac{\omega_s}{T} (\text{rad/sec.})$$

if T is not given, we assume it as $T=1\text{sec}$.

Step 3:- Replace log function by $\cosh^{-1}x$.

$$(for LP \& HP) \quad N > 1 \quad \frac{\cosh^{-1} \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\cosh^{-1}(\Omega_s/\Omega_p)} \quad \begin{array}{l} \neq \text{For BP \& BR denominator} \\ \text{is } \cosh^{-1}(\Omega_s) \\ \Omega_s \text{ is same as in Butterworth} \\ \text{case.} \end{array}$$

Step 4:- $S_k = a \cos \phi_k + j b \sin \phi_k$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \quad ; \quad k=1 \text{ to } N$$

$$a = \Omega_p \left(\frac{\mu^{1/N} - \mu^{-1/N}}{2} \right) \quad ; \quad b = \Omega_p \left(\frac{\mu^{1/N} + \mu^{-1/N}}{2} \right)$$

$$\therefore \mu = \epsilon^{-1} + \sqrt{\epsilon^{-2} + 1}$$

Step 5:- $H(s) = \frac{N(s)}{D(s)} \quad \& \quad D(s) = (s-s_1)(s-s_2)\dots(s-s_N)$

According to Chebyshev Procedure

$$N\text{-odd} \Rightarrow N(s) = D(s)|_{s=0}$$

$$N\text{-Even} \Rightarrow N(s) = \frac{D(s)|_{s=0}}{\sqrt{1+\epsilon^2}}$$

$$H_a(s) = H(s)|_{s \rightarrow \text{Analog domain transformation.}}$$

(a) If BLT is used, replace $s \rightarrow \frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}} \right)$ in $H_a(s)$

(b) If IIT is used, Convert $H_a(s)$ in to sequence of partial fractions in the form $H_a(s) = \sum_{i=1}^M \frac{A_i}{s-p_i}$ and

apply transformations $\frac{1}{s-p_i} \rightarrow \frac{1}{1 - e^{p_i T} z^{-1}}$