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UNIT - IV

Filters are frequency selective devices. An LTI S/m performs discrimination or filtering among the various freq components present in i/p. The nature of filtering action is determined by freq response characteristics $H(e^{j\omega})$, which in turn depends on choice of the system parameters like co-efficients 'a_k' and 'b_k' in the difference equation, that describes the DTS. So, by proper selection of the co-efficients, freq selective filters can be designed that pass desired freq components and attenuates undesired freq components.

Advantages of FIR filters

1. FIR filters with exactly Linear phase can be designed easily
2. Efficient realization of FIR filter exists for both recursive and non-recursive structures.
3. FIR filters realized nonrecursively i.e by direct convolution are always stable
4. Roundoff noise which is seen in finite precision arithmetic can be easily reduced for nonrecursive realization of FIR filters.

Disadvantages :

1. The duration of impulse response should be large to approximate sharp cutoff filter. (N should be large)

Hence large amount of processing is required to realize such filters when realized via slow convolution.

2. Delay of linear phase FIR filters need not always be an integer no of samples. This non-integer delay can lead to signal processing applications.

3. Memory requirement and execution time are very high.

Linear phase FIR filters

Transfer function of a FIR causal filter is given by

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n} \quad \text{--- (1)}$$

[For a causal system $h(n) = 0$ for $n < 0$.]

Fourier transform of $h(n)$ is given by

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) \cdot e^{-j\omega n} \quad \text{--- (2)}$$

F.T is periodic in freq with period 2π.

$$H(e^{j\omega}) = \pm \left| H(e^{j\omega}) \right| e^{j\theta(\omega)}$$

$|H(e^{j\omega})|$ is magnitude response

$\theta(\omega)$ is phase response

Magnitude of freq response is constant and its phase is a linear function of freq. If phase is linear we call linear phase filter phase $\theta(\omega) = -\alpha\omega \quad ; -\pi \leq \omega \leq \pi$

In general any deviation of freq response characteristics of a linear filter from ideal results in signal distoration. If the filter has a freq variable magnitude response characteristics in the pass band then the filter introduces amplitude distortion. If the phase characteristics is not linear within the desired band, the signal undergoes phase distortion.

To examine the linear and non linear phase characteristics, two delay functions are defined

phase delay and group delay.

phase delay $\tau_p = \dfrac{-\theta(\omega)}{\omega}$

group delay $\tau_g = \dfrac{-d\,\theta(\omega)}{d\omega}$

as $\theta(\omega) = -\alpha\omega \Rightarrow \tau_p = \alpha$ & $\tau_g = \alpha$

i.e for linear phase filter, the delay is constant means it is independent of freq.

$\therefore$ a filter that causes phase distortion has variable phase delay and filter with linear phase has a constant delay within the desired freq range. If the delay is not constant in desired freq range, the filter introduces delay distortion also called phase distortion.

Characteristics of FIR filters with Linear phase

Let $h(n)$ is a causal finite duration sequence defined over $0 \le n \le N-1$ and samples $h(n)$ are real.

The F.T of $h(n)$ is

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) \cdot e^{-j\omega n}$$

$H(e^{j\omega})$ is periodic with period 2π

$$\therefore H(e^{j\omega}) = H(e^{j\omega + 2\pi m}) ; m = 0, \pm 1, \pm 2 \ldots$$

$H(e^{j\omega})$ is complex it can be expressed as

$$H(e^{j\omega}) = \pm |H(e^{j\omega})| \, e^{j\theta(\omega)}$$

$\pm |H(e^{j\omega})|$ is amplitude function

$\angle H(e^{j\omega})$ & $\theta(\omega)$ is phase function

$|H(e^{j\omega})|$ is Magnitude function.

When $h(n)$ is real, Magnitude function is symmetric function and phase function is antisymmetric function

$$\therefore |H(e^{j\omega})| = |H(e^{-j\omega})|$$

$$|\theta(\omega)| = -|\theta(-\omega)|$$

(i) Linear phase & Symmetrical impulse response

Linear phase is $\theta(\omega) \propto \omega \Rightarrow \theta(\omega) = -\alpha\omega$

'α' is constant phase delay in samples for $-\pi \le \omega \le \pi$

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) \cdot e^{-j\omega n}$$

$$H(e^{j\omega}) = \pm \left| H(e^{j\omega}) \right| e^{j\theta(\omega)}$$

$$\Rightarrow \sum_{n=0}^{N-1} h(n) \cdot e^{-j\omega n} = \pm \left| H(e^{j\omega}) \right| \cdot e^{j\theta(\omega)}$$

$$\Rightarrow \sum_{n=0}^{N-1} h(n) \left[\cos\omega n - j \sin\omega n \right] = \pm \left| H(e^{j\omega}) \right| \left[\cos\theta(\omega) + j \sin\theta(\omega) \right]$$

Equating real and imaginary parts on both sides and substituting $\theta(\omega) = -\alpha\omega$

$$\Rightarrow \sum_{n=0}^{N-1} h(n) \cos\omega n = \pm \left| H(e^{j\omega}) \right| \cos\alpha\omega$$

$$-\sum_{n=0}^{N-1} h(n) \sin\omega n = \pm \left| H(e^{j\omega}) \right| \sin\alpha\omega$$

dividing above imaginary with real parts on both sides.

$$\frac{\sum_{n=0}^{N-1} h(n) \sin\omega n}{\sum_{n=0}^{N-1} h(n) \cos\omega n} = \frac{\sin\alpha\omega}{\cos\alpha\omega}$$

$$\Rightarrow \sum_{n=0}^{N-1} h(n) \left(\sin\alpha\omega \cos\omega n - \cos\alpha\omega \sin\omega n \right) = 0$$

$$\Rightarrow \boxed{\sum_{n=0}^{N-1} h(n) \sin(\alpha - n)\omega = 0} \quad -③$$

Solution to eq ③ exists when $\alpha = \frac{N-1}{2}$ & $h(n) = h(N-1-n)$ for $0 \le n \le N-1$

proof: $h(n)\sin(\alpha-n)\omega = h(n)\sin\left(\left(\frac{N-1}{2}-n\right)\omega\right)$; $\alpha = \frac{N-1}{2}$

$$= h(n)\sin\left[\left(\frac{N-1-2n}{2}\right)\omega\right]$$

$$= h(n)\sin\left\{\left(\frac{N-1-n-n}{2}\right)\omega\right]$$

$$= h(n)\sin\left(\frac{m-n}{2}\right)\omega \;;\; m = N-1-n$$

$$= \underline{\underline{0}}$$

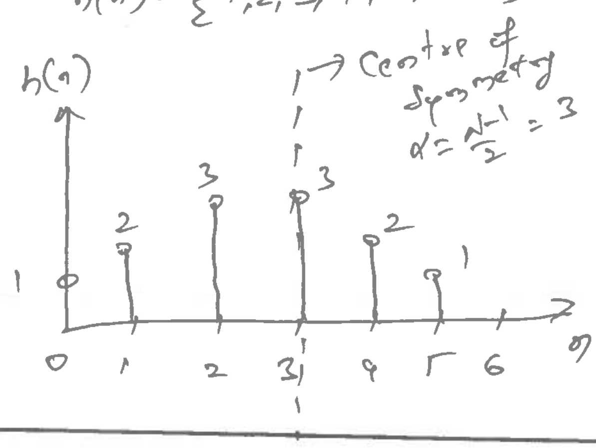
From condition $\alpha = \frac{N-1}{2}$, we can say that there is only one value of phase delay 'α' for which linear phase can be obtained. for every value of N there

From condition $h(n) = h(N-1-n)$, we can say that for $\alpha = \frac{N-1}{2}$, $h(n)$ has a special kind of symmetry

For $\alpha = \frac{N-1}{2}$

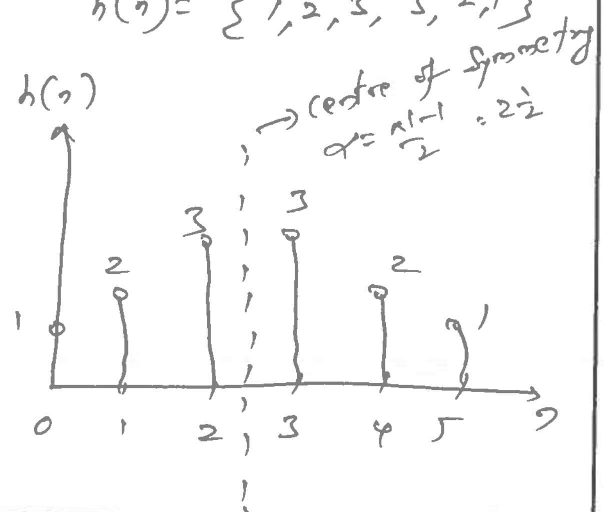
$\underline{N = \text{odd}}$ $N=7$

$h(n) = \{1, 2, 3, 4, 3, 2, 1\}$



$\underline{N = \text{even}}$ $N=6$

$h(n) = \{1, 2, 3, 3, 2, 1\}$



② Linear phase and Antisymmetric Impulse Response

Linear phase filter with phase $\theta(\omega) = -\alpha\omega$ requires the filter to have both constant group delay and constant phase delay.

If only const group delay is required another type of filter with linear phase is defined in which phase $\theta(\omega)$ is piece-wise linear function of ω. $\theta(\omega)$ is given by $\theta(\omega) = \beta - \alpha\omega$

$$\Rightarrow \quad H(e^{j\omega}) = \pm |H(e^{j\omega})| \, e^{j(\beta - \alpha\omega)} \quad \text{— ⑤}$$

$$\Rightarrow \quad \sum_{n=0}^{N-1} h(n) e^{-j\omega n} = \pm |H(e^{j\omega})| \, e^{j(\beta - \alpha\omega)}$$

$$\Rightarrow \quad \sum_{n=0}^{N-1} h(n) \left[\cos\omega n - j\sin\omega n\right] = \pm |H(e^{j\omega})| \left(\cos(\beta-\alpha\omega) - j\sin(\beta-\alpha\omega)\right)$$

Equating real and imaginary parts on both sides.

$$\Rightarrow \quad \sum_{n=0}^{N-1} h(n) \cos\omega n = \pm |H(e^{j\omega})| \cos(\beta - \alpha\omega)$$

$$-\sum_{n=0}^{N-1} h(n) \sin\omega n = \pm |H(e^{j\omega})| \sin(\beta - \alpha\omega)$$

dividing imaginary with real parts of above eqns

$$\frac{-\sum_{n=0}^{N-1} h(n) \sin\omega n}{\sum_{n=0}^{N-1} h(n) \cos\omega n} = \frac{\sin(\beta - \alpha\omega)}{\cos(\beta - \alpha\omega)}$$

$$\Rightarrow \boxed{\sum_{n=0}^{N-1} h(n)\,\sin\big(\beta-(\alpha-n)\omega\big) = 0} \quad \text{--- (5)}$$

Eq (5) $\beta = \pi/2$

$$\Rightarrow \boxed{\sum_{n=0}^{N-1} h(n)\,\cos(\alpha-n)\omega = 0} \quad \text{--- (6)}$$

Eq (6) is satisfied when $\alpha = \frac{N-1}{2}$ & $h(n) = -h(N-1-n)$ for $0 \le n \le N-1$

The filter that satisfy above conditions have a delay of $\frac{N-1}{2}$ samples but their impulse responses are antisymmetric around centre of sequence

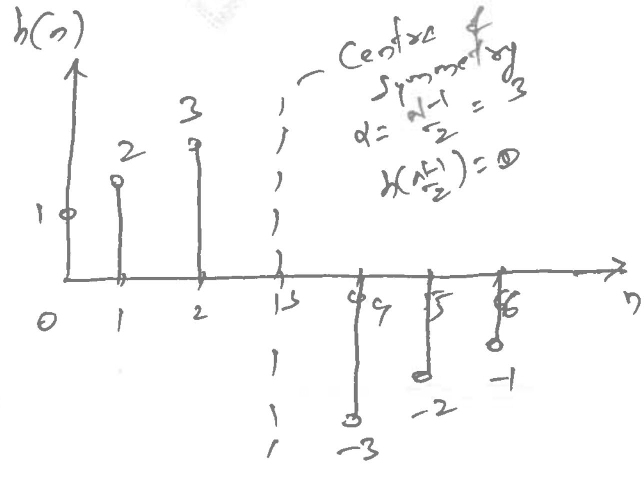
$$\beta = \pm \pi/2 \;;\; \alpha = \frac{N-1}{2} \;;\; h(n) = -h(N-1-n)$$

for $0 \le n \le N-1$

N – odd

$N = 7$

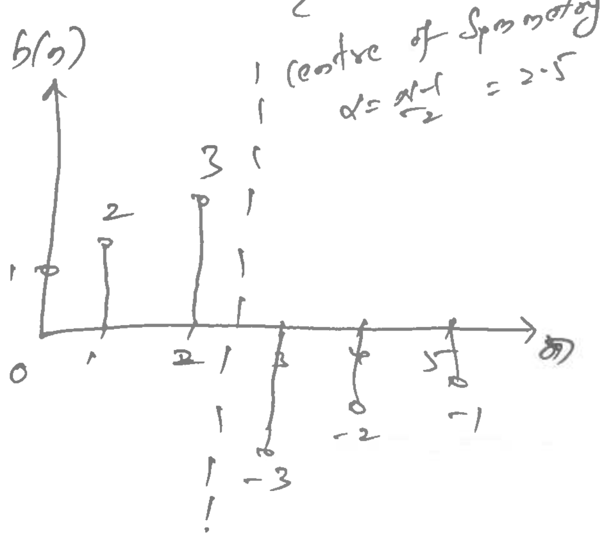
$h(n) = \{1, 2, 3, 4, -3, -2, -1\}$



N – even

$N = 6$

$h(n) = \{1, 2, 3, -3, -2, -1\}$



Frequency Response of linear phase FIR filters

Case-i :- Symmetrical impulse response (N-odd)

Freq response of impulse response is given by

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) \cdot e^{-j\omega n}$$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n) \cdot e^{-j\omega n} + h\left(\frac{N-1}{2}\right) \cdot e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n) \cdot e^{-j\omega n}$$

In 3^{rd} term let $m = N-1-n \Rightarrow n = N-1-m$

when $n = \frac{N+1}{2} \Rightarrow m = N-1-\left(\frac{N+1}{2}\right) = \frac{N-3}{2}$

when $n = N-1 \Rightarrow m = N-1-(N-1) = 0$

$$\Rightarrow H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n) \cdot e^{-j\omega n} + h\left(\frac{N-1}{2}\right) \cdot e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{m=0}^{\frac{N-3}{2}} h(N-1-m) \cdot e^{-j\omega(N-1-m)}$$

(put $m = n$)

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n) \cdot e^{-j\omega n} + h\left(\frac{N-1}{2}\right) \cdot e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(N-1-n) \cdot e^{-j\omega(N-1-n)}$$

for symmetric response $h(N-1-n) = h(n)$

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega n} + h\left(\frac{N-1}{2}\right) \cdot e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n) \cdot e^{-j\omega(N-1-n)}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n) \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)} \right] \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} 2\, h(n) \cos\omega\left(\frac{N-1}{2}-n\right) \right]$$

let $\frac{N-1}{2} - n = p$ then when $n=0 \Rightarrow p = \frac{N-1}{2}$

$n = \frac{N-3}{2} \Rightarrow p = \frac{N-1}{2} - \left(\frac{N-3}{2}\right) = 1$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right) + \sum_{p=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-p\right)\cos\omega p\right]$$

put $p=n$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-n\right)\cos\omega n\right]$$

$$\boxed{H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} \sum_{n=0}^{\frac{N-1}{2}} a(n)\cos\omega n}$$

where $a(0) = h\left(\frac{N-1}{2}\right)$; $a(n) = 2h\left(\frac{N-1}{2}-n\right)$

We can write $H(e^{j\omega}) = \underbrace{\bar{H}(e^{j\omega})}_{\text{amplitude function}} \cdot e^{j\overbrace{\theta(\omega)}^{\text{phase function}}}$

$$\Rightarrow \quad \bar{H}(e^{j\omega}) = \sum_{n=0}^{\frac{N-1}{2}} a(n)\cos\omega n$$

$$\theta(\omega) = -\alpha\omega = -\left(\frac{N-1}{2}\right)\omega$$

Amplitude function $\bar{H}(e^{j\omega})$ is real and even function of ω.

Magnitude function $|H(e^{j\omega})| = |\bar{H}(e^{j\omega})|$

phase function $\theta(\omega) = -\alpha\omega$; $\alpha = \frac{N-1}{2}$

i.e $\theta(\omega) = -\alpha\omega$ for $\bar{H}(e^{j\omega}) \geq 0$

$= -\alpha\omega + \pi$ for $\bar{H}(e^{j\omega}) < 0$

$= -\alpha\omega$ for $\bar{H}(e^{j\omega}) < 0$

Amplitude function $\bar{H}(e^{j\omega})$ is also called as zero phase frequency response.

$\bar{H}(e^{j\omega})$ can take both positive and negative values

$|\bar{H}(e^{j\omega})|$ - Magnitude response is non negative.

Frequency response of symmetrical impulse response for N-odd is show below. It can be observed that the amplitude function $\bar{H}(e^{j\omega})$ is symmetric with $\omega = \pi$ when impulse response is symmetric and N is odd the freq response is non zero at $\omega = 0$ and $\omega = \pi$. So, this freq response can be used to design LPF, HPF, BPF and BRF.

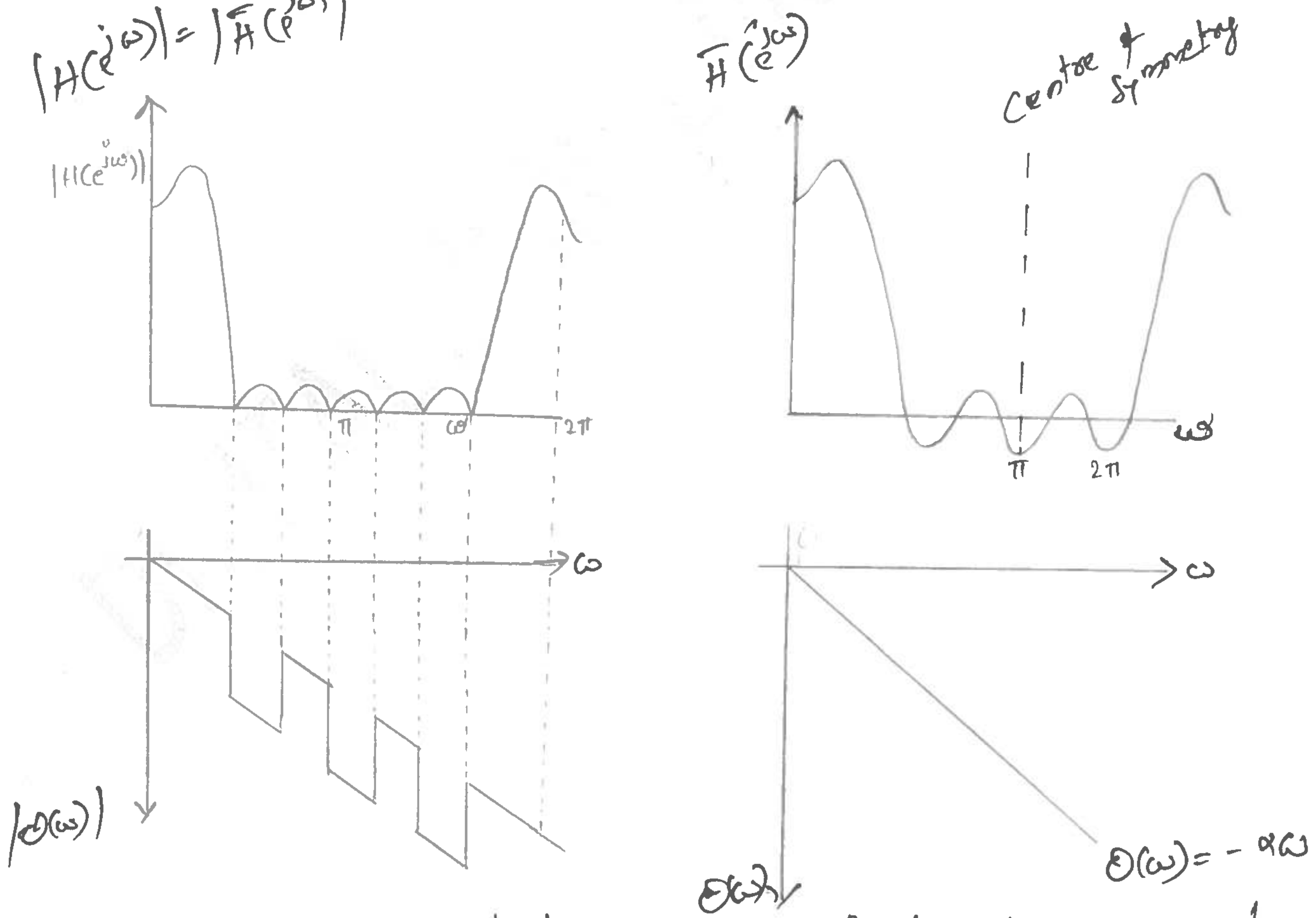


fig: Relation between magnitude response $|H(e^{j\omega})|$ and the zero phase response $\bar{H}(e^{j\omega})$ and between $\angle H(e^{j\omega})$ and $\theta(\omega)$.

Case ii : Symmetric impulse response for N-even

F.T of $h(n)$ is given by $H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n).e^{-j\omega n}$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N}{2}-1} h(n).e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{N-1} h(n).e^{-j\omega n}$$

In 2nd term

let $m = N-1-n \Rightarrow n = N-1-m$

when $n = \frac{N}{2} \rightarrow m = N-1-\frac{N}{2} = \frac{N}{2}-1$

when $n = N-1 \Rightarrow m = N-1-(N-1) = 0.$

$$\Rightarrow H(e^{j\omega}) = \sum_{n=0}^{\frac{N}{2}-1} h(n).e^{-j\omega n} + \sum_{m=0}^{\frac{N}{2}-1} h(N-1-m).e^{-j\omega(N-1-m)}$$

put $n = m$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n).e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} h(N-1-n).e^{-j\omega(N-1-n)}$$

for symmetric impulse response
$h(n) = h(N-1-n)$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n).e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} h(n).e^{-j\omega(N-1-n)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n)\left[e^{-j\omega n} + e^{-j\omega(N-1-n)}\right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left[\sum_{n=0}^{\frac{N}{2}-1} h(n)\left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right]\right]$$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left[\sum_{n=0}^{\frac{N}{2}-1} 2\,h(n)\cos\omega\left(\frac{N-1}{2}-n\right)\right]$$

let $k = \frac{N}{2} - n \Rightarrow n = \frac{N}{2} - k$

when $n=0 \Rightarrow k = \frac{N}{2}$

when $n = \frac{N}{2} - 1 \Rightarrow k = \frac{N}{2} - \left(\frac{N}{2} - 1\right) = 1$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \left[\sum_{k=1}^{\frac{N}{2}} 2h\left(\frac{N}{2} - k\right) \cos\left(\omega\left(k - \frac{1}{2}\right)\right) \right]$$

put $k = n$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \left[\sum_{=1}^{\frac{N}{2}} 2h\left(\frac{N}{2} - n\right) \cos\omega\left(n - \frac{1}{2}\right) \right]$$

$$\boxed{H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=1}^{\frac{N}{2}} b(n) \cos\left(n - \tfrac{1}{2}\right)\omega \quad \text{where} \quad b(n) = 2h\left(\tfrac{N}{2} - n\right)}$$

We can write $H(e^{j\omega}) = \bar{H}(e^{j\omega}) \cdot e^{j\theta(\omega)}$

where amplitude function $\bar{H}(e^{j\omega}) = \sum_{n=1}^{\frac{N}{2}} b(n) \cos\left(n - \frac{1}{2}\right)\omega$

$b(n) = 2h\left(\frac{N}{2} - n\right)$

phase function $\theta(\omega) = -\alpha\omega = -\left(\frac{N-1}{2}\right)\omega$

Magnitude function $|H(e^{j\omega})| = |\bar{H}(e^{j\omega})|$

$$= \left| \sum_{n=1}^{N/2} b(n) \cos\left(n - \tfrac{1}{2}\right)\omega \right|$$

The freq response of linear phase filter with symmetrical impulse response for N-even is shown below.

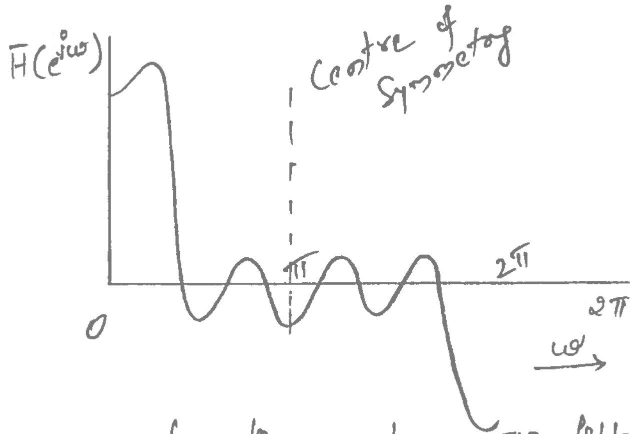


fig: Frequency response for linear phase FIR filter, Symmetric impulse response N even.

It can be observed that amplitude function of $H(e^{j\omega})$ (i.e $\bar{H}(e^{j\omega})$) is antisymmetric with $\omega = \pi$

when impulse response is symmetric and N is even the freq response is non-zero at $\omega = 0$ and at $\omega = \pi$ and so this freq response can be used to design LPF and BPF and cannot be used to design HPF and BRF filters.

<u>Case iii</u>:- Antisymmetric impulse response for N-odd.

For this type of $h(n)$, centre of symmetry is at $\frac{N-1}{2}$ and $h\left(\frac{N-1}{2}\right) = 0$.

The F.T of $h(n)$ is given by $H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) \cdot e^{-j\omega n}$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n) \cdot e^{-j\omega n} + h\left(\frac{N-1}{2}\right) \cdot e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n) \cdot e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n) \cdot e^{-j\omega n} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n) \cdot e^{-j\omega n} \quad \left(\because h\left(\frac{N-1}{2}\right) = 0\right)$$

In 2^{nd} term let $m = N-1-n \Rightarrow n = N-1-m$

when $n = \frac{N+1}{2} \Rightarrow m = N-1-\left(\frac{N+1}{2}\right) = \frac{N-3}{2}$

$n = N-1 \Rightarrow m = N-1-(N-1) = 0$

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n) \cdot e^{-j\omega n} + \sum_{m=0}^{\frac{N-3}{2}} h(N-1-m) \cdot e^{-j\omega(N-1-m)}$$

put $n = m$ dummy variable

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n) \cdot e^{-j\omega n} + \sum_{n=0}^{\frac{N-3}{2}} h(N-1-n) \cdot e^{-j\omega(N-1-n)}$$

For antisymmetric $h(N-1-n) = -h(n)$

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n) \cdot e^{-j\omega n} + \sum_{n=0}^{\frac{N-3}{2}} (-h(n)) \cdot e^{-j\omega(N-1-n)}$$

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n) \left[e^{-j\omega n} - e^{-j\omega(N-1-n)} \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[\sum_{n=0}^{\frac{N-3}{2}} h(n) \left[e^{-j\omega n} \cdot e^{j\omega\left(\frac{N-1}{2}\right)} - e^{-j\omega(N-1-n)} \cdot e^{j\omega\left(\frac{N-1}{2}\right)} \right] \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[\sum_{n=0}^{\frac{N-3}{2}} h(n) \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} - e^{-j\omega\left(\frac{N-1}{2}-n\right)} \right] \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[\sum_{n=0}^{\frac{N-3}{2}} 2j\, h(n) \cdot \sin\left(\frac{N-1}{2} - n\right)\omega \right] \qquad \left(\because j = e^{j\pi/2}\right)$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot e^{j\pi/2} \left[\sum_{n=0}^{\frac{N-3}{2}} 2h(n) \sin\left(\frac{N-1}{2} - n\right)\omega \right]$$

let $p = \frac{N-1}{2} - n \Rightarrow n = \frac{N-1}{2} - p$

when $n = 0 \Rightarrow p = \frac{N-1}{2} - 0 = \frac{N-1}{2}$

$n = \frac{N-3}{2} \Rightarrow p = \frac{N-1}{2} - \left(\frac{N-3}{2}\right) = 1$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot e^{j\pi/2} \left[\sum_{p=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - p\right) \cdot \sin \omega p\right]$$

$(p = n)$

$$\boxed{H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot e^{j\pi/2} \sum_{n=1}^{\frac{N-1}{2}} c(n) \sin \omega n \quad \text{where } c(n) = 2h\left(\frac{N-1}{2} - n\right)}$$

$$H(e^{j\omega}) = \underbrace{\bar{H}(e^{j\omega})}_{\text{Amplitude } f^n} \cdot e^{j \overbrace{\theta(\omega)}^{\text{phase } f^n}}$$

Amplitude function $\bar{H}(e^{j\omega}) = \sum_{n=1}^{\frac{N-1}{2}} c(n) \sin \omega n$

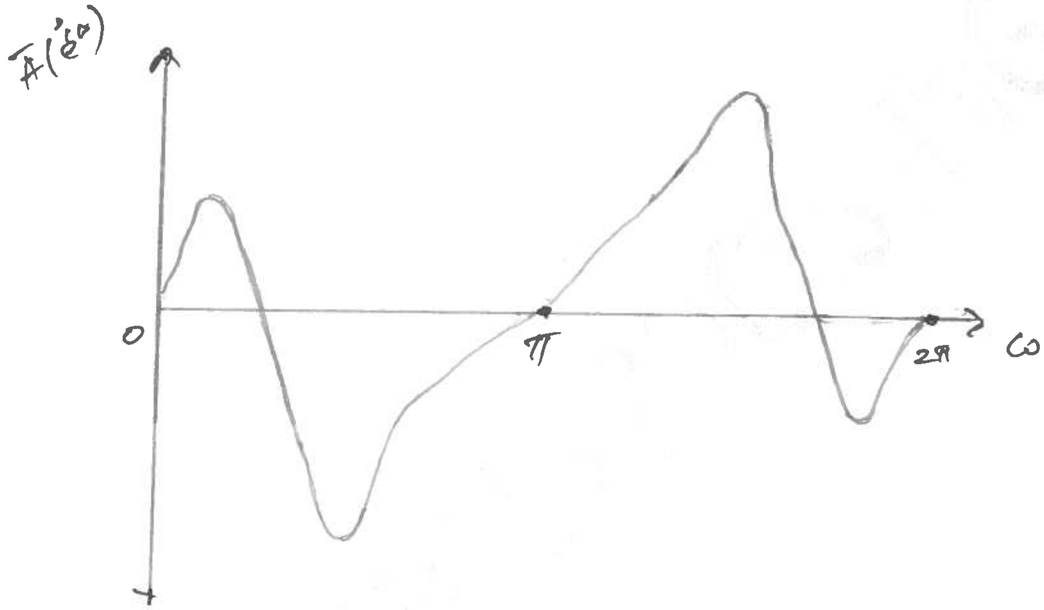
phase function $\theta(\omega) = \frac{\pi}{2} - \left(\frac{N-1}{2}\right)\omega = \beta - \alpha\omega$

$\beta = \pi/2 \; ; \; \alpha = \frac{N-1}{2}$

Magnitude function $|H(e^{j\omega})| = |\bar{H}(e^{j\omega})| = \left|\sum_{n=1}^{\frac{N-1}{2}} c(n) \sin \omega n\right|$

The frequency response of linear phase FIR filter for antisymmetric sequence $h(n)$ with N-odd is shown below. It can be observed that the amplitude function is antisymmetric with $\omega = \pi$.

The term $e^{j\pi/2}$ makes the frequency response imaginary. Hence this freq response is suitable for designing Hilbert transformers and differentiators.



Case iv :- Antisymmetric impulse response for N-even.

Freq response is the F.T of impulse response $h(n)$ given by $H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$. (centre of symmetry lies b/n $n = \frac{N-1}{2}$ and $\frac{N}{2}$)

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N}{2}-1} h(n).\, e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{N-1} h(n).\, e^{-j\omega n}$$

In 2nd term let $m = N-1-n \Rightarrow n = N-1-m$

when $n = \frac{N}{2} \Rightarrow m = N-1-\left(\frac{N}{2}\right) = \frac{N}{2}-1$

$n = N-1 \Rightarrow m = N-1-(N-1) = 0$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n).\, e^{-j\omega n} + \sum_{m=0}^{\frac{N}{2}-1} h(N-1-m).\, e^{-j\omega(N-1-m)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n) \cdot e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} h(N-1-n) \cdot e^{-j\omega(N-1-n)}$$ put $n = m$; dummy variable

for antisymmetric $h(N-1-n) = -h(n)$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n) \cdot e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} (-h(n))\, e^{-j\omega(N-1-n)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n) \left[e^{-j\omega n} - e^{-j\omega(N-1-n)} \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{\frac{N}{2}-1} h(n) \left[e^{-j\omega n} \cdot e^{j\omega\left(\frac{N-1}{2}\right)} - e^{-j\omega(N-1-n)} \cdot e^{j\omega\left(\frac{N-1}{2}\right)} \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{\frac{N}{2}-1} h(n) \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} - e^{-j\omega\left(\frac{N-1}{2}-n\right)} \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{\frac{N}{2}-1} 2j\, h(n) \sin\left(\frac{N-1}{2}-n\right)\omega \qquad \left(j = e^{j\pi/2}\right)$$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot e^{j\pi/2} \sum_{n=0}^{\frac{N}{2}-1} 2\, h(n) \sin\left(\frac{N}{2}\right.$$

let $\frac{N}{2} - n = p \Rightarrow p = \frac{N}{2} - n$

when $n = 0 \Rightarrow p = \frac{N}{2} - 0 = \frac{N}{2}$

$n = \frac{N}{2} - 1 \Rightarrow p = N-1-\left(\frac{N}{2}-1\right) =$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot e^{j\pi/2} \sum_{p=1}^{\frac{N}{2}} 2h\left(\frac{N}{2}-p\right) \cdot \sin\left(p-\tfrac{1}{2}\right)\omega$$

$p = n$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot e^{j\pi/2} \sum_{n=1}^{\frac{N}{2}} 2h\left(\frac{N}{2}-n\right) \sin\left(n-\tfrac{1}{2}\right)\omega$$

$$\boxed{H(e^{j\omega}) = e^{j\left(\frac{\pi}{2} - \left(\frac{N-1}{2}\right)\omega\right)} \cdot \sum_{n=1}^{\frac{N}{2}} d(n)\, \sin(n-1/2)\omega \quad \text{where} \quad d(n) = 2\, h\left(\frac{N}{2} - n\right)}$$

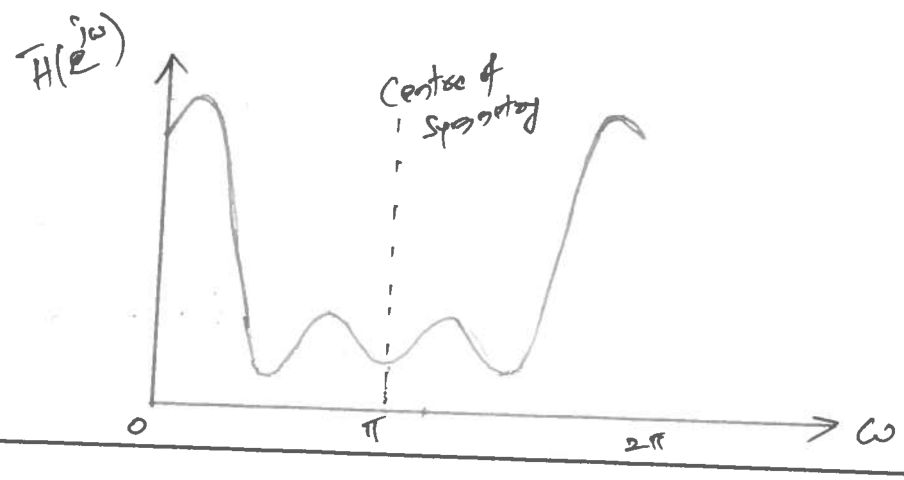
$$H(e^{j\omega}) = \underbrace{\bar{H}(e^{j\omega})}_{\text{Amplitude } f^n} \cdot e^{\overbrace{j\theta(\omega)}^{\text{phase } f^n}}$$

Amplitude function $\bar{H}(e^{j\omega}) = \sum_{n=1}^{\frac{N}{2}} d(n) \cdot \sin(n-1/2)\omega$

phase function $\theta(\omega) = \frac{\pi}{2} - \left(\frac{N-1}{2}\right)\omega$

Magnitude function $|H(e^{j\omega})| = |\bar{H}(e^{j\omega})| = \left|\sum_{n=1}^{\frac{N}{2}} d(n)\, \sin(n-1/2)\omega\right|$

The freq response of Linear phase FIR filter for antisymmetric impulse response with N-even is shown below. It can be observed that the amplitude function of $H(e^{j\omega})$ is symmetric with $\omega = \pi$.
The term $e^{j\pi/2}$ makes the freq response imaginary. Hence this freq response is suitable for designing Hilbert transformers and differentiators.



Case V: Symmetric impulse response, N is odd and centre of symmetry is at $n=0$.

Freq response is given by $H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n).\, e^{-j\omega n}$

$$H(e^{j\omega}) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} h(n).\, e^{-j\omega n}$$

$h(n)$ is defined for $n = \left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$ (as centre of symmetry is at $n=0$)

$$= \sum_{n=-\left(\frac{N-1}{2}\right)}^{-1} h(n).\, e^{-j\omega n} + h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n).\, e^{-j\omega n}$$

$$= \sum_{n=1}^{\frac{N-1}{2}} h(-n).\, e^{j\omega n} + h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n).\, e^{-j\omega n}$$

Symmetry Condition $h(-n) = h(n)$

$$= \sum_{n=1}^{\frac{N-1}{2}} h(n).\, e^{j\omega n} + h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n).\, e^{-j\omega n}$$

$$= h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\left(e^{j\omega n} + e^{-j\omega n}\right)$$

$$\boxed{H(e^{j\omega}) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2h(n) \cos \omega n}$$

$$H(e^{j\omega}) = \bar{H}(e^{j\omega}).\, e^{j\theta(\omega)}$$

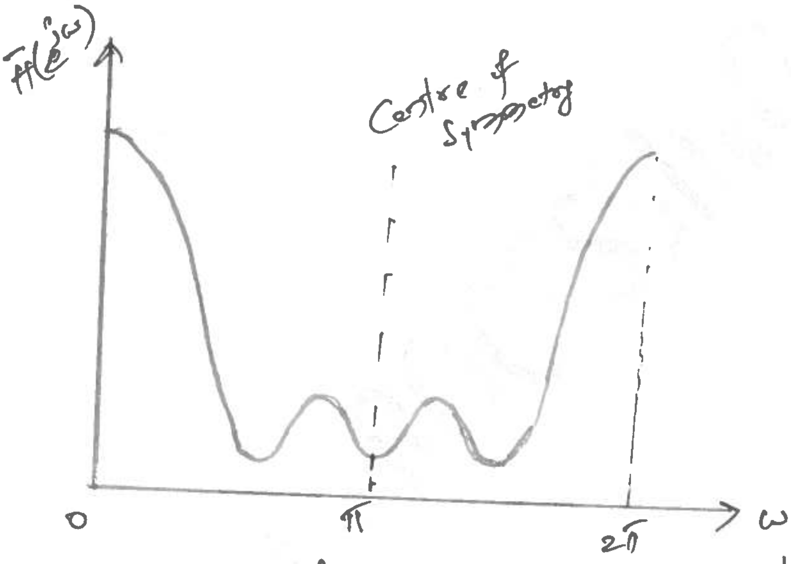
Amplitude function $\bar{H}(e^{j\omega}) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2h(n) \cos \omega n$

phase function $\theta(\omega) = 0$

Magnitude function $|H(e^{j\omega})| = |\bar{H}(e^{j\omega})| = \left| h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2h(n) \cos \omega n \right|$

Amplitude response of freq response is shown below. It can be observed that amplitude function of $H(e^{j\omega})$ is symmetric with $\omega = \pi$.

When impulse response is symmetric and N is odd, the freq response is non-zero at $\omega=0$ and $\omega=\pi$. So, this freq response can be used to design LPF, HPF, BPF and BSF.



Case vi:- Antisymmetric impulse response and N is odd with centre of antisymmetry at $n=0$.

$h(n)$ is defined for $n = -\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$

freq response is $H(e^{j\omega}) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} h(n) \cdot e^{-j\omega n}$

$$= \sum_{n=-\left(\frac{N-1}{2}\right)}^{-1} h(n)\, e^{-j\omega n} + h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) \cdot e^{-j\omega n}$$

$$= \sum_{n=1}^{\frac{N-1}{2}} h(-n) \cdot e^{j\omega n} + h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) \cdot e^{-j\omega n}$$

Centre of symmetry is at $n=0$ and $h(0)=0$

$$= \sum_{n=1}^{\frac{N-1}{2}} h(-n) \cdot e^{j\omega n} + \sum_{n=1}^{\frac{N-1}{2}} h(n) \cdot e^{-j\omega n}$$

Symmetric Condition $h(-n) = -h(n)$

$$H(e^{j\omega}) = \sum_{n=1}^{\frac{N-1}{2}} -h(n) \cdot e^{j\omega n} + \sum_{n=1}^{\frac{N-1}{2}} h(n) \cdot e^{-j\omega n}$$

$$= -\sum_{n=1}^{\frac{N-1}{2}} h(n) \left[e^{j\omega n} - e^{-j\omega n} \right]$$

$$= -\sum_{n=1}^{\frac{N-1}{2}} 2j\, h(n) \sin \omega n$$

$$= -j \sum_{n=1}^{\frac{N-1}{2}} 2h(n) \sin \omega n \qquad \left(-j = e^{-j\pi/2}\right)$$

$$\boxed{H(e^{j\omega}) = e^{-j\pi/2} \sum_{n=1}^{\frac{N-1}{2}} 2h(n) \sin \omega n}$$

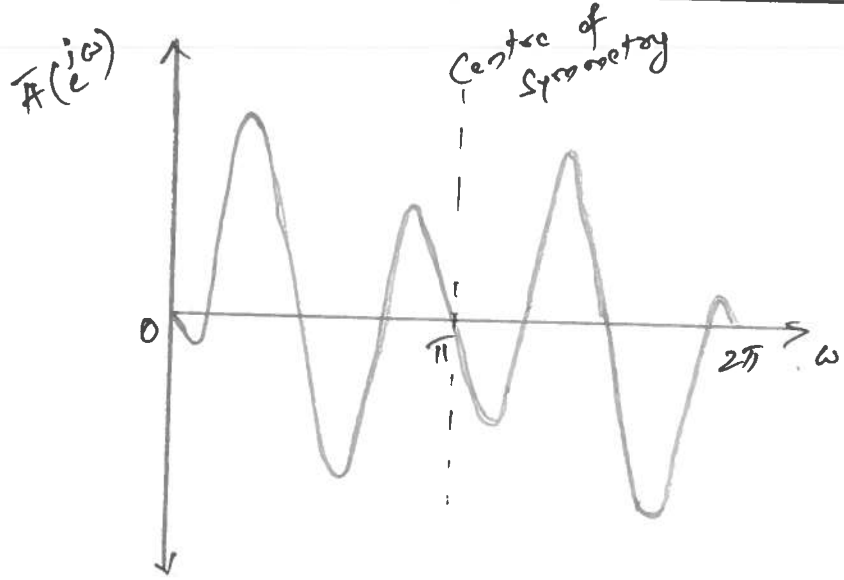
$$H(e^{j\omega}) = \bar{H}(e^{j\omega}) \cdot e^{j\theta(\omega)}$$

Amplitude function $\bar{H}(e^{j\omega}) = \sum_{n=1}^{\frac{N-1}{2}} 2h(n) \sin \omega n$

phase function $\theta(\omega) = -\pi/2$

Magnitude function $|H(e^{j\omega})| = |\bar{H}(e^{j\omega})| = \left| \sum_{n=1}^{\frac{N-1}{2}} 2h(n) \sin \omega n \right|$

Amplitude response of freq response is shown below. It can be observed that the amplitude function is antisymmetric with $\omega = \pi$. The term $e^{-j\pi/2}$ makes freq response imaginary. Hence this freq response is suitable for designing Hilbert transformers and differentiators.


$A(e^{j\omega})$
Centre of Symmetry
0
π
2π
ω

Summary of the characteristics of Linear phase FIR filters

Type	Frequency response $H(e^{j\omega})$	Magnitude response $\lvert \bar{H}(e^{j\omega}) \rvert$	Phase response $\angle H(e^{j\omega})$	Applications.
Symmetrical Impulse response N = odd.	$e^{-j\omega\frac{(N-1)}{2}}\left[\sum_{n=0}^{\frac{N-1}{2}} a(n)\cos\omega n\right]$ $a(0) = h\left(\frac{N-1}{2}\right)$ $a(n) = 2h\left[\left(\frac{N-1}{2}\right) - n\right]$	$\left\lvert \sum_{n=0}^{\frac{N-1}{2}} a(n)\cos\omega n \right\rvert$	$-\alpha\omega + \theta$ wher $\theta = 0$ for $\bar{H}(e^{j\omega}) > 0$ $\theta = \pi$ for $\bar{H}(e^{j\omega}) < 0$	Low pass, High pass, Band pass, Band stop
Symmetrical Impulse response N = even.	$e^{-j\omega\frac{(N-1)}{2}}\left[\sum_{n=1}^{\frac{N}{2}} b(n)\cos\left(n-\frac{1}{2}\right)\omega\right]$ $b(n) = 2h\left(\frac{N}{2} - n\right)$	$\left\lvert \sum_{n=1}^{N/2} b(n)\cos\left(n-\frac{1}{2}\right)\omega \right\rvert$	$-\alpha\omega + \theta$. where $\theta = 0$ for $\bar{H}(e^{j\omega}) > 0$ $\theta = \pi$ for $\bar{H}(e^{j\omega}) < 0$	Low pass, Band pass

Summary of Characteristics of Linear phase FIR filters.

Type	Frequency response $H(e^{j\omega})$	Magnitude response $\lvert\bar{H}(e^{j\omega})\rvert$	Phase response $\angle H(e^{j\omega})$	Applications.
Antisymmetrical Impulse response N=odd.	$e^{j\pi/2} \cdot e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot \sum_{n=1}^{\left(\frac{N-1}{2}\right)} c(n)\sin\omega n$ $c(n) = 2h\left(\frac{N-1}{2} - n\right)$	$\left\lvert \sum_{n=1}^{\frac{N-1}{2}} c(n)\sin\omega n \right\rvert$	$-\alpha\omega + \pi/2 + \theta$ where $\theta = 0$ for $\bar{H}(e^{j\omega}) > 0$ $\theta = \pi$ for $\bar{H}(e^{j\omega}) < 0$	Differentiator, Hilbert Transformer
Antisymmetrical Impulse response N=even	$e^{j\pi/2} \cdot e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=1}^{N/2} d(n)\sin\omega(n-1/2)$ $d(n) = 2h\left(\frac{N}{2} - n\right)$	$\left\lvert \sum_{n=1}^{N/2} d(n)\sin\omega(n-1/2) \right\rvert$	$-\alpha\omega + \frac{\pi}{2} + \theta$ where $\theta = 0$ for $\bar{H}(e^{j\omega}) > 0$ $\theta = \pi$ for $\bar{H}(e^{j\omega}) < 0$	Hilbert Transformer, Differentiator

Type	Frequency response $H(e^{j\omega})$	Magnitude response $\lvert H(e^{j\omega})\rvert$	Phase response $\angle H(e^{j\omega})$	Applications.
Symmetric Impulse response N=odd Centre of symmetry at n=0.	$H(e^{j\omega}) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2h(n)\cos\omega n$.	$\lvert H(e^{j\omega})\rvert = \left\lvert h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2h(n)\cos\omega n \right\rvert$	$\theta(\omega) = 0$.	LPF, HPF, BPF, BSF/BRF
Antisymmetric Impulse response N=odd with centre of antisymmetry at n=0.	$e^{j\pi/2} \sum_{n=1}^{\frac{N-1}{2}} 2h(n)\sin\omega n$.	$\left\lvert \sum_{n=1}^{\frac{N-1}{2}} 2h(n)\sin\omega n \right\rvert$	$\theta(\omega) = -\pi/2$	Hilbert transformers and differentiators

Prob: Determine the frequency response of FIR filter defined by $y(n) = 0.25x(n) + x(n-1) + 0.25x(n-2)$. Calculate the phase delay and Group delay.

Sol: Given $y(n) = 0.25x(n) + x(n-1) + 0.25x(n-2)$

Taking fourier transform on both sides

$$Y(e^{j\omega}) = 0.25\, X(e^{j\omega}) + e^{-j\omega} X(e^{j\omega}) + 0.25 \cdot e^{-2j\omega} X(e^{j\omega})$$

$$Y(e^{j\omega}) = (0.25 + e^{-j\omega} + 0.25 e^{-j2\omega})\, X(e^{j\omega})$$

$$\Rightarrow H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = 0.25 + e^{-j\omega} + 0.25 e^{-2j\omega}.$$

$$= e^{-j\omega}\left[0.25 e^{j\omega} + 1 + 0.25 e^{-j\omega}\right]$$

$$= e^{-j\omega}\left[0.25 (e^{j\omega} + e^{-j\omega}) + 1\right]$$

$$= e^{-j\omega}\left[0.25[2\cos\omega] + 1\right]$$

$$H(e^{j\omega}) = e^{-j\omega}(1 + 0.5\cos\omega] \longrightarrow (1)$$

W.K.T $\quad H(e^{j\omega}) = \overline{H}(e^{j\omega})\, e^{+j\theta(\omega)} \longrightarrow (2)$

Comparing eqn(1) and (2) $\quad \theta(\omega) = -\omega.$

The phase delay $\displaystyle T_p = \frac{-\theta(\omega)}{\omega} = \frac{-(-\omega)}{\omega} = 1$

The group delay $\displaystyle = -\frac{d\,\theta(\omega)}{d\omega} = \frac{-d\,(-\omega)}{d\omega} = -(-1) = 1$

Prob: The Impulse response of a causal LTI FIR system is given by $h(n) = a_0\delta(n) + a_1\delta(n-1) + a_2\delta(n-2) + a_3\delta(n-3) + a_4\delta(n-4) + a_5\delta(n-5) + a_6\delta(n-6)$ for what values of the impulse response samples will its frequency response $H(e^{j\omega})$ has linear phase?

Sol: A causal LTI system has Linear phase if its Impulse response $h(n)$ is symmetrical (or) Assymmetrical about $n = \frac{N-1}{2}$ satisfying $h(n) = h(N-1-n)$ (or) $h(n) = -h(N-1-n)$

$\Rightarrow$ The given impulse is of length 7.

$\therefore$ For linear phase the impulse response is either symmetrical or antisymmetrical about $n=3$.

for symmetrical impulse response

$$a_0 = a_6, \quad a_1 = a_5, \quad a_2 = a_4.$$

for antisymmetrical impulse response.

$$a_0 = -a_6; \quad a_1 = -a_5, \quad a_2 = -a_4, \quad a_3 = 0.$$

Prob: The impulse response of the non causal LTI-FIR DTS is given by

$$h(n) = a_0 \delta(n+2) + a_1 \delta(n+1) + a_2 \delta(n) + a_3 \delta(n-1) + a_4 \delta(n-2)$$

Find the values of the impulse response samples for which its frequency response $H(e^{j\omega})$ have a Zero phase.

Sol: A non causal LTI-FIR system have Zerophase if its impulse response $h(n)$ is symmetrical (or) antisymmetrical about $n=0$

$\Rightarrow$ The given Impulse response is of length 5

for symmetrical impulse response $a_0 = a_4$, $a_1 = a_3$

for antisymmetrical impulse response $a_2 = 0$, $a_3 = -a_1$,

$$a_4 = -a_0.$$

Design Techniques for Linear phase FIR filters

1. Fourier Series Method

Freq response of digital filter is periodic with period '2π'. From Fourier series analysis, any periodic function can be expressed as linear combination of complex exponentials. Hence the desired freq response $H_d(e^{j\omega})$ of FIR digital filter can be represented by Fourier series given by

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n)\, e^{-j\omega n} \quad -①$$

Fourier co-efficients $h_d(n)$ are the desired impulse response sequence of the filter.

By applying Inverse F.T to $H_d(e^{j\omega})$ we get

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) \cdot e^{j\omega n} \cdot d\omega \quad -②$$

Impulse response $h_d(n)$ is an infinite duration sequence. For FIR filters we truncate this infinite impulse response to a finite duration sequence of length 'N'

$$h(n) = h_d(n) \quad ; |n| \le \frac{N-1}{2} \text{ (or) } n = -\left(\tfrac{N-1}{2}\right) \text{ to } \left(\tfrac{N-1}{2}\right)$$

$$= 0 \quad ; \text{otherwise} \quad ③$$

Z.T is given by $H(z) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} h(n) \cdot z^{-n} \quad -④$

The transfer function $H(z)$ represents a non-causal FIR filter.

$$H(z) = h\left(-\left(\frac{N-1}{2}\right)\right) \cdot z^{\left(\frac{N-1}{2}\right)} + \dots + h(-2) \cdot z^{2} + h(-1)\, z^{1} + h(0)$$
$$+\, h(1) z^{-1} + h(2)\, z^{-2} + \dots + h\left(\frac{N-1}{2}\right) \cdot z^{-\left(\frac{N-1}{2}\right)}$$

$$= h(0) + \sum_{n=1}^{\frac{N-1}{2}} \left[h(n)\, z^{-n} + h(-n) \cdot z^{n}\right]$$

For symmetrical impulse response having centre of symmetry at $n=0$; $h(n) = h(-n)$

$$\Rightarrow H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\left(z^{n} + z^{-n}\right) \quad \text{— (5)}$$

$H(z)$ is non-causal digital FIR filter and hence it is not physically realizable.

$\therefore$ the above filter can be realized by multiplying eq (5) by $z^{-\left(\frac{N-1}{2}\right)}$, where $\frac{N-1}{2}$ is delay in samples. (or) delay factor.

$$\therefore H'(z) = z^{-\left(\frac{N-1}{2}\right)} \cdot H(z)$$

$$\boxed{H'(z) = z^{-\left(\frac{N-1}{2}\right)}\left[h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\left(z^{n} + z^{-n}\right)\right]}$$

The modification obtained by multiplying transfer function by delay factor $z^{-\left(\frac{N-1}{2}\right)}$ does not affect the amplitude response of the filter. But the abrupt truncation of Fourier series (eq-4) results in oscillations in passband and stop band. These oscillations are due to slow convergence of Fourier series, particularly near the points of discontinuity. This effect is called Gibbs phenomenon. These unwanted oscillations can be reduced by multiplying the desired impulse response co-efficients by an appropriate window function.

Design procedure for FIR filter by Fourier Series Method

1. Specifications of digital FIR filter are
   i) Desired freq response $H_d(e^{j\omega})$
   ii) Cutoff freq ω_c for Lowpass and highpass and ω_{c1} and ω_{c2} for bandpass and bandstop filters.

If analog filter cutoff freq F_c and sampling freq F_s are given, then calculate the cutoff freq of digital filter ω_c using the equation $\omega_c = \frac{2\pi F_c}{F_s}$

   iii) Number of samples of impulse response 'N'.

2. Determine desired impulse response $h_d(n)$ by taking inverse F.T of desired freq response $H_d(e^{j\omega})$

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) \cdot e^{j\omega n} \cdot d\omega$$

3. Calculate 'N' Samples $h_d(n)$ for $n = -\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$ and form the impulse response $h(n)$ of an FIR filter.

$\therefore$ Impulse response $h(n) = h_d(n)$ for $|n| \le \frac{N-1}{2}$
$= 0$ otherwise

As impulse response is symmetric with $n=0$, so $h(-n) = h(n)$. Hence it is sufficient to calculate $h(n)$ for $n = 0$ to $\left(\frac{N-1}{2}\right)$

4. Apply Z-T of impulse response to get non-causal transfer function of FIR filter $H(z)$

$$H(z) = Z\{h(n)\} = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} h(n) \cdot z^{-n}$$

5. Convert non causal transfer function $H(z)$ to causal transfer function $H'(z)$ by multiplying $H(z)$ by $z^{-\left(\frac{N-1}{2}\right)}$

$$H'(z) = z^{-\left(\frac{N-1}{2}\right)} \cdot H(z) = z^{-\left(\frac{N-1}{2}\right)} \cdot \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n) \cdot z^{-n}$$

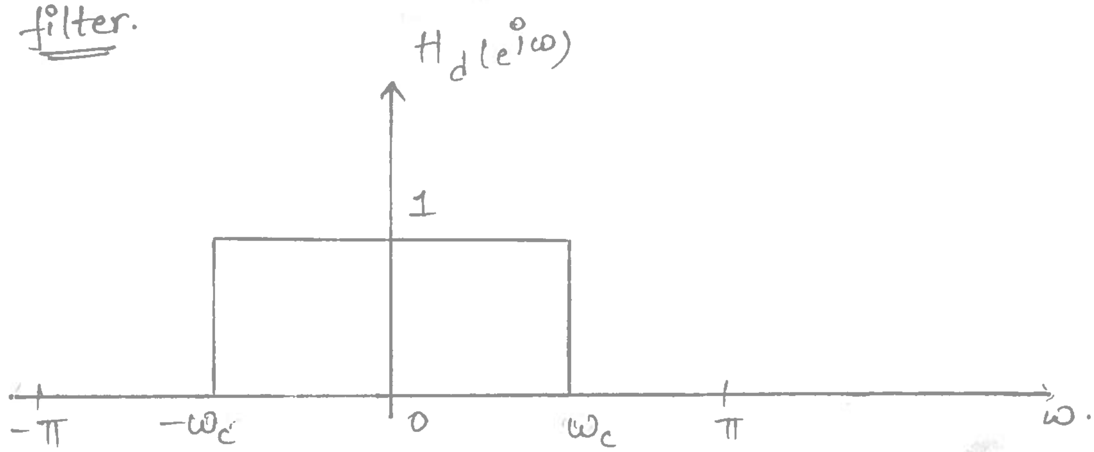
$$H'(z) = z^{-\left(\frac{N-1}{2}\right)} \left[h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) \left(z^{n} + z^{-n} \right) \right]$$

6. Draw a suitable structure for realization of FIR filter designed.

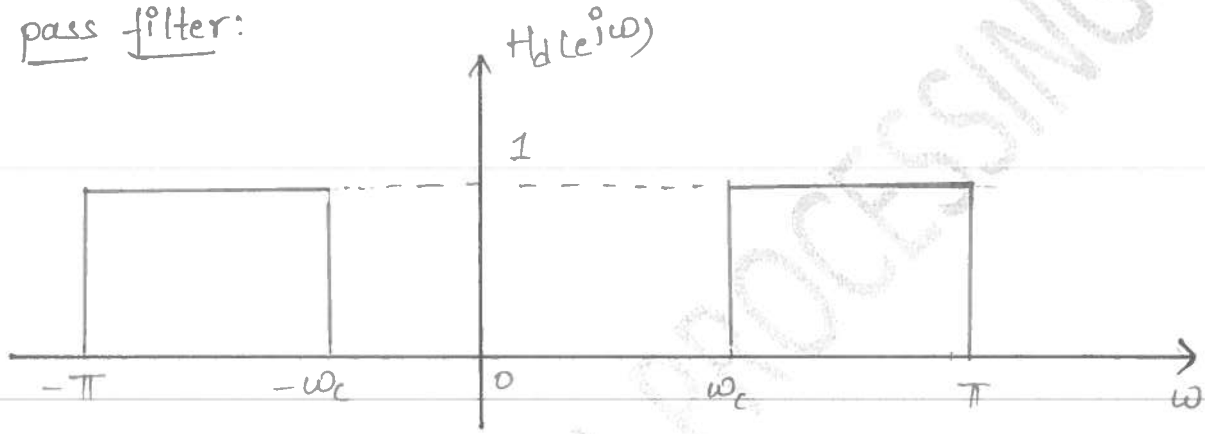
7. Draw freq response $H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$

ie $|H(e^{j\omega})|$ Vs ω (0 to π) & $\bar{H}(e^{j\omega})$ Vs ω (0 to π)
(Magnitude) (Amplitude)

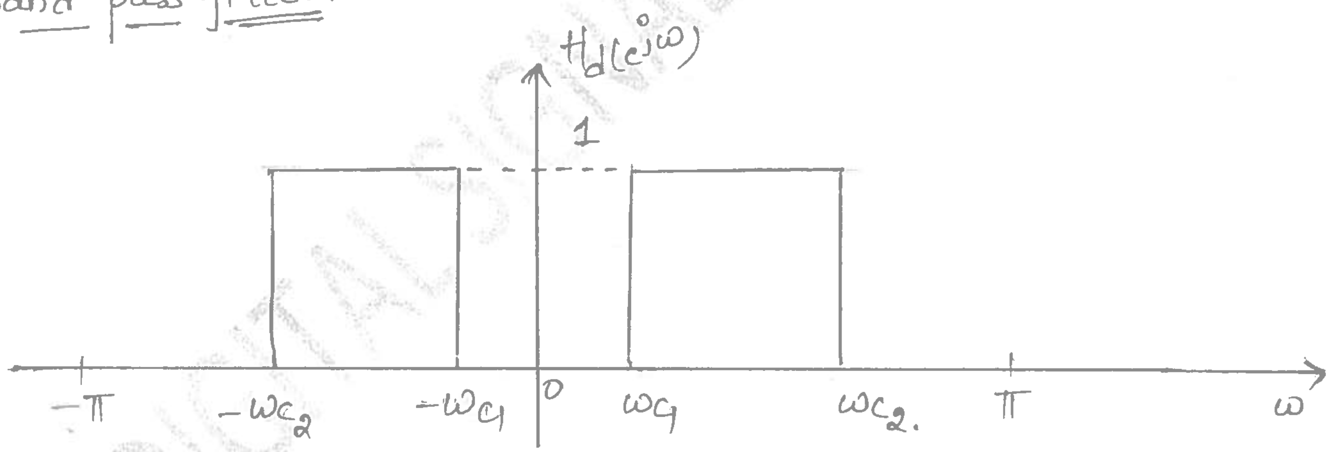
Lowpass filter.



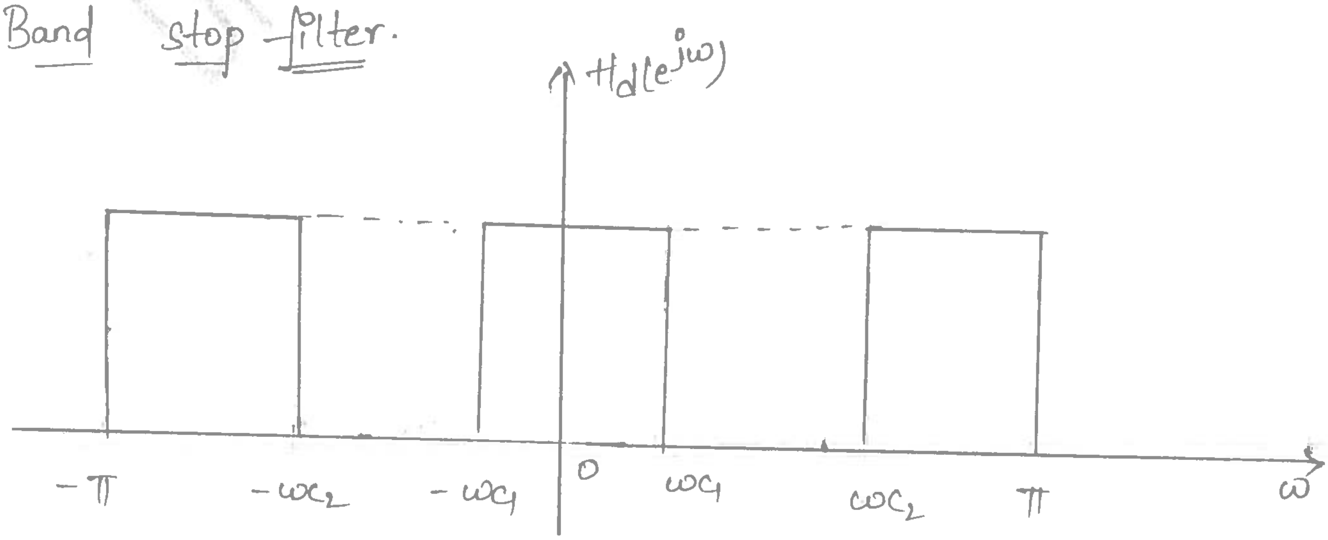
High pass filter:



Band pass filter:



Band stop filter.



The Normalised Ideal (Desired) Freq. response & Impulse response for FIR filter design using Windows:

Type of filter	Ideal (Desired) freq. response	Ideal (Desired) Impulse response.
Low pass	$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} ; & -\omega_c \le \omega \le +\omega_c \\ 0 ; & -\pi \le \omega < -\omega_c \\ 0 ; & \omega_c < \omega \le \pi \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\alpha} e^{j\omega n} d\omega.$ $[\because H_d(e^{j\omega}) = 0$ in range $-\pi \le \omega < -\omega_c$ and $\omega_c < \omega \le \pi]$
High pass	$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} ; & -\pi \le \omega \le -\omega_c \\ e^{-j\omega\alpha} ; & \omega_c \le \omega \le \pi \\ 0 ; & -\omega_c < \omega < \omega_c \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{-j\omega\alpha} \cdot e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{-j\omega\alpha} e^{j\omega n} d\omega$ $[\because H_d(e^{j\omega}) = 0$ in range $-\omega_c < \omega < +\omega_c]$
Band pass	$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} ; & -\omega_{c2} \le \omega \le -\omega_{c1} \\ e^{-j\omega\alpha} ; & \omega_{c1} \le \omega \le \omega_{c2} \\ 0 ; & -\pi \le \omega < -\omega_{c2} \\ 0 ; & -\omega_{c1} < \omega < \omega_{c1} \\ 0 ; & \omega_{c2} < \omega \le \pi \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\omega_{c2}}^{-\omega_{c1}} e^{-j\omega\alpha} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\omega_{c1}}^{\omega_{c2}} e^{-j\omega\alpha} \cdot e^{j\omega n} d\omega$ $[H_d(e^{j\omega}) = 0$ in range $-\pi \le \omega < -\omega_{c2}$; $-\omega_{c1} < \omega < \omega_{c1}$ and $\omega_{c2} < \omega \le \pi]$
Band Stop	$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} ; & -\pi \le \omega \le -\omega_{c2} \\ e^{-j\omega\alpha} ; & -\omega_{c1} \le \omega \le \omega_{c1} \\ e^{-j\omega\alpha} ; & \omega_{c2} \le \omega \le \pi \\ 0 ; & -\omega_{c2} < \omega < -\omega_{c1} \\ 0 ; & \omega_{c1} < \omega < \omega_{c2} \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_{c2}} e^{j\omega\alpha} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{-\omega_{c1}}^{\omega_{c1}} e^{-j\omega\alpha} \cdot e^{j\omega n} d\omega$ $+ \frac{1}{2\pi}\int_{\omega_{c2}}^{\pi} e^{-j\omega\alpha} e^{j\omega n} d\omega.$ $[\because H_d(e^{j\omega}) = 0$ in range $-\omega_{c2} < \omega < -\omega_{c1}$ & $\omega_{c1} < \omega < +\omega_{c2}]$

Specification and desired Impulse response for FIR filter design by Fourier Series Method.

Type of filter	Specifications	Impulse response.
1. Low pass	$H_d(e^{j\omega}) = \begin{cases} 1; & -\omega_c \le \omega \le +\omega_c \\ 0; & -\pi \le \omega \le -\omega_c \\ 0; & \omega_c \le \omega \le \pi \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{+\omega_c} e^{j\omega n} d\omega$ $[\because H_d(e^{j\omega}) = 0$ in range $-\pi \le \omega < -\omega_c$ and $+\omega_c < \omega \le \pi]$
2. High pass	$H_d(e^{j\omega}) = \begin{cases} 1; & -\pi \le \omega \le -\omega_c \\ 1; & \omega_c \le \omega \le \pi \\ 0; & -\omega_c < \omega < +\omega_c \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{j\omega n} d\omega.$ $[\because H_d(e^{j\omega}) = 0$ in range $-\omega_c < \omega < +\omega_c.$
3. Band pass	$H_d(e^{j\omega}) = \begin{cases} 1; & -\omega_{c_2} \le \omega \le -\omega_{c_1} \\ 1; & \omega_{c_1} \le \omega \le \omega_{c_2} \\ 0; & -\pi \le \omega < -\omega_{c_2} \\ 0; & -\omega_{c_1} < \omega < +\omega_{c_1} \\ 0; & \omega_{c2} < \omega \le \pi \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\omega_{c_2}}^{-\omega_{c_1}} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\omega_{c_1}}^{\omega_{c_2}} e^{j\omega n} d\omega$ $[\because H_d(e^{j\omega}) = 0$ in the range $-\pi \le \omega < -\omega_{c_2}$; $-\omega_{c_1} < \omega < +\omega_{c_1}$ and $+\omega_{c_2} < \omega \le \pi$
4. Band stop.	$H_d(e^{j\omega}) = \begin{cases} 1; & -\pi \le \omega \le -\omega_{c_2} \\ 1; & -\omega_{c_1} \le \omega \le +\omega_{c_1} \\ 1; & \omega_{c_2} \le \omega \le \pi \\ 0; & -\omega_{c_2} < \omega < -\omega_{c_1} \\ 0; & \omega_{c_1} < \omega < \omega_{c_2}. \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_{c_2}} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{-\omega_{c_1}}^{\omega_{c_1}} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\omega_{c_2}}^{\pi} e^{j\omega n} d\omega.$ $[\because H_d(e^{j\omega}) = 0$ in range $-\omega_{c_2} < \omega < -\omega_{c_1}$ and $\omega_{c_1} < \omega < \omega_{c_2}$

Prob: Design a FIR LPF with cutoff freq of 1kHz and sampling freq of 4 kHz with 11 samples using Fourier series method. Determine the freq response and verify the design by sketching the Magnitude response.

Sol: Given $F_c = 1\,kHz$; $F_s = 4\,kHz$

$$\omega_c = \frac{2\pi F_c}{F_s} = \Omega_c T = \frac{\Omega_c}{F_s}$$

$$\omega_c = \frac{2\pi F_c}{F_s} = \frac{2\pi \times 1 \times 10^3}{4 \times 10^3} = 0.5\pi \text{ rad/sample}$$

Desired freq response $H_d(e^{j\omega})$ of LPF is

$$H_d(e^{j\omega}) = 1 \quad \text{for} \quad -\omega_c \leq \omega \leq \omega_c$$
$$= 0 \quad \text{for} \quad -\pi \leq \omega \leq -\omega_c \ \& \ \omega_c < \omega \leq \pi$$

Desired impulse response $h_d(n)$ of LPF is

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) \cdot e^{j\omega n} \cdot d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1 \times e^{j\omega n} \cdot d\omega$$

$$= \frac{1}{2\pi}\left(\frac{e^{j\omega n}}{jn}\right)\Bigg|_{-\omega_c}^{+\omega_c} = \frac{1}{2\pi}\left[\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn}\right]$$

$$= \frac{1}{\pi n}\left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right] = \frac{1}{\pi n}\sin\omega_c n \,; \ \forall n$$

except at $n=0$

when $n=0$; $h_d(n) = h_d(0) = \lim_{n\to 0} \frac{\sin\omega_c n}{\pi n}$

$$= \frac{1}{\pi}\lim_{n\to 0}\frac{\sin\omega_c n}{n} = \frac{\omega_c}{\pi}$$

Impulse response $h(n)$ of FIR filter is obtained by truncating $h_d(n)$ to 11 samples.

$\therefore \quad h(n) = h_d(n) = \frac{\sin\omega_c n}{n\pi}$; for $n = -\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$

except at $n=0$

$$= \frac{\omega_c}{\pi} \quad ; \quad \text{for } n=0$$

$N=11 \Rightarrow \frac{N-1}{2} = 5$

Calculate $h(n)$ for $n = -5$ to 5

$h(n)$ satisfies symmetry about $n=0$ i.e $h(n) = h(-n)$

So we can calculate $h(n)$ for $n=0$ to 5

($\sin\theta$ - radians)

$n=0 \Rightarrow h(0) = \frac{\omega_c}{\pi} = 0.5$

$n=1 \Rightarrow h(1) = \sin\left(\frac{0.5\pi\times 1}{\pi\times 1}\right) = 0.3183$

$n=2 \Rightarrow h(2) = \sin\left(\frac{0.5\pi\times 2}{\pi\times 2}\right) = 0$

$n=3 \Rightarrow h(3) = \sin\left(\frac{0.5\pi\times 3}{\pi\times 3}\right) = -0.1061$

$n=4 \Rightarrow h(4) = \sin\left(\frac{0.5\pi \times 4}{\pi \times 4}\right) = 0$

$n=5 \Rightarrow h(5) = \sin\left(\frac{0.5\pi \times 5}{\pi \times 5}\right) = 0.0637$

using symmetry property

$n=1 \Rightarrow h(1) = h(-1) = 0.3183$

$n=-2 \Rightarrow h(2) = h(-2) = 0$

$n=-3 \Rightarrow h(3) = h(-3) = -0.1061$

$n=-4 \Rightarrow h(4) = h(-4) = 0$

$n=5 \Rightarrow h(-5) = h(5) = 0.0637$

Transfer function $H(z)$ of digital LPF is given by

$$H(z) = \sum_{n=-5}^{5} h(n) \cdot z^{-n}$$

$$= h(-5)\, z^{5} + h(-4)\, z^{4} + h(-3)\, z^{3} + h(-2)\, z^{2} + h(-1)\, z^{1}$$
$$+ h(0) + h(1)\, z^{-1} + h(2)\, z^{-2} + h(3)\, z^{-3} + h(4)\, z^{-4} + h(5)\, z^{-5}$$

using symmetry property $h(n) = h(-n)$

$$H(z) = h(0) + h(1)\left[z + z^{-1}\right] + h(2)\left(z^{2} + z^{-2}\right) + h(3)\left(z^{3} + z^{-3}\right)$$
$$+ h(4)\left(z^{4} + z^{-4}\right) + h(5)\left(z^{5} + z^{-5}\right)$$

$H(z)$ is a non-causal FIR filter and to convert it to causal FIR filter multiply $H(z)$ by z^{-5}

i.e $\boxed{H'(z) = z^{-5} H(z)}$

$$H'(z) = h(0)\cdot z^{-5} + h(1)\left[z^{-4} + z^{-6}\right] + h(2)\left(z^{-3} + z^{-7}\right)$$
$$+ h(3)\left[z^{-2} + z^{-8}\right] + h(4)\left[z^{-1} + z^{-9}\right] + h(5)\left(z^{0} + z^{-10}\right)$$

$$H'(z) = 0.5\, z^{-5} + 0.3183\left(z^{-4} + z^{-6}\right) - 0.1061\left(z^{-2} + z^{-8}\right)$$
$$+ 0.0637\left(1 + z^{-10}\right) \qquad \begin{pmatrix} h(2) = 0 \\ h(4) = 0 \end{pmatrix}$$

Frequency response (There are two Methods)

(1) Method-1

When impulse response is symmetric and N is odd with centre of symmetry at $n=0$, the magnitude response $|H(e^{j\omega})|$ is given by $|\bar{H}(e^{j\omega})|$

$$\bar{H}(e^{j\omega}) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2\, h(n) \cos \omega n$$

$$= h(0) + \sum_{n=1}^{5} 2 h(n) \cos \omega n$$

$$= h(0) + 2h(1) \cos \omega + 2h(2) \cos 2\omega$$
$$+ 2h(4) \cos 3\omega + 2h(5) \cos 5\omega$$

$$= 0.5 + 2\times 0.3183 \cos \omega + 2\times 0 \cos 2\omega + 2\times -0.1061 \cos 3\omega$$
$$+ 2\times 0 \cos 4\omega + 2\times 0.0637 \cos 5\omega$$

$$= 0.5 + 0.6366 \cos \omega - 0.2122 \cos 3\omega + 0.1274 \cos 5\omega$$

using above equation, the amplitude response $\bar{H}(e^{j\omega})$ and magnitude function $|H(e^{j\omega})| = |\bar{H}(e^{j\omega})|$ are calculated for different values of 'ω'.

ω	$0\times\pi/16$	$1\times\pi/16$	$2\times\pi/16$	$3\times\frac{\pi}{16}$	$4\times\frac{\pi}{16}$	$5\times\frac{\pi}{16}$
$\bar{H}(e^{j\omega})$	1.0578	1.0187	0.9581	0.9457	1.0101	1.0866
$\lvert H(e^{j\omega})\rvert = \lvert\bar{H}(e^{j\omega})\rvert$	1.0578	1.0187	0.9581	0.9457	1.0101	1.0866

$6\times\frac{\pi}{16}$	$7\times\frac{\pi}{16}$	$8\times\frac{\pi}{16}$	$9\times\frac{\pi}{16}$	$10\times\frac{\pi}{16}$	$11\times\frac{\pi}{16}$	$12\times\frac{\pi}{16}$
1.0573	0.8480	0.5	0.1520	−0.0574	−0.0866	−0.0101
1.0573	0.8480	0.5	0.1520	0.0574	0.0866	0.0101

$13\times\frac{\pi}{16}$	$14\times\frac{\pi}{16}$	$15\times\frac{\pi}{16}$	$16\times\frac{\pi}{16}$
0.0542	0.418	−0.0187	−0.0578
0.0542	0.418	0.0187	0.0578

(ii) Method-2

Frequency Response $H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$

$$H(e^{j\omega}) = \Big[0.5\, z^{-5} + 0.3183\,(z^{-4} + z^{-6}) - 0.1061\,(z^{-2} + z^{-8}) + 0.0637\,(1 + z^{-10})\Big]\Big|_{z=e^{j\omega}}$$

$$= 0.5\, e^{-j5\omega} + 0.3183\,(e^{-j4\omega} + e^{-j6\omega}) - 0.1061\,(e^{-j2\omega} + e^{-j8\omega}) + 0.0637\,(1 + e^{-j10\omega})$$

$$= 0.5(\cos 5\omega - j\sin 5\omega) + 0.3183(\cos 4\omega - j\sin 4\omega + \cos 6\omega - j\sin 6\omega) - 0.1061(\cos 2\omega - j\sin 2\omega + \cos 8\omega - j\sin 8\omega) + 0.0637(1 + \cos 10\omega - j\sin 10\omega)$$

$$= \big[0.5\cos 5\omega + 0.3183\cos 4\omega + 0.3183\cos 6\omega - 0.1061\cos 2\omega - 0.1061\cos 8\omega + 0.0637 + 0.0637\cos 10\omega\big] + j\big[-0.5\sin 5\omega - 0.3183\sin 4\omega - 0.3183\sin 6\omega + 0.1061\sin 2\omega + 0.1061\sin 8\omega - 0.0637\sin 10\omega\big]$$

using above equation freq response $H(e^{j\omega})$ and magnitude function $|H(e^{j\omega})|$ of LPF are calculated for various values of ω

ω	$H(e^{j\omega})$	$\lvert H(e^{j\omega})\rvert$
$0 \times \frac{\pi}{16}$	$1.0578 + j0$	1.0578
$1 \times \frac{\pi}{16}$	$0.5659 - j\,0.8970$	1.0186
$2 \times \frac{\pi}{16}$	$-0.3667 - j0.8852$	0.9581

$3\times\frac{\pi}{16}$	$-0.9276 - j0.1895$	0.9457
$4\times\frac{\pi}{16}$	$-0.7143 - j0.7143$	1.0102
$5\times\frac{\pi}{16}$	$0.2120 + j1.0657$	1.0866
$6\times\frac{\pi}{16}$	$0.9769 + j0.4046$	1.0574
$7\times\frac{\pi}{16}$	$0.7050 - j0.4711$	0.8479
$8\times\frac{\pi}{16}$	$0 - j0.5$	0.5
$9\times\frac{\pi}{16}$	$-0.1264 - j0.0844$	0.1520
$10\times\frac{\pi}{16}$	$0.0530 - j0.22$	0.0574
$11\times\frac{\pi}{16}$	$0.0169 - j0.0850$	0.0866
$12\times\frac{\pi}{16}$	$-0.0071 - j0.0071$	0.0100
$13\times\frac{\pi}{16}$	$0.0532 - j0.0106$	0.0542
$14\times\frac{\pi}{16}$	$0.0160 - j0.0386$	0.0418
$15\times\frac{\pi}{16}$	$0.0104 + 0.0155$	0.0186
$16\times\frac{\pi}{16}$	$0.0578 + j0$	0.0578

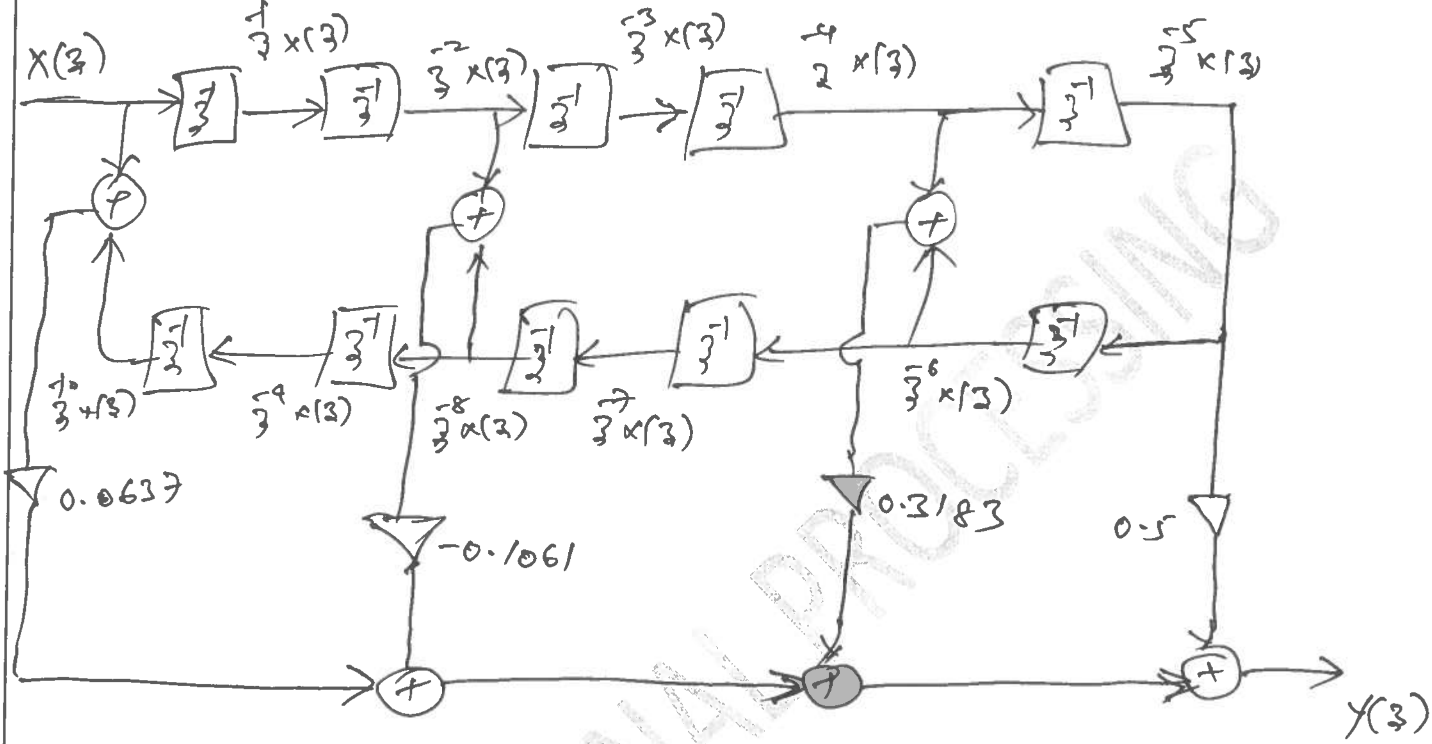
Realization Structure:

$$H(z) = \frac{Y(z)}{X(z)} = 0.5z^{-5} + 0.3183\left(z^{-4} + z^{-6}\right) - 0.1061\left(z^{-2} + z^{-8}\right)$$

$$+ 0.0637\left(1 + z^{-10}\right)$$

$$Y(z) = 0.5\, z^{-5} X(z) + 0.3183\left[z^{-4} X(z) + z^{-6} X(z)\right] - 0.1061\left(z^{-2} X(z) + z^{-8} X(z)\right) + 0.0637\left(X(z) + z^{-10} X(z)\right)$$

Linear phase structure of FIR LPF.

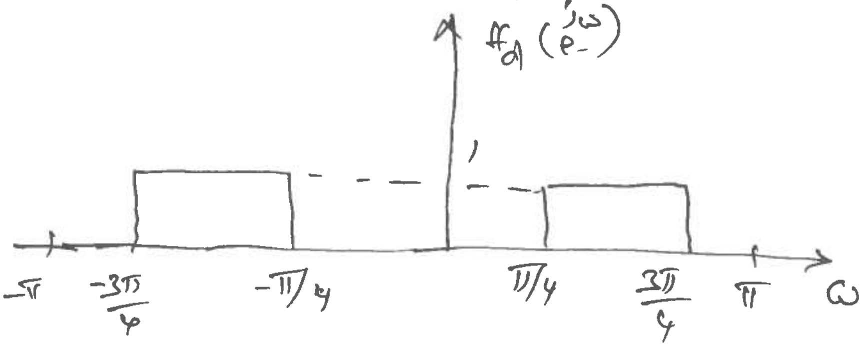


Prob: Design an ideal band pass filter with a frequency response

$$H_d(e^{j\omega}) = 1 \quad \text{for } \pi/4 \le |\omega| \le \frac{3\pi}{4}$$
$$= 0 \quad \text{otherwise}$$

Find the values of $h(n)$ for $N=11$ and plot the frequency response.

Sol: Ideal frequency response of the filter is shown below



$N=11$ given

desired impulse response $h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n} \cdot d\omega$

$$h_d(n) = \frac{1}{2\pi}\left[\int_{-\frac{3\pi}{4}}^{-\pi/4} e^{j\omega n} \cdot d\omega + \int_{\pi/4}^{\frac{3\pi}{4}} e^{j\omega n} \cdot d\omega\right]$$

$$= \frac{1}{2\pi j n}\left[e^{-\frac{j\pi n}{4}} - e^{-\frac{j3\pi n}{4}} + e^{\frac{j3\pi n}{4}} - e^{\frac{j\pi n}{4}}\right]$$

$$= \frac{1}{\pi n}\left(\sin\frac{3\pi}{4}n - \sin\frac{\pi}{4}n\right) ; \forall n$$

Impulse response $h(n)$ is obtained by truncating $h_d(n)$ for 11 samples.

$$h(n) = h_d(n) \quad \text{for} \quad -\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right)$$

$$= 0 \quad \text{otherwise}$$

for $n=0$; $h(0) = h_d(0)$ so we have to find in other way

$$h(0) = h_d(0) = \frac{1}{2\pi}\left(\int_{-\frac{3\pi}{4}}^{-\pi/4} d\omega + \int_{\pi/4}^{\frac{3\pi}{4}} d\omega\right)$$

$$= \frac{1}{2\pi}\left[-\frac{\pi}{4} + \frac{3\pi}{4} + \frac{3\pi}{4} - \frac{\pi}{4}\right]$$

$$= \frac{1}{2} = 0.5$$

The filters Co-efficients are symmetrical about $n=0$ Satisfying the condition $h(n) = h(-n)$. Hence we can find $h(n)$ for 1 to $\left(\frac{N-1}{2}\right)$ (i.e 1 to 5)

$n=1$; $h(1) = h_d(1) = \dfrac{\sin\frac{3\pi}{4} - \sin\frac{\pi}{4}}{\pi} = 0$

$n=2$; $h(2) = h_d(2) = \dfrac{\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}}{2\pi} = -0.3183$

$n=3$; $h(3) = h_d(3) = \dfrac{\sin\frac{9\pi}{4} - \sin\frac{3\pi}{4}}{3\pi} = 0$

$n=4$; $h(4) = h_d(4) = \dfrac{\sin 3\pi - \sin\pi}{4\pi} = 0$

$n=5$; $h(5) = h_d(5) = \dfrac{\sin\frac{15\pi}{4} - \sin\frac{5\pi}{4}}{5\pi} = 0$

$\therefore$ $h(1) = h(-1) = 0$

$h(2) = h(-2) = -0.3183$

$h(3) = h(-3) = 0$

$h(4) = h(-4) = 0$

$h(5) = h(-5) = 0$

Transfer function of FIR filter (non-causal) is

$$H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\left(z^{n} + z^{-n}\right)$$

$$H(z) = 0.5 - 0.3183\left(z^{2} + z^{-2}\right)$$

Transfer function of causal FIR filter that can be realized is $H'(z) = z^{-\left(\frac{N-1}{2}\right)} H(z)$

$$H'(z) = z^{-5}\left(0.5 - 0.3183\left(z^{2}+z^{-2}\right)\right)$$

$$= -0.3183\, z^{-3} + 0.5\, z^{-5} - 0.3183\, z^{-7}$$

Filter co-efficients of causal filter are

$h(0) = h(10) = h(1) = h(9) = h(2) = h(8) = h(4) = h(6) = 0$

$h(3) = h(7) = -0.3183$

$h(5) = 0.5$

Amplitude response $\bar{H}(e^{j\omega})$ is given by

$$\bar{H}(e^{j\omega}) = \sum_{n=1}^{\frac{N-1}{2}} a(n) \cos\omega n$$

$a(0) = h\left(\frac{N-1}{2}\right) = h(5) = 0.5$

$a(n) = 2h\left(\frac{N-1}{2} - n\right)$

$a(1) = 2h(5-1) = 2h(4) = 0$

$a(2) = 2h(5-2) = 2h(3) = -0.6366$

$a(3) = 2h(5-3) = 2h(2) = 0$

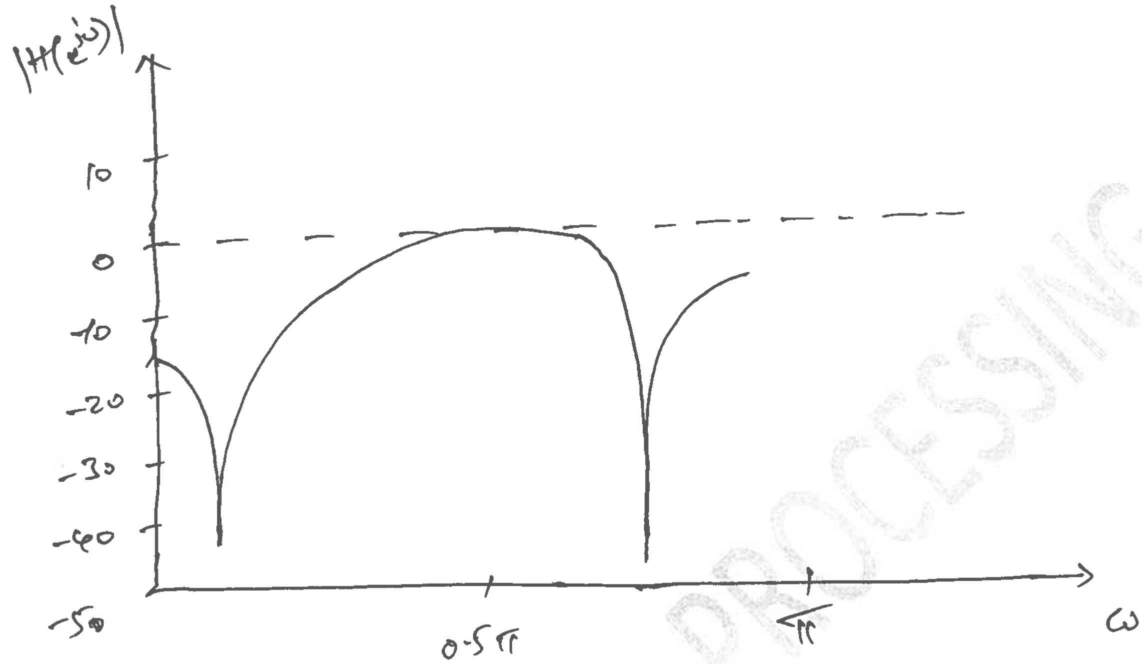
$a(4) = 2h(5-4) = 2h(1) = 0$

$a(5) = 2h(5-5) = 2h(0) = 0$

$$\therefore \boxed{\bar{H}(e^{j\omega}) = 0.5 - 0.6366 \cos 2\omega}$$

ω (degrees)	0	20	30	45	60	75	90
$\bar{H}(e^{j\omega})$	-0.1366	0.012	0.1817	0.5	0.818	1.05	1.1366
$\lvert H(e^{j\omega})\rvert_{dB}$	-17.3	-38.17	-14.8	-6.02	-1.74	0.4346	1.11

105	120	135	150	160	180
1.05	0.818	0.5	0.1817	0.012	−0.1366
0.4346	−1.74	−6.02	−14.8	−38.17	−17.3



Design of FIR filters Using Windows

In Fourier series method, Fourier Co-efficients have infinite length. To obtain FIR filter, we have to truncate the infinite Fourier series at $n = \pm\left(\frac{N-1}{2}\right)$.
But abrupt truncation of Fourier series results in oscillation in pass band and stop band. These oscillations are due to slow convergence of Fourier series and this effect is known as Gibbs phenomenon.
To reduce these ripples & oscillations, the Fourier Co-efficients $(h_d(n))$ of the filter are modified by multiplying the infinite impulse response with a finite weighing sequence $\omega(n)$ called a window function defined as

$$\boxed{\begin{aligned} \omega(n) &= \omega(-n) \neq 0; \text{ for } |n| \leq \left(\frac{N-1}{2}\right) \\ &= 0 \quad \text{for } |n| > \left(\frac{N-1}{2}\right) \end{aligned}}$$

The window function chosen for truncating the infinite impulse response should have some desirable characteristics. They are

i) Central lobe of freq response of window should contain most of energy and should be narrow.

ii) Highest side lobe level of the freq response should be small.

iii) Side lobes of freq response should decrease in energy rapidly as ω tends to π.

Reason :- Finite duration sequence $h(n)$ satisfies the desired magnitude response

$$h(n) = h_d(n)\, w(n) \quad \text{for all } |n| \le \left(\frac{N-1}{2}\right)$$

$$= 0 \quad \text{for } |n| < \left(\frac{N-1}{2}\right)$$

Frequency response $H(e^{j\omega})$ of filter can be obtained by convolution of $H_d(e^{j\omega})$ and $W(e^{j\omega})$, given by

$$H(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\theta}) \cdot W\left(e^{j(\omega-\theta)}\right) . d\theta$$

$$H(e^{j\omega}) = H_d(e^{j\omega}) * W(e^{j\omega})$$

From above equation we can see that the freq response of the filter $H(e^{j\omega})$ depends on freq response of the window $W(e^{j\omega})$.

Different types of are

1. Rectangular window
2. Triangular & Bartlett
3. Hanning window
4. Hamming window
5. Blackmann window
6. Kaiser window.

1. Rectangular window : N - point rectangular window $W_R(n)$ is defined as

$$w_R(n) = 1 \quad ; \; n = -\left(\frac{N-1}{2}\right) \text{ to } \frac{N-1}{2}$$
$$= 0 \quad ; \text{otherwise}$$

→ used for odd values of 'N' only

(or)

$$w_R(n) = 1 \quad ; \; n = 0 \text{ to } N-1$$
$$= 0 \quad ; \text{otherwise}$$

→ used for both odd and even values of 'N'

Spectrum of $w_R(n)$ is obtained by applying F.T

$$W_R(e^{j\omega}) = F\{w_R(n)\}$$

$$= \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} 1 \times e^{-j\omega n}$$

$$= e^{j\omega\left(\frac{N-1}{2}\right)} + \cdots + e^{j\omega} + 1 + e^{-j\omega} + e^{-j2\omega} + \cdots + e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$= e^{j\omega\left(\frac{N-1}{2}\right)}\left[1 + e^{-j\omega} + e^{-j2\omega} + \cdots + e^{-j\omega(N-1)}\right]$$

$$= e^{j\omega\left(\frac{N-1}{2}\right)}\left[\frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}\right] \qquad \left(1 + a + a^2 + \cdots + a^{N-1} = \frac{1-a^N}{1-a}\right)$$

$$= \frac{e^{j\omega\frac{N}{2}}\left(1 - e^{-j\omega N}\right)}{e^{j\omega/2}\left(1 - e^{-j\omega}\right)} = \frac{e^{j\omega\frac{N}{2}} - e^{-j\omega\frac{N}{2}}}{e^{j\omega/2} - e^{-j\omega/2}}$$

$$\boxed{W_R(e^{j\omega}) = \frac{\sin \omega\frac{N}{2}}{\sin\frac{\omega}{2}}}$$

The frequency spectrum for N=25 is shown below. (amplitude response, magnitude response and log magnitude response are shown below)

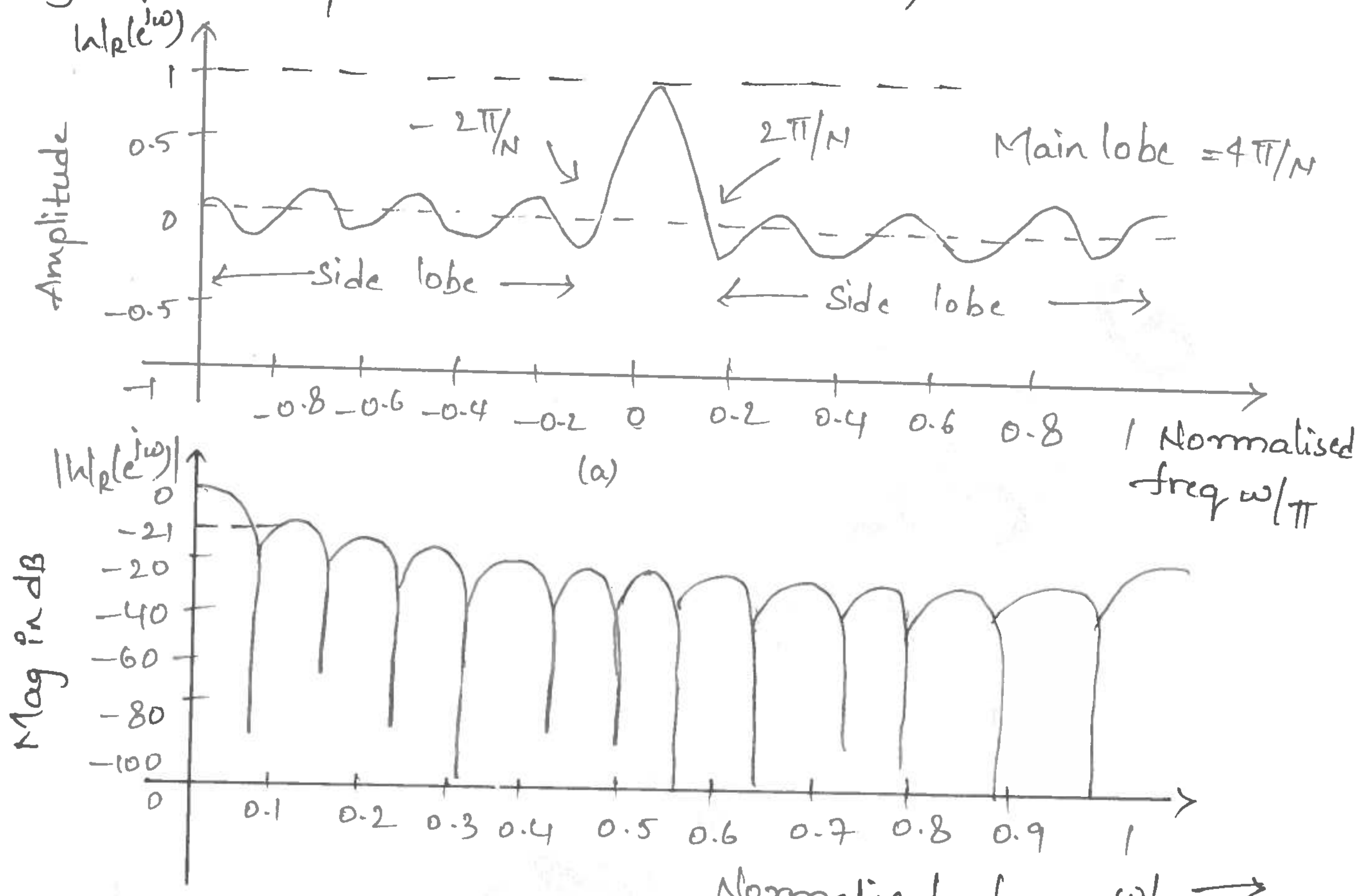


fig a: Frequency response of Rect window for N=25
b: Log magnitude response of Rect window for N=25

Frequency response is real and its zero occurs $\omega = \frac{2k\pi}{N}$
Response b/n $\omega = \frac{2\pi}{N}$ and $-\frac{2\pi}{N}$ is called main lobe and other lobes are sidelobes
Width of main lobe is $\frac{4\pi}{N}$. This causes smearing of desired transfer function features
Side lobes extended over a wide freq. range i.e., larger magnitude components become smeared over a ~~wig~~ wide range of frequencies in $H(e^{j\omega})$. In pass band these side lobes effects

results in over shoots and undershoots to desired response. In stop band these effects appear as a non-zero response. These sidelobes donot decrease significantly but they remain constant as duration of rectangular window increases.

The highest side lobe is appox 22% of main lobe amplitude (or) -13dB relative to the maximum value at $\omega = 0$.

To reduce the effect of the sidelobes, window sequence having spectrum exhibiting smaller sidelobes must be considered i.e., amplitudes of high freq components (i.e., side lobe level) can be reduced by replacing these sharp transitions by gradual ones. Triangular and Cosine windows are devolo-ped from this.

2. Triangular (or) Bartlett Window

N point triangular window $w_T(n)$ is defined as

$$w_T(n) = \begin{cases} 1 - \dfrac{2|n|}{N-1} & \text{for } |n| \le \left(\dfrac{N-1}{2}\right) \\ 0 & ; \text{otherwise} \end{cases}$$

→ Only for odd values of 'N'

(or)

$$w_T(n) = \begin{cases} 1 - \dfrac{2\left|n - \frac{N-1}{2}\right|}{N-1} & ; \text{for } 0 \text{ to } (N-1) \\ 0 & ; \text{otherwise} \end{cases}$$

→ for both odd and even values of 'N'

Freq response is given by $H_T(e^{j\omega}) = \left[\dfrac{\sin\omega\left(\frac{N+1}{4}\right)}{\sin\frac{\omega}{2}}\right]^2$

Amplitude, Magnitude and log Magnitude plots are shown below.

The side lobe level is smaller than that of the rectangular window being reduced from -13 dB to -25 dB. But main lobe width is $\frac{8\pi}{N}$ i.e., twice that of rectangular window.

Triangular window produces smoother magnitude response of $|H(e^{j\omega})|$ both in passband and stop band. But it has two disadvantages * In stop band response is smoother but attenuation is less than that produced by rectangular window. * Transition is more compared to that of rectangular window.

fig: Freq. response

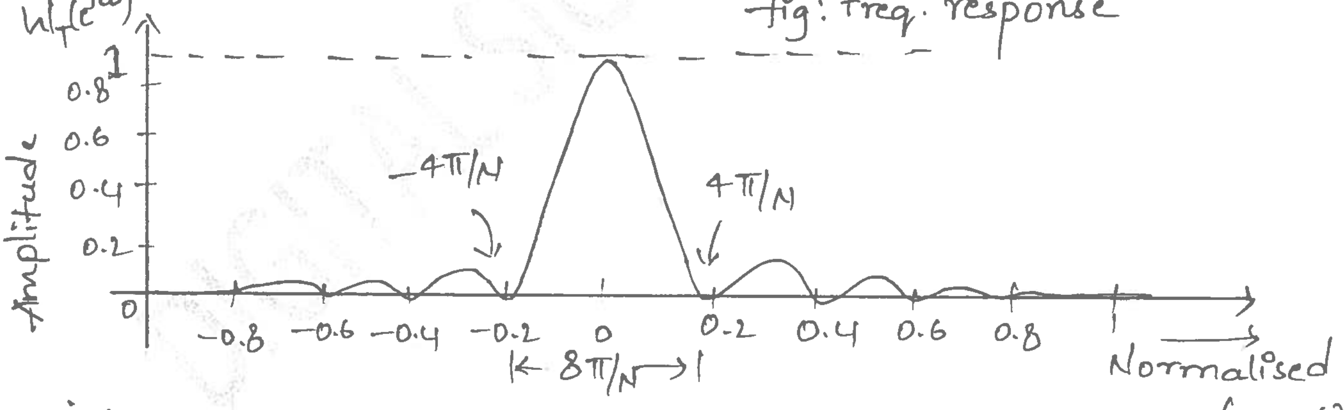
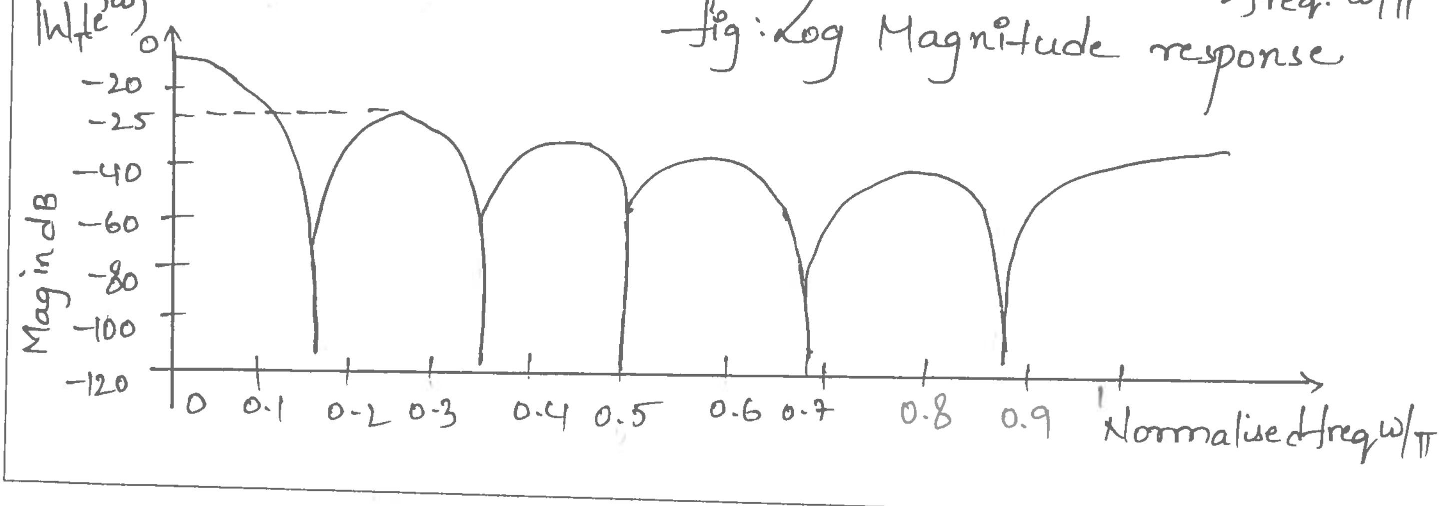


fig: Log Magnitude response



3. Raised Cosine window

N - point raised cosine window $w_{RC}(n)$ is defined as

$$w_\alpha(n) \text{ (or) } w_{RC}(n) = \alpha + (1-\alpha)\cos\frac{2\pi n}{N-1} \; ; \; n = -\left(\frac{N-1}{2}\right) \text{ to } \left(\frac{N-1}{2}\right)$$

$$= 0 \quad ; \text{ otherwise}$$

→ only for odd values of 'N'

(or)

$$w_\alpha(n) \text{ (or) } w_{RC}(n) = \alpha - (1-\alpha)\cos\frac{2\pi n}{N-1} \; ; \; n = 0 \text{ to } N-1$$

$$= 0 \quad ; \text{ otherwise}$$

→ for both odd and even values of 'N'

Freq Response $W_\alpha(e^{j\omega}) = F\{w_\alpha(n)\}$

$$W_\alpha(e^{j\omega}) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} \left[\alpha + (1-\alpha)\cos\frac{2\pi n}{N-1}\right] e^{-j\omega n}$$

$$= \alpha \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} e^{-j\omega n} + \left(\frac{1-\alpha}{2}\right) \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} \left[e^{+j\frac{2\pi n}{N-1}} + e^{-j\frac{2\pi n}{N-1}}\right] e^{-j\omega n}$$

$$= \underbrace{\alpha \cdot \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} e^{-j\omega n}}_{A} + \underbrace{\left(\frac{1-\alpha}{2}\right) \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} e^{-j\left(\omega - \frac{2\pi}{N-1}\right)n}}_{B} + \underbrace{\left(\frac{1-\alpha}{2}\right) \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} e^{-j\left(\omega + \frac{2\pi}{N-1}\right)n}}_{C}$$

$$A \Rightarrow \alpha \cdot \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} e^{-j\omega n} = \alpha\left[e^{j\omega\left(\frac{N-1}{2}\right)} + \dots + e^{j\omega} + 1 + e^{-j\omega} + e^{-j2\omega} + \dots + e^{-j\omega\left(\frac{N-1}{2}\right)} \right]$$

$$= \alpha \cdot e^{j\omega\left(\frac{N-1}{2}\right)} \left[\frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} \right]$$

$$= \alpha \cdot \left[\frac{e^{j\omega\frac{N}{2}} - e^{-j\omega\frac{N}{2}}}{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}} \right] = \alpha \cdot \frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}}$$

$$B \Rightarrow \left(\frac{1-\alpha}{2}\right) \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} e^{-j\left(\omega - \frac{2\pi}{N-1}\right)n}$$

$$= \left(\frac{1-\alpha}{2}\right)\left[e^{j\left(\omega-\frac{2\pi}{N-1}\right)\left(\frac{N-1}{2}\right)} + \dots + e^{j\left(\omega-\frac{2\pi}{N-1}\right)} + 1 + e^{-j\left(\omega-\frac{2\pi}{N-1}\right)} + e^{-j2\left(\omega-\frac{2\pi}{N-1}\right)} + \dots + e^{-j\left(\omega-\frac{2\pi}{N-1}\right)\left(\frac{N-1}{2}\right)} \right]$$

$$= \left(\frac{1-\alpha}{2}\right)\left[\frac{e^{j\left(\omega-\frac{2\pi}{N-1}\right)\left(\frac{N-1}{2}\right)} \left(1 - e^{-j\left(\omega-\frac{2\pi}{N-1}\right)N}\right)}{\left(1 - e^{-j\left(\omega-\frac{2\pi}{N-1}\right)}\right)} \right]$$

$$= \left(\frac{1-\alpha}{2}\right)\left[\frac{e^{j\left(\frac{\omega N}{2} - \frac{\pi N}{N-1}\right)} \left(1 - e^{-j\left(\omega-\frac{2\pi}{N-1}\right)N}\right)}{e^{j\left(\frac{\omega}{2} - \frac{\pi}{N-1}\right)} \left(1 - e^{-j\left(\omega-\frac{2\pi}{N-1}\right)}\right)} \right]$$

$$= \left(\frac{1-\alpha}{2}\right)\left[\frac{e^{j\left(\frac{\omega N}{2}-\frac{\pi N}{N-1}\right)} - e^{-j\left(\frac{\omega N}{2}-\frac{\pi N}{N-1}\right)}}{e^{j\left(\frac{\omega}{2}-\frac{\pi}{N-1}\right)} - e^{-j\left(\frac{\omega}{2}-\frac{\pi}{N-1}\right)}}\right]$$

$$= \left(\frac{1-\alpha}{2}\right)\left[\frac{\sin\left(\frac{\omega N}{2}-\frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2}-\frac{\pi}{N-1}\right)}\right]$$

Similarly $C = \left(\frac{1-\alpha}{2}\right)\left[\frac{\sin\left(\frac{\omega N}{2}+\frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2}+\frac{\pi}{N-1}\right)}\right]$

$$\therefore W_\alpha(e^{j\omega}) = \alpha\frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}} + \left(\frac{1-\alpha}{2}\right)\left(\frac{\sin\left(\frac{\omega N}{2}-\frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2}-\frac{\pi}{N-1}\right)}\right)$$

$$+\left(\frac{1-\alpha}{2}\right)\left(\frac{\sin\left(\frac{\omega N}{2}+\frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2}+\frac{\pi}{N-1}\right)}\right).$$

Raised cosine windows are smoother at ends, but closer to one at middle. Smoother ends and broader middle section produces less distortion of $h_d(n)$ around $n=0$.

4. **Hanning window :-** This a type of raised Cosine window. (for $\alpha = 0.5$)

$$\omega_{Hn}(n) = 0.5 + 0.5 \cos\frac{2\pi n}{N-1} \quad ; \text{ for } n = -\left(\frac{N-1}{2}\right) \text{ to } \left(\frac{N-1}{2}\right)$$

$$= 0 \quad \text{otherwise}$$

$\rightarrow$ N - odd only

(or)

$$\omega_{Hn}(n) = 0.5 - 0.5 \cos\frac{2\pi n}{N-1} \quad ; \text{ for } n = 0 \text{ to } N-1$$

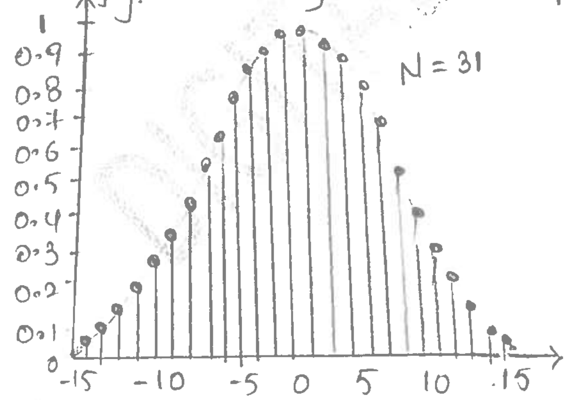
$$= 0 \quad \text{otherwise}$$

$\rightarrow$ N - odd/even both.

Freq response or freq spectrum of Hanning window $W_{Hn}(e^{j\omega})$ is obtained by taking F.T of Hanning window $\omega_{Hn}(n)$.

$$\therefore W_{Hn}(e^{j\omega}) = F\left[\omega_{Hn}(n)\right] = 0.5 \frac{\sin\frac{\omega N}{2}}{} + 0.25 \frac{\sin\left(\frac{\omega N}{2} - \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi}{N-1}\right)} + 0.25 \frac{\sin\left(\frac{\omega N}{2} + \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{N-1}\right)}$$

fig: Hanning window sequence



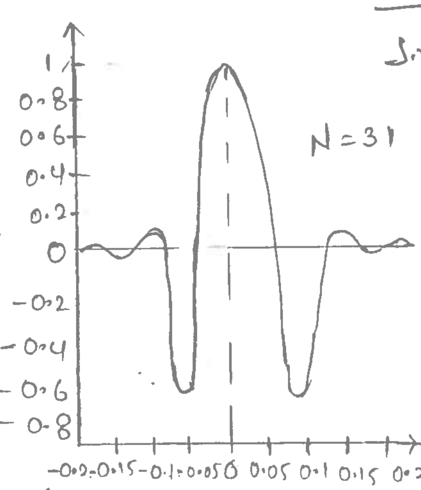


fig: Amplitude response

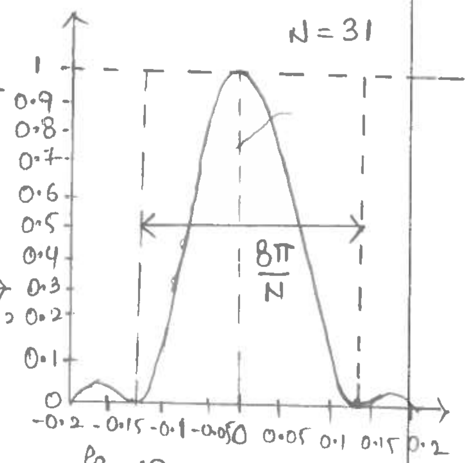
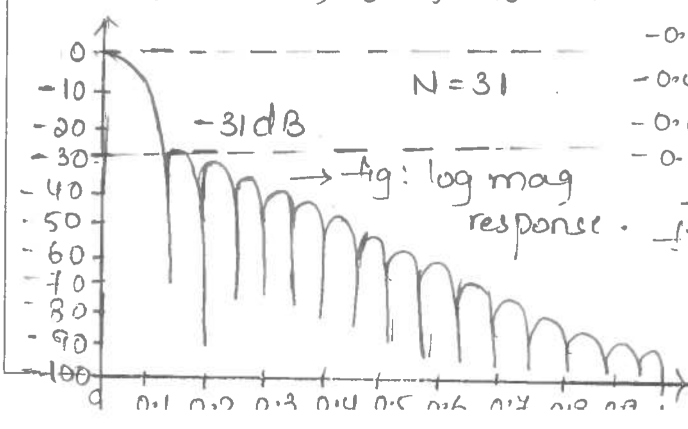


fig: Mag. response.



$\rightarrow$ fig: log mag response.

It can be observed that, the magnitude of 1st side lobe is −31 dB. There is 6 dB improvement over triangular window. But the main lobe width is same.

5. <u>Hamming window</u>: $\omega_{Hm}(n)$ is obtained by putting $\alpha = 0.54$ in raised cosine window.

$$\omega_{Hm}(n) = 0.54 + 0.46 \cos \frac{2\pi n}{N-1} \quad ; \text{for } n = -\left(\frac{N-1}{2}\right) \text{ to } \left(\frac{N-1}{2}\right)$$

$$= 0 \quad ; \text{otherwise}$$

→ only for N−odd

(or)

$$\omega_{Hm}(n) = 0.54 - 0.46 \cos \frac{2\pi n}{N-1} \quad ; \text{for } n = 0 \text{ to } N-1$$

$$= 0 \quad ; \text{otherwise}$$

→ for both N−odd & even

Freq response or freq spectrum is obtained by taking F.T of sequence. or by substituting $\alpha = 0.54$ in freq response of Raised cosine window.

$$W_{Hm}(e^{j\omega}) = F\{\omega_{Hm}(n)\} = 0.54 \frac{\sin \frac{\omega N}{2}}{\sin \frac{\omega}{2}} + 0.23 \frac{\sin\left(\frac{\omega N}{2} - \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi}{N-1}\right)}$$

$$+ 0.23 \frac{\sin\left(\frac{\omega N}{2} + \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{N-1}\right)}$$

Main lobe width is $\frac{8\pi}{N}$. The magnitude of 1st side lobe is reduced to −41 dB, an improvement of 10 dB relative to Hamming window. But this improvement is achieved at the expense of side-lobe magnitude at higher freq's which are almost constant with freq. (In Hanning window side lobe amplitudes decrease with freq)

fig.a: Hamming window sequence

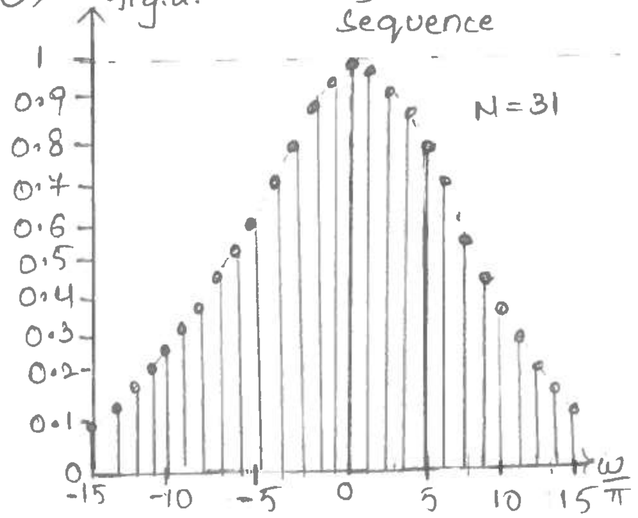


fig: Amplitude response

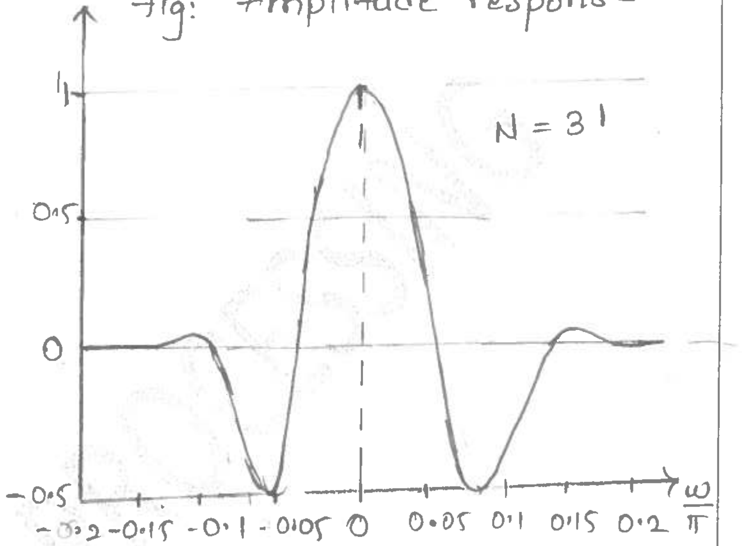


fig: Magnitude response

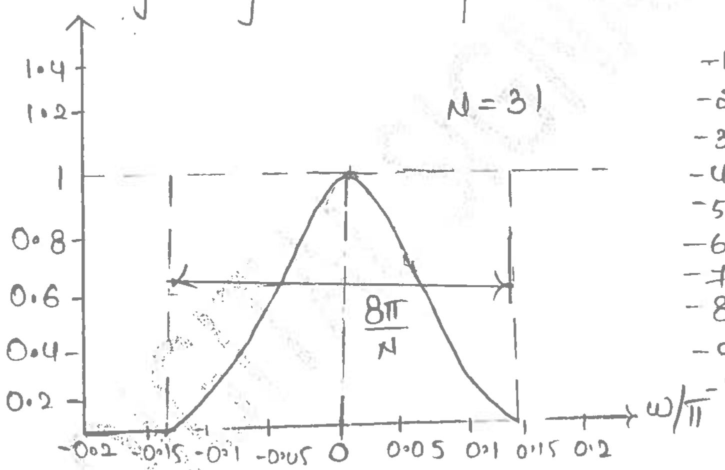
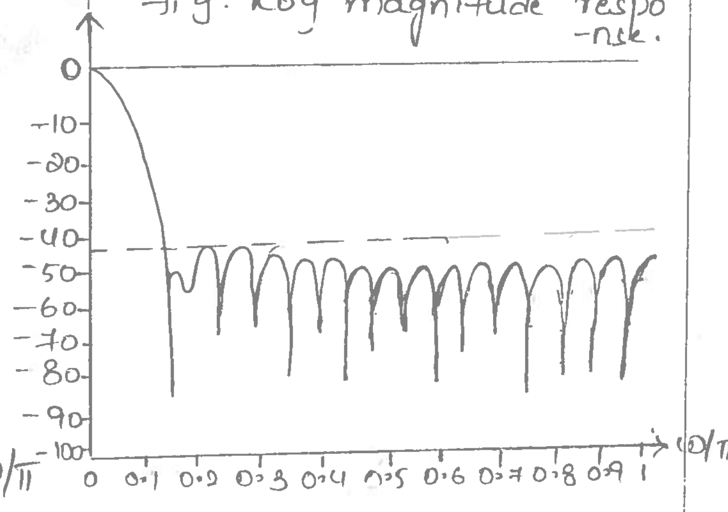


fig: Log magnitude response.



6. Blackman Window : Blackman window is another type of cosine window defined by

$$\omega_B(n) = 0.42 + 0.5\cos\frac{2\pi n}{N-1} + 0.08\cos\frac{4\pi n}{N-1}\ ; \text{ for } n = -\left(\frac{N-1}{2}\right) \text{ to } \left(\frac{N-1}{2}\right)$$

$$= 0 \qquad ; \text{ otherwise}$$

→ only for N-odd

(or)

$$\omega_B(n) = 0.42 - 0.5\cos\frac{2\pi n}{N-1} + 0.08\cos\frac{4\pi n}{N-1} \quad ; \text{ for } n = 0 \text{ to } N-1$$

$$= 0 \quad ; \text{ otherwise}$$

→ for both N – odd & even

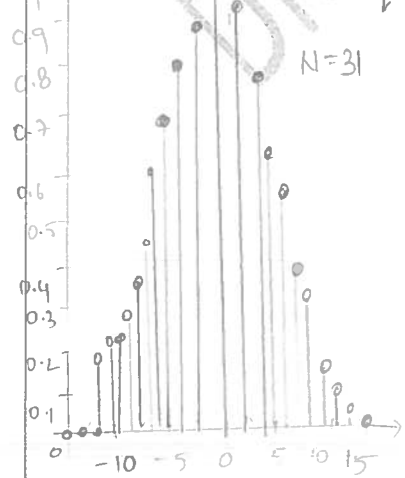
Freq response is given by

$$W_B(e^{j\omega}) = 0.42\frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}} + 0.25\frac{\sin\left(\frac{\omega N}{2} - \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi}{N-1}\right)} + 0.25\frac{\sin\left(\frac{\omega N}{2} + \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{N-1}\right)}$$

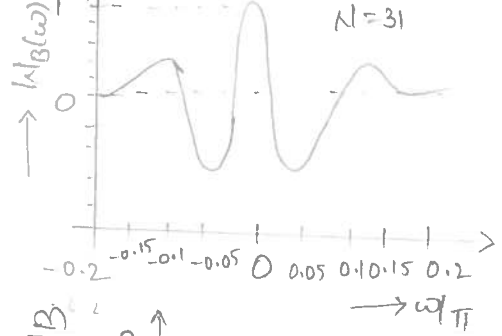
$$+ 0.04\frac{\sin\left(\frac{\omega N}{2} - \frac{2\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} - \frac{2\pi}{N-1}\right)} + 0.04\frac{\sin\left(\frac{\omega N}{2} + \frac{2\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} + \frac{2\pi}{N-1}\right)}$$

Width of mainlobe is $\frac{12\pi}{N}$. The magnitude of 1st side lobe is –58 dB and side lobe magnitude decreases with freq. This is a desirable feature achieved at the expense of increased mainlobe width.

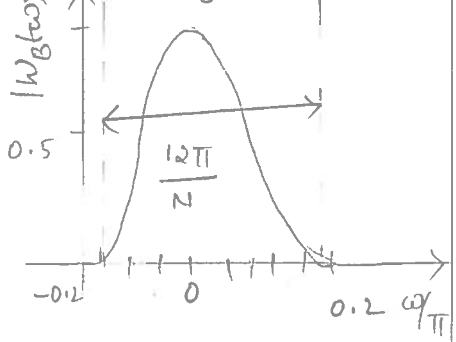
$\omega_B(n)$ Blackman Window seq.

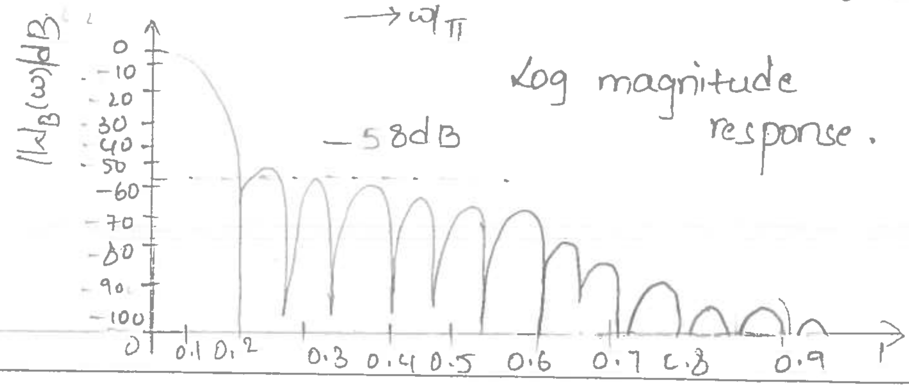


Amplitude response



Magnitude response





Log magnitude response.

Name of window	Window sequence
Rectangular	$w_R(n) = 1$; for $n = -\left(\frac{N-1}{2}\right)$ to $+\left(\frac{N-1}{2}\right)$ $= 0$; other n
	$w_R(n) = 1$; for $n = 0$ to $N-1$ $= 0$; other n
Triangular	$w_T(n) = 1 - \frac{2\lvert n \rvert}{N-1}$; for $n = -\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$ $= 0$; for other n
	$w_T(n) = 1 - \frac{2\lvert n-(N-1)/2 \rvert}{N-1}$; for $n = 0$ to $N-1$ $= 0$; other n
Hanning	$w_C(n) = 0.5 + 0.5\cos\frac{2\pi n}{N-1}$; for $n = -\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$ $= 0$; other n
	$w_C(n) = 0.5 - 0.5\cos\frac{2\pi n}{N-1}$; for $n = 0$ to $N-1$ $= 0$; other n
Hamming	$w_H(n) = 0.54 + 0.46\cos\frac{2\pi n}{N-1}$; for $n = -\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$ $= 0$; other n
	$w_H(n) = 0.54 - 0.46\cos\frac{2\pi n}{N-1}$; for $n = 0$ to $N-1$ $= 0$; other n

Blackman	$w_B(n) = 0.42 + 0.5\cos\frac{2\pi n}{N-1} + 0.08\cos\frac{4\pi n}{N-1}$; for $n = -\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$ $= 0$; other n
	$w_B(n) = 0.42 - 0.5\cos\frac{2\pi n}{N-1} + 0.08\cos\frac{4\pi n}{N-1}$; N= for $n = 0$ to $N-1$ $= 0$; other n
	$w_K(n) = \frac{I_0(\beta_1)}{I_0(a)}$; for $n = -\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$ $= 0$; for other n where $\beta_1 = a\left[1 - \left(\frac{2n}{N-1}\right)^2\right]^{0.5}$

NOTE :
In filter specifications gain & magnitude are same & will be in negative dB. The attenuation is inverse of gain & so it is negative of magnitude or gain in dB. Hence attenuation will be in positive dB.

Frequency - Domain characteristics of some window functions:

Type of window	Approximate width of main-lobe	Magnitude of first side-lobe.
Rectangular	$4\pi/N$	-13 dB
Bartlett	$8\pi/N$	-25 dB
Hanning	$8\pi/N$	-31 dB
Hamming	$8\pi/N$	-41 dB
Blackman	$12\pi/N$	-58 dB

FIR filter Design Using Windows :-

Method-1 : Symmetry Condition $h(N-1-n) = h(n)$

1. Specifications of digital FIR filter are
   * Desired freq response $H_d(e^{j\omega}) = C\, e^{-j\alpha\omega}$

     C - Const ; $\alpha = \frac{N-1}{2}$

   * Cutoff freq ω_c for LPF and HPF
     - ω_{c_1} & ω_{c_2} for BP and BR

   [If analog freq's are given then digital freq ω_c is calculated by $\omega_c = \frac{2\pi f_c}{f_s}$ f_c - cutoff freq ; f_s - sampling freq]

   * No. of samples of impulse response 'N'

2. Determine desired impulse response $h_d(n)$ by taking IFT of $H_d(e^{j\omega})$ $\Rightarrow$ $H_d(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) \cdot e^{j\omega n} \cdot d\omega$

3. Choose the window sequence $\omega(n)$ defined for $n = 0$ to $N-1$. Multiply $h_d(n)$ with $\omega(n)$ to get $h(n)$ of filter. Calculate N- samples of impulse response for $n = 0$ to $N-1$.

   Impulse response $h(n) = h_d(n) \times \omega(n)$; $n = 0$ to $N-1$

   Impulse response is symmetric with centre of symmetry at $\left(\frac{N-1}{2}\right)$ and so $h(N-1-n) = h(n)$. Hence it is sufficient if we calculate $h(n)$ for $n = 0$ to $\left(\frac{N-1}{2}\right)$

4. Take Z-Transform of $h(n)$ to get transfer f^n $H(z)$

of the filter.

$$H(z) = Z\{h(n)\} = \sum_{n=0}^{N-1} h(n) \cdot z^{-n}$$

5. Draw a suitable structure for realization of FIR filter

6. Determine freq response $H(e^{j\omega})$

* Choose a linear phase magnitude f^n $|H(e^{j\omega})|$. Using $h(n)$, obtain an equation for $|H(e^{j\omega})|$

(or)

* Freq response $|H(e^{j\omega})|$ can be obtained by replacing z by $e^{j\omega}$ in transfer f^n $H(z)$

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$$

Calculate freq response values for ω b/n 0 & π and sketch the magnitude response.

Method-2 : Symmetry condition $h(-n) = h(n)$

Step 1 and 2 are same as method-1

3. Choose desired window sequence $\omega(n)$ defined for $n = -\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$. Multiply $h_d(n)$ with $\omega(n)$ to get impulse response $h(n)$ of filter. Calculate N-samples of $h(n)$ for $n = -\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$.

Impulse response $h(n) = h_d(n) \times \omega(n)$; $n = -\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$

The impulse response is symmetric with centre of symmetry at $n=0$; so $h(-n) = h(n)$. Hence it is sufficient if $h(n)$ is calculated for $n = 0$ to $\frac{N-1}{2}$

4. Take Z-Transform of $h(n)$ to get non causal transfer function of FIR filter $H(z)$

$$H(z) = Z\left[h(n)\right] = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n) \cdot z^{-n}$$

5. Convert non causal transfer function $H(z)$ to causal transfer f^n $H'(z)$ by

$$H'(z) = z^{-\left(\frac{N-1}{2}\right)} \cdot H(z)$$

$$= z^{-\left(\frac{N-1}{2}\right)} \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n) \cdot z^{-n}$$

By applying symmetry $h(n) = h(-n)$

$$H'(z) = z^{-\left(\frac{N-1}{2}\right)} \left[h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\left(z^{n} + z^{-n}\right) \right]$$

Step 6 and 7 are same as method-1.

3. Design of FIR filters by Frequency Sampling method

Let $h(n)$ be filter co-efficients of an FIR filter and $H(k)$ is DFT of $h(n)$. Then DFT pair is given by

$$h(n) = \frac{1}{N}\sum_{k=0}^{N-1} H(k)\, e^{\frac{j2\pi kn}{N}} \quad ; \quad n = 0, 1, \ldots N-1$$

$$H(k) = \sum_{n=0}^{N-1} h(n) \cdot e^{\frac{-j2\pi kn}{N}} \quad ; \quad k = 0 \text{ to } N-1$$

DFT samples $H(k)$ for an FIR sequence can be considered as samples of the filter Z-Transform evaluated at N-points equally spaced around unit circle.

$$H(k) = H(z)\Big|_{z = e^{\frac{j2\pi k}{N}}}$$

Transfer f^n $H(z)$ of an FIR filter with impulse response is given by

$$H(z) = \sum_{n=0}^{N-1} h(n) \cdot z^{-n}$$

$$\Rightarrow H(z) = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} H(k) \cdot e^{\frac{j2\pi kn}{N}}\right] z^{-n}$$

$$= \sum_{k=0}^{N-1} \frac{H(k)}{N} \sum_{n=0}^{N-1}\left(e^{\frac{j2\pi k}{N}} \cdot z^{-1}\right)^{n}$$

$$= \sum_{k=0}^{N-1} \frac{H(k)}{N}\left(\frac{1-\left(e^{\frac{j2\pi k}{N}} \cdot z^{-1}\right)^{N}}{1-e^{\frac{j2\pi k}{N}} \cdot z^{-1}}\right)$$

$$\boxed{H(z) = \frac{1-z^{-N}}{N} \sum_{k=0}^{N-1} \frac{H(k)}{1-e^{\frac{j2\pi k}{N}} z^{-1}}}$$

We know $H(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}} = H\left(e^{\frac{j2\pi k n}{N}}\right) = H(k)$

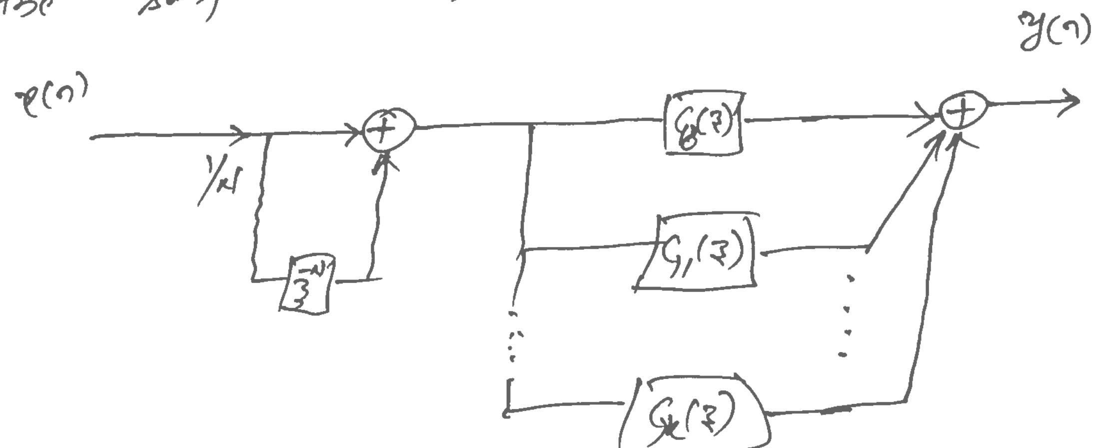
$H(k)$ is k^{th} DFT Component obtained by sampling the freq response $H(e^{j\omega})$. Hence this approach is Called frequency sampling method.

Realization

$$H(z) = \frac{1-z^{-N}}{N} \sum_{k=0}^{N-1} G_k(z)$$

$$G_k(z) = \frac{H(k)}{1-e^{\frac{j2\pi k}{N}} z^{-1}} \qquad 0 \le k \le N-1$$

$G_k(z)$ is transfer function of 1st order FIR filter where poles lie on unit circle at equidistant points. $G_k(z)$ is Called resonant filters as they are Resonant at the sample value of k^{th} freq.



Procedure for Type-1 design :-

1. Choose the ideal (desired) freq response $H_d(e^{j\omega})$
2. Sample $H_d(e^{j\omega})$ at N-points by taking $\omega = \omega_k = \frac{2\pi k}{N}$, $k = 0$ to $N-1$ to generate the sequence $H(k)$

$$H(k) = H_d(e^{j\omega}) \Big|_{\omega = \frac{2\pi k}{N}} \quad ; \; k = 0 \text{ to } N-1$$

3. Compute N-samples of impulse response $h(n)$

* When N is odd

impulse response $h(n) = \frac{1}{N}\left[H(0) + 2 \sum_{k=1}^{\frac{N-1}{2}} \text{Real}\left[H(k) \cdot e^{\frac{j2\pi k n}{N}} \right] \right]$

* When N is even

impulse response $h(n) = \frac{1}{N}\left[H(0) + 2 \sum_{k=1}^{\frac{N}{2}-1} \text{Real}\left[H(k) \cdot e^{\frac{j2\pi k n}{N}} \right] \right]$

$\left(H\left(\frac{N}{2}\right) = 0 \right)$

4. Apply z-Transform of $h(n)$ to obtain filter transfer function $H(z) = \sum_{n=0}^{N-1} h(n) \cdot z^{-n}$

Procedure for Type-2 design :-

1. Choose the ideal (desired) freq response $H_d(e^{j\omega})$
2. Sample $H_d(e^{j\omega})$ at N-points by taking $\omega = \omega_k = \frac{2\pi(2k+1)}{2N}$, $k = 0$ to $N-1$ to generate $H(k)$

$$H(k) = H_d(e^{j\omega}) \Big|_{\omega = \frac{2\pi(2k+1)}{2N}} \quad \text{for } k = 0 \text{ to } N-1$$

3. Compute N samples of impulse response $h(n)$ using

* when N is odd

impulse response $h(n) = \frac{2}{N} \sum_{k=0}^{\frac{N-3}{2}} Re\left[H(k) \cdot e^{\frac{jn\pi(2k+1)}{N}}\right]$

$$\left(H\left(\frac{N-1}{2}\right) = 0 \right)$$

* when N is even

impulse response $h(n) = \frac{2}{N} \times 2 \sum_{k=0}^{\frac{N}{2}-1} Re\left[H(k) \cdot e^{\frac{jn\pi(2k+1)}{N}}\right]$

4. Apply Z-Transform of $h(n)$ to obtain filter transfer function $H(z)$

$$\therefore H(z) = \sum_{n=0}^{N-1} h(n) \cdot z^{-n}$$

Design a linear phase FIR bandpass filter to pass frequencies in the range 0.4π to 0.65π rad/sample by taking 7 samples of hanning window sequence.

Let us choose symmetric impulse response with symmetry condition $h(N-1-n) = h(n)$.

$\therefore$ The desired ideal frequency response $H_d(e^{j\omega})$ for BPF is,

$$H_d(e^{j\omega}) = e^{-j\omega\alpha} \quad ; -\omega_{c2} \le \omega \le -\omega_{c1} \ \& \ -\omega_{c1} \le \omega \le +\omega_{c2}$$
$$= 0 \quad ; \text{otherwise}$$

The desired impulse response $h_d(n)$ is obtained by taking inverse fourier transform of $H_d(e^{j\omega})$.

By defination of IFT,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_{c2}}^{-\omega_{c1}} e^{-j\omega\alpha}\, e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_{c1}}^{\omega_{c2}} e^{-j\omega\alpha}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_{c2}}^{-\omega_{c1}} e^{j\omega(n-\alpha)}\, d\omega + \frac{1}{2\pi}\int_{\omega_{c1}}^{\omega_{c2}} e^{j\omega(n-\alpha)}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\omega_{c2}}^{-\omega_{c1}} + \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{\omega_{c1}}^{\omega_{c2}}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega_{c1}(n-\alpha)} - e^{-j\omega_{c2}(n-\alpha)}}{j(n-\alpha)}\right] + \frac{1}{2\pi}\left[\frac{e^{j\omega_{c2}(n-\alpha)} - e^{j\omega_{c1}(n-\alpha)}}{j(n-\alpha)}\right]$$

$$= \frac{1}{\pi(n-\alpha)}\left[\frac{e^{j\omega_{c2}(n-\alpha)} - e^{-j\omega_{c2}(n-\alpha)}}{2j} - \frac{e^{j\omega_{c1}(n-\alpha)} - e^{-j\omega_{c1}(n-\alpha)}}{2j}\right]$$

$$= \frac{\sin\omega_{c2}(n-\alpha) - \sin\omega_{c1}(n-\alpha)}{\pi(n-\alpha)} \quad ; \text{for all } n \text{ except } n=\alpha$$

When $n=\alpha$, $h_d(n)$ becomes $0/0$, which is indeterminate.

When $n=\alpha$; $h_d(n) = \underset{(n-\alpha)\to 0}{Lt} \dfrac{\sin\omega_{c2}(n-\alpha) - \sin\omega_{c1}(n-\alpha)}{\pi(n-\alpha)}$

$$= \frac{1}{\pi}\left[\underset{(n-\alpha)\to 0}{Lt} \frac{\sin\omega_{c2}(n-\alpha)}{n-\alpha} - \underset{(n-\alpha)\to 0}{Lt} \frac{\sin\omega_{c}(n-\alpha)}{n-\alpha}\right]$$

Using L' Hospital rule

$$\underset{\theta\to 0}{Lt} \frac{\sin A\theta}{\theta} = A$$

$$= \frac{\omega_{c2}-\omega_{c1}}{\pi}$$

The impulse response $h(n)$ of FIR filter is obtained by multiplying $h_d(n)$ by window sequence.

The hanning window sequence $w_c(n)$ is given by,

$$w_c(n) = 0.5 - 0.5 \cos\frac{2\pi n}{N-1} \quad ; \text{ for } n=0 \text{ to } (N-1)$$

$$= 0 \quad ; \text{ Otherwise}$$

$$h(n) = h_d(n)\, w_c(n)$$

$$= \left[\frac{\sin\omega_{c2}(n-\alpha) - \sin\omega_{c1}(n-\alpha)}{\pi(n-\alpha)}\right]\left[0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right)\right] \quad ; \text{ for } n \neq \alpha$$

$$= \left(\frac{\omega_{c2}-\omega_{c1}}{\pi}\right)\left[0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right)\right] \quad ; \text{ for } n=\alpha$$

Given that $N=7$; $\omega_{c1} = 0.4\pi$ rad/sample & $\omega_{c2} = 0.65\pi$ rad/sample.

Here $\alpha = \dfrac{N-1}{2} = \dfrac{7-1}{2} = 3$; $N-1=6$

Hence calculate $h(n)$ for $n=0$ to 6.

Since, $h(n)$ satisfies the symmetry condition, $h(N-1-n) = h(n)$,

Calculate $h(n)$ for $n=0$ to 3.

$$\therefore h(n) = \frac{\left[\sin\omega_{c2}(n-3) - \sin\omega_{c1}(n-3)\right]\left[0.5 - 0.5\cos\frac{n\pi}{3}\right]}{\pi(n-3)} \quad ; \text{ for } n \neq 3$$

$$= \left(\frac{\omega_{c2}-\omega_{c1}}{\pi}\right)\left(0.5 - 0.5\cos\frac{n\pi}{3}\right) \quad ; \text{ for } n=3$$

When $n=0$, $h(0) = \frac{\left[\sin(0.65\pi(0-3)) - \sin(0.4\pi(0-3))\right]\left[0.5-0.5\cos\frac{0\times\pi}{3}\right]}{\pi(0-3)} = 0$

$n=1$; $h(1) = \frac{\left[\sin(0.65\pi(1-3)) - \sin(0.4\pi(1-3))\right]\left[0.5-0.5\cos\frac{1\times\pi}{3}\right]}{\pi(1-3)} = -0.0556$

$n=2$; $h(2) = \frac{\left[\sin(0.65\pi(2-3)) - \sin(0.4\pi(2-3))\right]\left(0.5-0.5\cos\frac{2\times\pi}{3}\right]}{\pi(2-3)} = -0.0143$

$n=3$; $h(3) = \cancel{\left[\sin(0.65\pi(3-3)) - \sin(\right.} \left(\frac{0.65\pi - 0.4\pi}{\pi}\right)\left(0.5-0.5\cos\frac{3\pi}{3}\right) = 0.25$

$n=4$; $h(4) = h(6-4) = h(2) = -0.0143$ Using symmetry condition

$n=5$; $h(5) = h(6-5) = h(1) = -0.0556$ $h(N-1-n) = h(n)$

$n=6$; $h(6) = h(6-6) = h(0) = 0$ $\Rightarrow h(6-n) = h(n)$

The transfer function $H(z)$ of FIR BPF is given by

$$H(z) = Z\{h(n)\} = \sum_{n=0}^{N-1} h(n)z^{-n} = \sum_{n=0}^{6} h(n)z^{-n}$$

$$= h(0)+h(1)z^{-1}+h(2)z^{-2}+h(3)z^{-3}+h(4)z^{-4}+h(5)z^{-5}+h(6)z^{-6}$$

$$= h(0)+h(1)z^{-1}+h(2)z^{-2}+h(3)z^{-3}+h(2)z^{-4}+h(1)z^{-5}+h(0)z^{-6}$$

$$= h(0)\left[1+z^{-6}\right]+h(1)\left[z^{-1}+z^{-5}\right]+h(2)\left[z^{-2}+z^{-4}\right]+h(3)z^{-3}$$

$$= 0\times\left[1+z^{-6}\right]-0.0556\left[z^{-1}+z^{-5}\right]-0.0143\left[z^{-2}+z^{-4}\right]+0.25z^{-3}$$

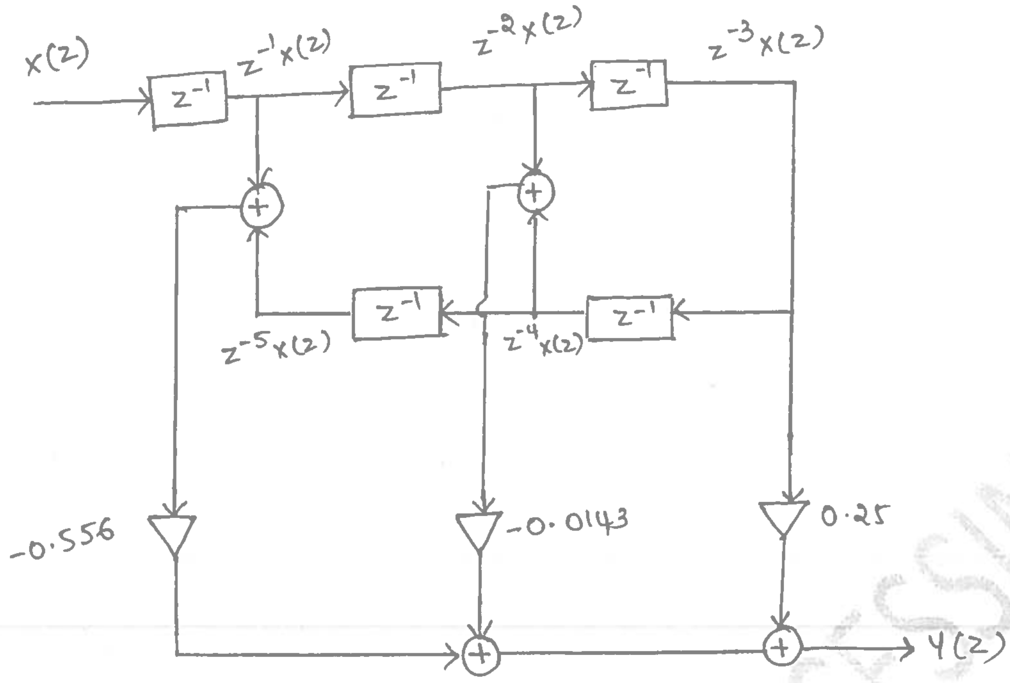
$$= -0.0556\left[z^{-1}+z^{-5}\right]-0.0143\left[z^{-2}+z^{-4}\right]+0.25z^{-3}$$

Structure:

Let $H(z) = \frac{Y(z)}{X(z)} = -0.0556\left[z^{-1}+z^{-5}\right]-0.0143\left[z^{-2}+z^{-4}\right]+0.25z^{-3}$

$$\therefore Y(z) = -0.0556\left[z^{-1}X(z)+z^{-5}X(z)\right]-0.0143\left[z^{-2}X(z)+z^{-4}X(z)\right]$$
$$+0.25z^{-3}X(z)$$

6.70



Frequency Response:

When impulse response is symmetric & N is odd with centre of symmetry at $(N-1)/2$ the magnitude response $|H(e^{j\omega})|$ is given by $|A(\omega)|$,

$$\text{where } A(\omega) = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right)\cos\omega n$$

$$\therefore A(\omega) = h(3) + \sum_{n=1}^{3} 2h(3-n)\cos\omega n$$

$$= h(3) + 2h(2)\cos\omega + 2h(1)\cos 2\omega + 2h(0)\cos 3\omega$$

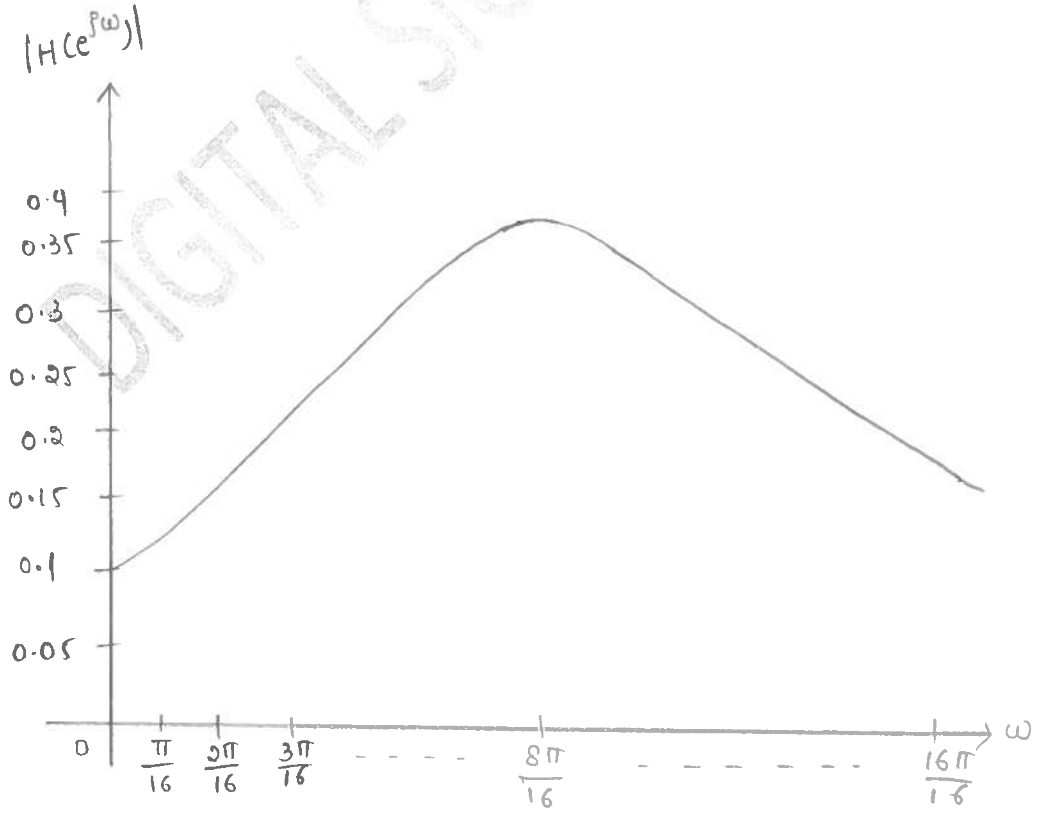
$$= 0.25 + 2\times(-0.0143)\cos\omega + 2\times(-0.0556)\cos 2\omega$$

$$+ 2\times 0 \cos 3\omega$$

$$= 0.25 - 0.0286\cos\omega - 0.1112\cos 2\omega$$

Table: $A(\omega)$ and $|H(e^{j\omega})|$ for various values of ω.

ω	$A(\omega)$	$\lvert H(e^{j\omega})\rvert = \lvert A(\omega)\rvert$
$0\times\pi/16$	0.1102	0.1102
$1\times\pi/16$	0.1192	0.1192
$2\times\pi/16$	0.1449	0.1449
$3\times\pi/16$	0.1836	0.1836
$4\times\pi/16$	0.2297	0.2297
$5\times\pi/16$	0.2766	0.2766
$6\times\pi/16$	0.3176	0.3176
$7\times\pi/16$	0.3471	0.3471
$8\times\pi/16$	0.3612	0.3612
$9\times\pi/16$	0.3583	0.3583
$10\times\pi/16$	0.3395	0.3395
$11\times\pi/16$	0.3084	0.3084
$12\times\pi/16$	0.2702	0.2702
$13\times\pi/16$	0.2312	0.2312
$14\times\pi/16$	0.1977	0.1977
$15\times\pi/16$	0.1753	0.1753
$16\times\pi/16$	0.1674	0.1674



Design a linear phase FIR bandstop filter to reject frequencies in the range 0.4π to 0.65π rad/sample using rectangular window, by taking 7 samples of window sequence.

Let us choose symmetric impulse response with symmetry condition, $h(N-1-n) = h(n)$. $\therefore$ the desired ideal frequency response $H_d(e^{j\omega})$ for bandstop filter is

$$H_d(e^{j\omega}) = e^{-j\omega\alpha} \; ; \quad -\pi \le \omega \le -\omega_{c2} \;\&$$
$$-\omega_{c1} \le \omega \le +\omega_{c1} \;\&$$
$$+\omega_{c2} \le \omega \le +\pi$$
$$= 0 \; ; \quad \text{otherwise}$$

The desired impulse response $h_d(n)$ obtained by taking IFT of $H_d(e^{j\omega})$.

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\omega_{c2}} e^{-j\omega\alpha}\cdot e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{-\omega_{c1}}^{\omega_{c1}} e^{-j\omega\alpha}\, e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_{c2}}^{\pi} e^{-j\omega\alpha}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\omega_{c2}} e^{j\omega(n-\alpha)}\, d\omega + \frac{1}{2\pi}\int_{-\omega_{c1}}^{\omega_{c1}} e^{j\omega(n-\alpha)}\, d\omega + \frac{1}{2\pi}\int_{\omega_{c2}}^{\pi} e^{j\omega(n-\alpha)}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\pi}^{-\omega_{c2}} + \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\omega_{c1}}^{\omega_{c1}} + \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{\omega_{c2}}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega_{c2}(n-\alpha)} - e^{-j\pi(n-\alpha)}}{j(n-\alpha)} + \frac{e^{j\omega_{c1}(n-\alpha)} - e^{-j\omega_{c1}(n-\alpha)}}{j(n-\alpha)} + \frac{e^{j\pi(n-\alpha)} - e^{j\omega_{c2}(n-\alpha)}}{j(n-\alpha)}\right]$$

$$= \frac{1}{2\pi(n-\alpha)}\left[\frac{e^{j\omega_{c1}(n-\alpha)} - e^{-j\omega_{c1}(n-\alpha)}}{2j} + \frac{e^{j\pi(n-\alpha)} + e^{-j\pi(n-\alpha)}}{2j} - \frac{e^{j\omega_{c2}(n-\alpha)} - e^{-j\omega_{c2}(n-\alpha)}}{2j}\right]$$

$$= \frac{\sin\omega_{c1}(n-\alpha) + \sin\pi(n-\alpha) - \sin\omega_{c2}(n-\alpha)}{\pi(n-\alpha)} \; ; \quad \text{for all } n, \text{ except } n = \alpha$$

When $n = \alpha$;

$$h_d(n) = \underset{(n-\alpha)\to 0}{Lt} \frac{\sin\omega_{c1}(n-\alpha) + \sin\pi(n-\alpha) - \sin\omega_{c2}(n-\alpha)}{\pi(n-\alpha)}$$

$$= \frac{1}{\pi}\left\{\underset{(n-\alpha)\to 0}{Lt} \frac{\sin\omega_{c1}(n-\alpha)}{n-\alpha} + \underset{(n-\alpha)\to 0}{Lt} \frac{\sin\pi(n-\alpha)}{n-\alpha} - \underset{(n-\alpha)\to 0}{Lt} \frac{\sin\omega_{c2}(n-\alpha)}{(n-\alpha)}\right]$$

$$= \frac{1}{\pi}\left[\omega_{c1} + \pi - \omega_{c2}\right]$$

Using L' Hospital rule $\underset{\theta\to 0}{Lt} \frac{\sin A\theta}{\theta} = A$

$$= 1 - \left(\frac{\omega_{c2} - \omega_{c1}}{\pi}\right)$$

The impulse response $h(n)$ of FIR filter is obtained by multiplying $h_d(n)$ by window sequence

The window sequence for rectangular window $w_R(n)$ is

$$w_R(n) = 1 \quad ; \text{ for } n = 0 \text{ to } N-1$$
$$= 0 \quad ; \text{ otherwise}$$

$$\therefore h(n) = h_d(n) \times w_R(n) = h_d(n)$$

$$\Rightarrow h_d(n) = \frac{\sin\omega_{c1}(n-\alpha) + \sin\pi(n-\alpha) - \sin\omega_{c2}(n-\alpha)}{\pi(n-\alpha)} \quad ; \text{ for } n \neq \alpha$$

$$= 1 - \left(\frac{\omega_{c2} - \omega_{c1}}{\pi}\right) \quad ; \text{ for } n = \alpha$$

Given that $N = 7$; $\omega_{c1} = 0.4\pi$ rad/sample ; $\omega_{c2} = 0.65\pi$ rad/sample

$$\therefore \alpha = \frac{N-1}{2} = \frac{7-1}{2} = 3 \quad ; \quad N-1 = 6$$

Hence calculate $h(n)$ for $n = 0$ to 6.

$\because h(n)$ satisfies symmetry condition, $h(N-1-n) = h(n)$, calculate $h(n)$ for $n = 0$ to 3.

$$\therefore h(n) = \frac{\sin\omega_{c1}(n-3) - \sin\omega_{c2}(n-3)}{\pi(n-3)} \quad ; \text{ for } n \neq 3$$

$$= 1 - \frac{\omega_{c2} - \omega_{c1}}{\pi} \quad ; \text{ for } n = 3$$

since n, α are integers, $\sin(n-\alpha)\pi = 0$

When $n=0$; $h(0) = \dfrac{\sin(0.4\pi(0-3)) - \sin(0.65\pi(0-3))}{\pi \times (0-3)} = -0.0458$

$n=1$; $h(1) = \dfrac{\sin(0.4\pi(1-3)) - \sin(0.65\pi(1-3))}{\pi \times (1-3)} = 0.2223$

$n=2$; $h(2) = \dfrac{\sin(0.4\pi(2-3)) - \sin(0.65\pi(2-3))}{\pi \times (2-3)} = 0.0191$

$n=3$; $h(3) = 1 - \left(\dfrac{0.65\pi - 0.4\pi}{\pi}\right) = 0.75$

$n=4$; $h(4) = h(6-4) = h(2) = 0.0191$

$n=5$; $h(5) = h(6-5) = h(1) = 0.2223$

$n=6$; $h(6) = h(6-6) = h(0) = -0.0458$

$\because h(N-1-n) = h(n)$
$\Rightarrow h(6-n) = h(n)$

The transfer function $H(z)$ of FIR bandstop filter is

$$H(z) = Z\{h(n)\} = \sum_{n=0}^{N-1} h(n) z^{-n} = \sum_{n=0}^{6} h(n) z^{-n}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(2)z^{-4} + h(1)z^{-5} + h(0)z^{-6}$$

$\{\because h(N-1-n) = h(n)\}$

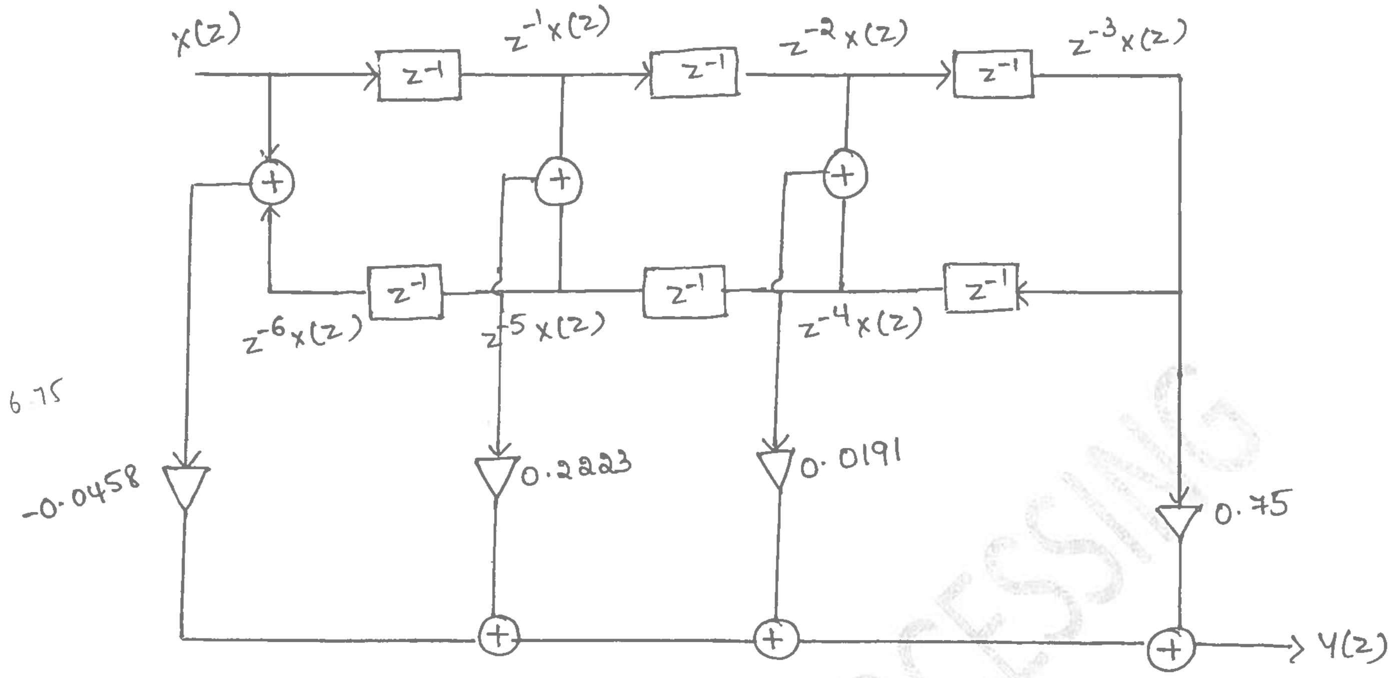
$$= h(0)[1+z^{-6}] + h(1)[z^{-1}+z^{-5}] + h(2)[z^{-2}+z^{-4}] + h(3)z^{-3}$$

$$= -0.0458[1+z^{-6}] + 0.2223[z^{-1}+z^{-5}] + 0.0191[z^{-2}+z^{-4}] + 0.75z^{-3}$$

Structure :

Let $H(z) = \dfrac{Y(z)}{X(z)} = -0.0458[1+z^{-6}] + 0.223[z^{-1}+z^{-5}] + 0.0191[z^{-2}+z^{-4}] + 0.75z^{-3}$

$$Y(z) = -0.0458[X(z) + z^{-6}X(z)] + 0.223[z^{-1}X(z) + z^{-5}X(z)] + 0.0191[z^{-2}X(z) + z^{-4}X(z)] + 0.75z^{-3}X(z)$$



Frequency response :

When impulse response is symmetric & N is odd with centre of symmetry at $(N-1)/2$ the magnitude response $|H(e^{j\omega})|$ is given by $|A(\omega)|$,

where $A(\omega) = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-n\right)\cos\omega n$

$$\therefore \; A(\omega) = h(3) + \sum_{n=1}^{3} 2h(3-n)\cos\omega n$$

$$= h(3) + 2h(2)\cos\omega + 2h(1)\cos 2\omega + 2h(0)\cos 3\omega$$

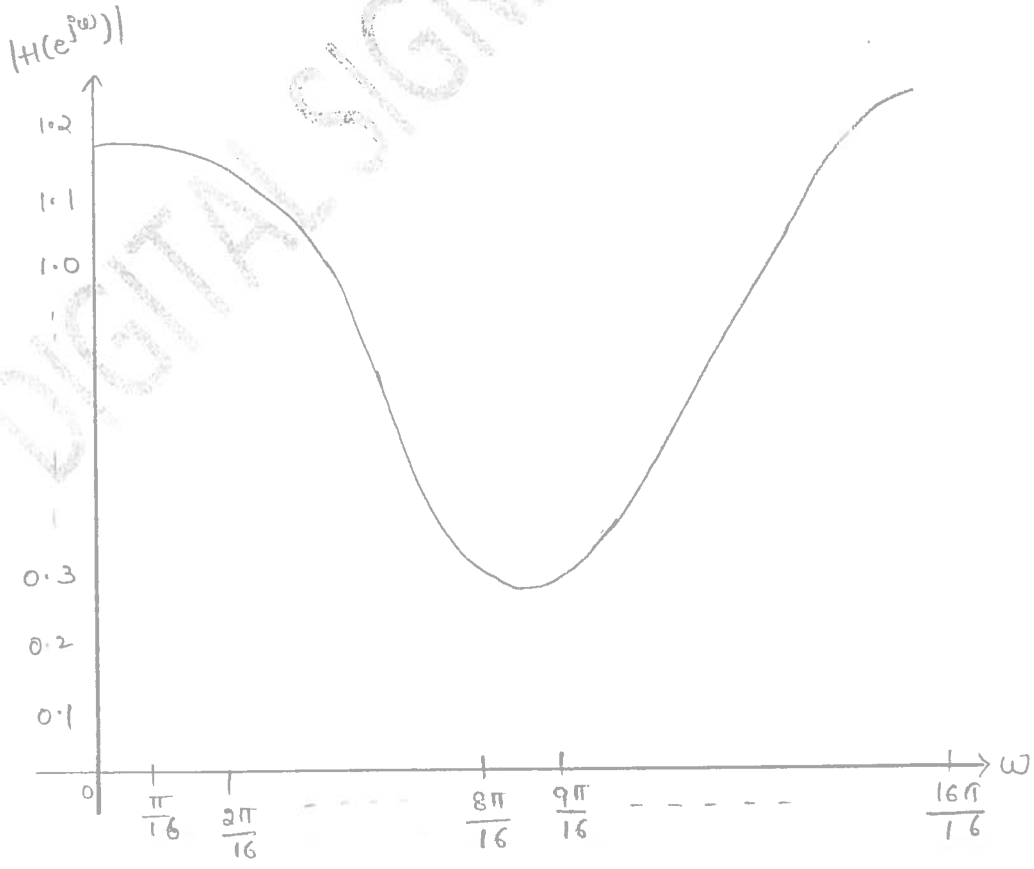
$$= 0.75 + 2\times 0.0191\cos\omega + 2\times 0.2223\cos 2\omega + 2\times(-0.0458)\cos 3\omega$$

$$= 0.75 + 0.0382\cos\omega + 0.4446\cos 2\omega - 0.0914\cos 3\omega$$

Table : $A(\omega)$ & $|H(e^{j\omega})|$ for various values of ω

ω	$A(\omega)$	$\lvert H(e^{j\omega})\rvert = \lvert A(\omega)\rvert$
$0\times\pi/16$	1.1414	1.1414
$1\times\pi/16$	1.1223	1.1223
$2\times\pi/16$	1.0646	1.0646
$3\times\pi/16$	0.9697	0.9697
$4\times\pi/16$	0.8416	0.8416
$5\times\pi/16$	0.6907	0.6907
$6\times\pi/16$	0.5346	0.5346
$7\times\pi/16$	0.3974	0.3974
$8\times\pi/16$	0.3054	0.3054

ω	$A(\omega)$	$\lvert H(e^{j\omega})\rvert = \lvert A(\omega)\rvert$
$9\times\pi/16$	0.2810	0.2810
$10\times\pi/16$	0.3365	0.3365
$11\times\pi/16$	0.4689	0.4689
$12\times\pi/16$	0.6583	0.6583
$13\times\pi/16$	0.8705	0.8705
$14\times\pi/16$	1.0640	1.0640
$15\pi/16$	1.1992	1.1992
$16\pi/16$	1.2478	1.2478

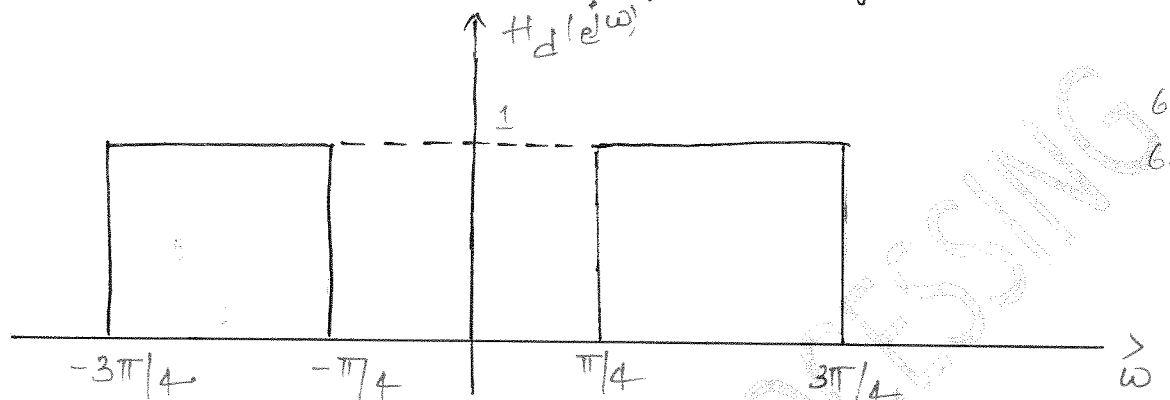


Prob: Design Ideal band pass filter with a frequency response

$H_d(e^{j\omega}) = 1$ for $\frac{\pi}{4} \le |\omega| \le \frac{3\pi}{4}$

0 otherwise . Find the values of $h(n)$ for $N = 11$ and plot the frequency response.

Sol: The Ideal frequency response of filter is given as



6.29
6.39

$$h_d(n) = IFT[H_d(e^{j\omega})]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-3\pi/4}^{-\pi/4} e^{j\omega n}\, d\omega + \int_{\pi/4}^{3\pi/4} e^{j\omega n}\, d\omega\right]$$

$$= \frac{1}{2\pi j n}\left[\left(e^{j\omega n}\right)_{-3\pi/4}^{-\pi/4} + \left(e^{j\omega n}\right)_{\pi/4}^{3\pi/4}\right]$$

$$= \frac{1}{2\pi j n}\left[e^{-j\pi n/4} - e^{-j3\pi/4} + e^{\frac{j3\pi n}{4}} - e^{\frac{j\pi n}{4}}\right]$$

$$\Rightarrow \frac{1}{\pi n}\left[\frac{e^{\frac{-j\pi n}{4}} - e^{\frac{j\pi n}{4}}}{2j} + \frac{e^{\frac{j3\pi n}{4}} - e^{\frac{-j3\pi n}{4}}}{2j}\right]$$

$$h_d(n) \Rightarrow \frac{1}{\pi n}\left[-\left(\frac{e^{\frac{+j\pi n}{4}} - e^{\frac{-j\pi n}{4}}}{2j}\right) + \left(\frac{e^{\frac{-j3\pi n}{4}} - e^{\frac{j3\pi n}{4}}}{2j}\right)\right]$$

$$h_d(n) = \frac{1}{\pi n}\left[\sin\frac{3\pi}{4}n - \sin\frac{\pi}{4}n\right] \quad ; \quad -\infty \le n \le \infty$$

Trunkating $h_d(n)$ to 11 samples, we have

$$h(n) = \begin{cases} h_d(n) \; ; \text{for } |n| \le 5 \\ 0 \; ; \text{otherwise.} \end{cases}$$

The filter coefficients are symmetrical about $n=0$ Satisfying the condition $h(n) = h(-n)$

for $n=0$

$$h(0) = \lim_{n\to 0}\left[\frac{\sin\frac{3\pi}{4}n}{\pi n} - \frac{\sin\frac{\pi}{4}n}{\pi n}\right]$$

$$h(0) = \frac{3}{4} - \frac{1}{4} = 2/4 = 1/2.$$

$$n=1 \Rightarrow h(1) = h(-1) = \frac{\sin\frac{3\pi}{4} - \sin\frac{\pi}{4}}{\pi} = 0.$$

$$n=2 \Rightarrow h(2) = h(-2) = \frac{\sin\frac{3\pi}{2} - \sin\pi/2}{2\pi} = \frac{-2}{2\pi} = -0.3183$$

$$n=3 \Rightarrow h(3) = h(-3) = \frac{\sin\frac{9\pi}{4} - \sin\frac{3\pi}{4}}{3\pi} = 0.$$

$$n=4 \Rightarrow h(4) = h(-4) = \frac{\sin 3\pi - \sin\pi}{4\pi} = 0.$$

$$n=5 \Rightarrow h(5) = h(-5) = \frac{\sin\frac{15\pi}{4} - \sin\frac{5\pi}{4}}{5\pi} = 0.$$

The transfer function of the filter is

$$H(z) = h(0) + \sum_{n=1}^{N-1/2} \left[h(n)\left(z^{-n} + z^{n}\right)\right]$$

$$= 0.5 - 0.3183\left(z^{-2} + z^{2}\right)$$

$N = 11$

$\because \frac{N-1}{2} = \frac{11-1}{2} = 5$

The Transfer function of the realisable filter is $H'(z) = z^{-\left(\frac{N-1}{2}\right)} H(z)$

$$H'(z) = z^{-5}\left[0.5 - 0.3183\left(z^{-2} + z^{2}\right)\right]$$

$$\Rightarrow -0.3183z^{-3} + 0.5z^{-5} - 0.3183z^{-7}$$

$\therefore$ The filter coefficients of the causal filter are

$h(0) = h(10) = h(1) = h(9) = h(2) = h(8) = h(4) = h(6) = 0$.

$h(3) = h(7) = -0.3183$.

$h(5) = 0.5$

$$\bar{H}(e^{j\omega}) = \sum_{n=1}^{\left(\frac{N-1}{2}\right)} a(n) \cos \omega n = \sum_{n=1}^{5} a(n) \cos \omega n \longrightarrow (1)$$

$N = 11$

$$a(0) = h\left(\frac{N-1}{2}\right) = h\left(\frac{11-1}{2}\right) = h(5) = 0.5$$

$$a(n) = 2\,h\left(\frac{N-1}{2} - n\right)$$

$a(1) = 2\,h(5-1) = 2h(4) = 0$

$a(2) = 2\,h(5-2) = 2h(3) = 2 \times (-0.3183) = -0.6366$

$a(3) = 2\,h(5-3) = 2h(2) = 0$.

$a(4) = 2h(5-4) = 2h(1) = 0$.

$a(5) = 2h(5-5) = 2h(0) = 0$.

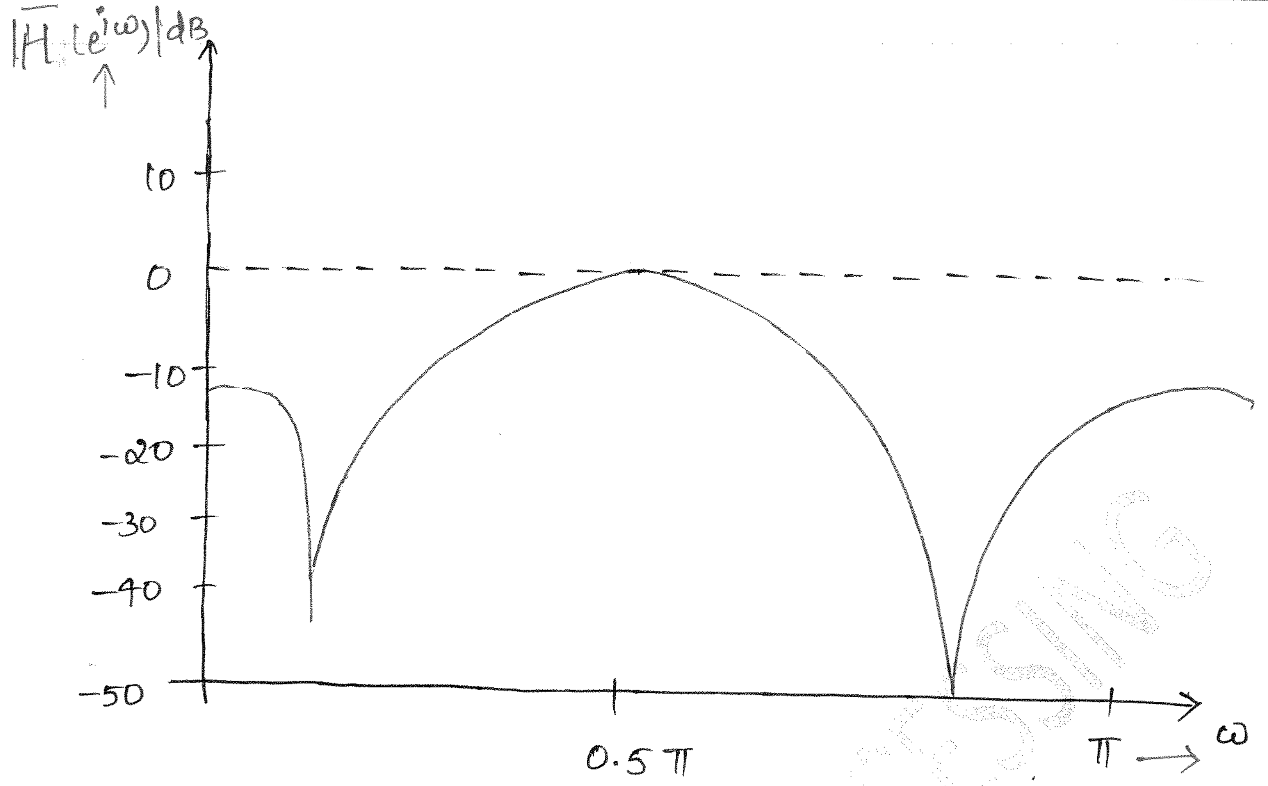
from eqn (1)

$$\bar{H}(e^{j\omega}) = a(1)\cos\omega + a(2)\cos 2\omega + a(3)\cos 3\omega + a(4)\cos 4\omega + a(5)\cos 5\omega.$$

$\bar{H}(e^{j\omega}) = 0.5 - 0.6366\cos 2\omega.$

ω (degress)	$\bar{H}(e^{j\omega})$	$\lvert H(e^{j\omega})\rvert$ dB $20\log\lvert\bar{H}(e^{j\omega})\rvert$
0	−0.1366	−17.3
20	0.012	−38.17
30	0.1817	−14.8
45	0.5	−6.02
60	0.818	−1.74
75	1.05	0.4346
90	1.1366	1.11
105	1.05	0.4346
120	0.818	−1.74
135	0.5	−6.02
150	0.1817	−14.8
160	0.012	−38.17
180	−0.1366	−17.3

The frequency response of the Band pass filter is shown in the figure.

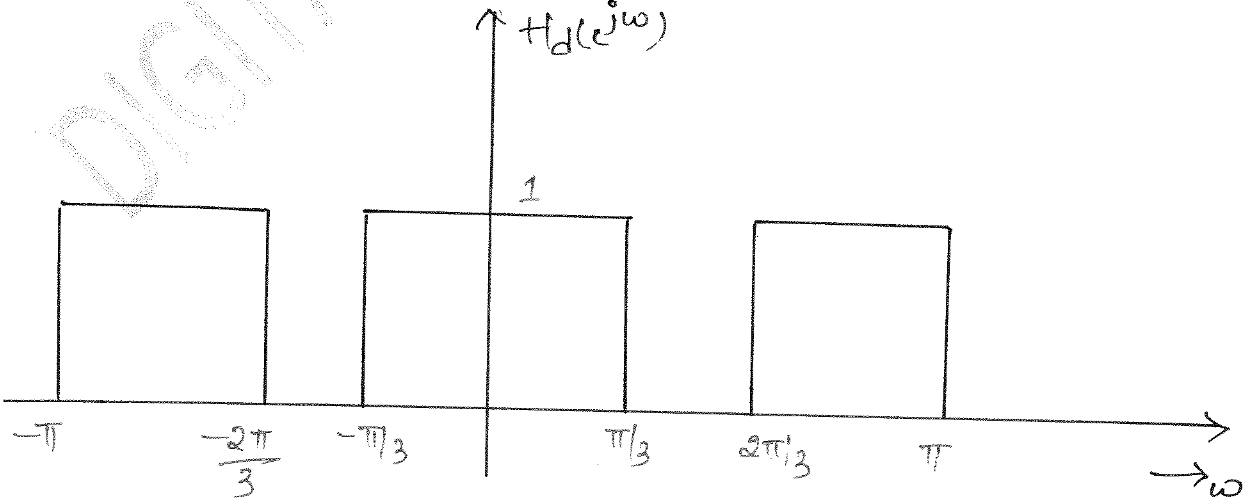


Prob: Design Ideal Band reject filter with the desired frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega| \le \pi/3 \text{ and } |\omega| \ge 2\pi/3 \\ 0 & ;\ \text{otherwise.} \end{cases}$$

Find the value of $h(n)$ for $N=11$. Find the $H(z)$ and Plot the magnitude response.

Sol: The desired frequency response is shown in below fig.



$$h_d(n) = IFT[H_d(e^{j\omega})]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-2\pi/3} e^{j\omega n}\, d\omega + \int_{-\pi/3}^{\pi/3} e^{j\omega n}\, d\omega + \int_{2\pi/3}^{\pi} e^{j\omega n}\, d\omega\right]$$

$$= \frac{1}{2\pi}\left\{\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-2\pi/3} + \left[\frac{e^{j\omega n}}{jn}\right]_{-\pi/3}^{\pi/3} + \left[\frac{e^{j\omega n}}{jn}\right]_{2\pi/3}^{\pi}\right\}$$

$$\Rightarrow \frac{1}{2\pi jn}\left[e^{-j2\pi n/3} - e^{-j\pi n} + e^{j\pi n/3} - e^{-j\pi n/3} + e^{j\pi n} - e^{j2\pi n/3}\right]$$

$$h_d(n) = \frac{1}{\pi n}\left[\frac{e^{j\pi n} - e^{-j\pi n} + e^{j\pi n/3} - e^{-j\pi n/3} - (e^{j2\pi n/3} - e^{-j2\pi n/3})}{2j}\right]$$

$$h_d(n) = \frac{1}{\pi n}\left[\sin\pi n + \sin\frac{\pi}{3}n - \sin\frac{2\pi}{3}n\right]; \quad -\infty \le n \le \infty$$

Truncating $h_d(n)$ to 11 samples we have

$$h(n) = \begin{cases} h_d(n) & \text{for } |n| \le 5 \\ 0 & ;\ \text{otherwise} \end{cases}$$

The filter coefficients are symmetrical about $n=0$ satisfying the condition $h(n) = h(-n)$

for $n=0$; $h(0) = \lim_{n\to 0}\left[\frac{\sin\pi n}{\pi n} + \frac{\sin\pi/3\, n}{\pi n} - \frac{\sin 2\pi/3\, n}{\pi n}\right]$

$$= 1 + 1/3 - 2/3 = 0.667$$

$n=1 \Rightarrow h(1) = h(-1) = \dfrac{\sin\pi + \sin\pi/3 - \sin 2\pi/3}{\pi} = 0$

$n=2 \Rightarrow h(2) = h(-2) = \dfrac{\sin 2\pi + \sin 2\pi/3 - \sin 4\pi/3}{2\pi} = 0.2757$

$n=3 \Rightarrow h(3) = h(-3) = \dfrac{\sin 3\pi + \sin\pi - \sin 2\pi}{3\pi} = 0$

$n=4 \Rightarrow h(4) = h(-4) = \dfrac{\sin 4\pi + \sin 4\pi/3 - \sin 8\pi/3}{4\pi} = -0.1378$

$n=5 \Rightarrow h(5) = h(-5) = \dfrac{\sin 5\pi + \sin 5\pi/3 - \sin 10\pi/3}{5\pi} = 0$

The transfer function of the filter is

$$H(z) = h(0) + \sum_{n=1}^{N-1/2} [h(n)[z^{-n} + z^{n}]]$$

$N = 11$

$$H(z) = h(0) + \sum_{n=1}^{5} h(n)[z^{-n} + z^{n}]$$

$$H(z) = 0.667 + 0 + 0.2757(z^{2} + z^{-2}) + 0 - 0.1378(z^{4} + z^{-4}) + 0$$

$$H(z) = 0.667 + 0.2757(z^{2} + z^{-2}) - 0.1378(z^{4} + z^{-4})$$

The transfer function of the realisable filter

$$\underline{H'(z) = z^{-\left(\frac{N-1}{2}\right)} H(z)}$$

$$H'(z) = z^{-5} H(z)$$

$$H'(z) = -0.1378 z^{-1} + 0.2757 z^{-3} + 0.667 z^{-5} + 0.2757 z^{-7} - 0.1378 z^{-9}$$

From above eqn the filter coefficients of causal filter are $h(0) = h(10) = h(2) = h(8) = h(4) = h(6) = 0$

$h(1) = h(9) = -0.1378$

$h(3) = h(7) = 0.2757$

$h(5) = 0.667$

$$\bar{H}(e^{j\omega}) = \sum_{n=0}^{N-1/2} a(n) \cos\omega n \quad ; \quad N=11$$

$$= \sum_{n=0}^{5} a(n) \cos\omega n. = a(0) + a(1)\cos\omega + a(2)\cos 2\omega + a(3)\cos 3\omega + a(4)\cos 4\omega + a(5)\cos 5\omega \rightarrow (a)$$

$$a(0) = h\left(\frac{N-1}{2}\right) = h\left(\frac{11-1}{2}\right) = h(5) = 0.667$$

$$a(n) = 2h\left[\left(\frac{N-1}{2}\right) - n\right]$$

$$a(1) = 2h(5-1) = 2h(4) = 0.$$

$$a(2) = 2h(5-2) = 2h(3) = 0.5514$$

$$a(3) = 2h(5-3) = 2h(2) = 0$$

$$a(4) = 2h(5-4) = 2h(1) = -0.2756$$

$$a(5) = 2h(5-5) = 2h(0) = 0.$$

Sub above values in eqn (a) we get

$$\bar{H}(e^{j\omega}) = 0.667 + 0.5514\cos 2\omega - 0.2756\cos 4\omega.$$

ω (in degrees)	0	15	30	45	60	75
$\bar{H}(e^{j\omega})$	0.9428	1.0867	1.08	0.9426	0.529	0.0516
$\lvert H(e^{j\omega})\rvert$ dB	-0.5	0.058	0.67	-0.513	-5.53	-25.7

ω (in degrees)	90	105	120	135	150	165	180
$\bar{H}(e^{j\omega})$	16	0.0516	0.529	0.9426	1.08	1.0067	0.9428
$\lvert \bar{H}(e^{j\omega})\rvert$ dB	-15.9	-25.7	-5.53	-0.513	0.67	0.058	-0.5

fig: frequency response of the Bandreject filter.

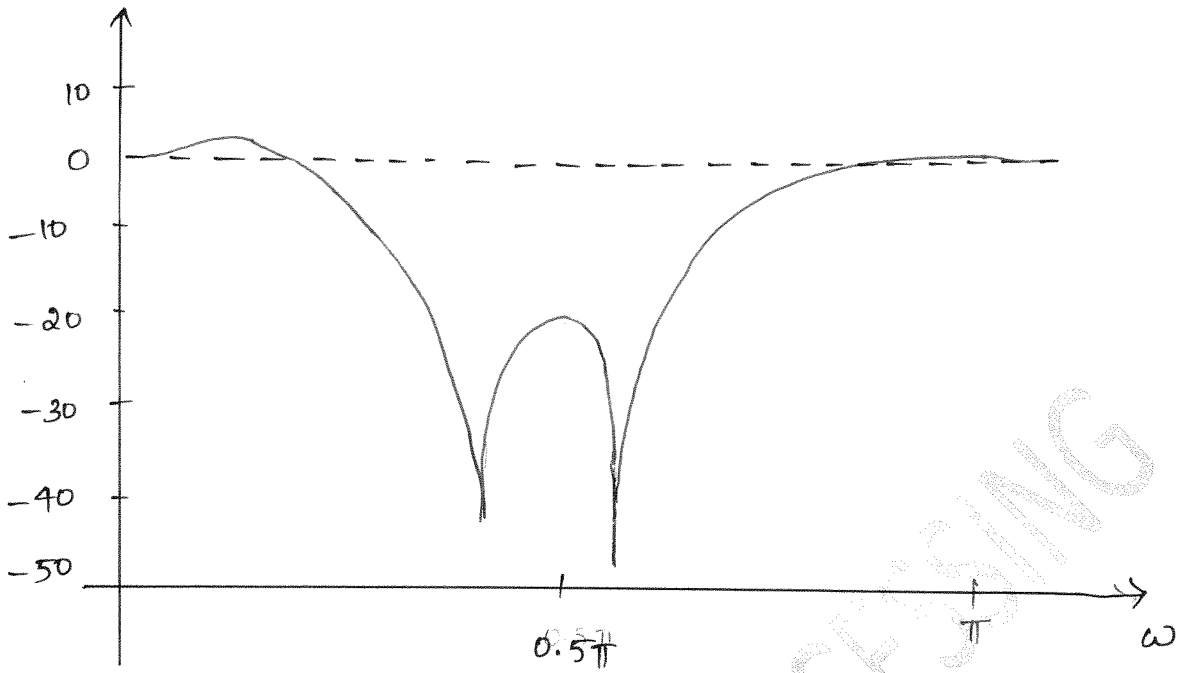


fig: The magnitude v/s ω plot of a Lowp Band reject fiter

Design a FIR high pass filter with cutoff frequency of 1.5khz and sampling frequency of 5khz with 7 samples using Fourier series method. Determine the frequency response & verify the design by sketching the magnitude response.

Given that, $f_c = 1.5\,khz$; $f_s = 5\,khz$

$$\omega_c = \Omega_c T = \frac{\Omega_c}{f_s} = \frac{2\pi f_c}{f_s} = \frac{2\pi \times 1.5\times 10^3}{5\times 10^3} = 0.6\pi \text{ rad/sample}$$

The desired frequency response $H_d(e^{j\omega})$ of HPF is

$$H_d(e^{j\omega}) = 1 \;;\; \text{for } -\pi \le \omega \le -\omega_c \;\&\; \omega_c \le \omega \le \pi$$
$$= 0 \;;\; \text{otherwise}$$

The desired impulse response $h_d(n)$ of HPF is

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} 1\times e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} 1\times e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-\omega_c} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\omega_c}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega_c n}}{jn} - \frac{e^{-j\pi n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\pi n}}{jn} - \frac{e^{j\omega_c n}}{jn}\right]$$

$$= \frac{1}{\pi n}\left(\frac{e^{j\pi n} - e^{-j\pi n}}{2j} - \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right)$$

$$= \frac{1}{\pi n}\left[\sin \pi n - \sin \omega_c n\right] \;;\; \text{for all } n, \text{ except } n=0$$

When $n=0$; $h_d(n) = h_d(0) = \underset{n\to 0}{Lt}\left[\frac{\sin\pi n - \sin\omega_c n}{\pi n}\right]$

When $n=0$; $h_d(n)$ becomes $0/0$, which is indeterminate.

$$= \underset{n\to 0}{Lt}\frac{\sin \pi n}{\pi n} - \underset{n\to 0}{Lt}\frac{\sin\omega_c n}{\pi n}$$

Using L'Hospital rule

$$\underset{\theta\to 0}{Lt}\frac{\sin A\theta}{\theta} = A$$

$$= \frac{1}{\pi}\underset{n\to 0}{Lt}\frac{\sin \pi n}{\pi} - \frac{1}{\pi}\underset{n\to 0}{Lt}\frac{\sin\omega_c n}{n}$$

$$= \frac{1}{\pi}\times\pi - \frac{1}{\pi}\times\omega_c$$

$$= 1 - \frac{\omega_c}{\pi}$$

The impulse response $h(n)$ of FIR filter is obtained by truncating $h_d(n)$ to 7 samples

$$\therefore h(n) = h_d(n) = \frac{\sin \pi n - \sin \omega_c n}{\pi n} = \frac{-\sin \omega_c n}{\pi n} \; ; \text{ for } n = -\left(\frac{N-1}{2}\right) \text{ to } \left(\frac{N-1}{2}\right)$$

except $n=0$

$$= 1 - \frac{\omega_c}{\pi} \; ; \text{ for } n=0 \qquad \left\{ \because \sin n\pi = 0 \right\}$$

Here, $N=7$ $\therefore \frac{N-1}{2} = \frac{7-1}{2} = 3$

Hence calculate $h(n)$ for $n=-3$ to 3

since, the impulse response $h(n)$ satisfies the symmetry condition, $h(-n) = h(n)$, calculate $h(n)$ for $n=0$ to 3

When $n=0$, $h_d(0) = \underset{n\to 0}{Lt}\left[-\frac{\sin \omega_c n}{\pi n}\right]$

$$= -\frac{1}{\pi} \underset{n\to 0}{Lt}\left[\frac{-\sin \omega_c n}{n}\right] = 1 - \frac{\omega_c}{\pi} = 0.4$$

when $n=1$, $h(1) = -\frac{\sin(0.6\pi \times 1)}{\pi \times 1} = -0.3027$

when $n=2$, $h(2) = \frac{-\sin(0.6\pi \times 2)}{\pi \times 2} = 0.0935$

When $n=3$; $h(3) = \frac{-\sin(0.6\pi \times 3)}{\pi \times 3} = 0.0623$

When $n=-1$; $h(-1) = h(1) = -0.3027$

When $n=-2$; $h(-2) = h(2) = 0.0935$

When $n=-3$; $h(-3) = h(3) = 0.0623$

Using Symmetry condition, $h(-n) = h(n)$

The transfer function $H(z)$ of the digital HPF is given by

$$H(z) = z^{-\left(\frac{N-1}{2}\right)} Z\{h(n)\}$$

$$= z^{-\frac{(N-1)}{2}} \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n)\, z^{-n}$$

$$= z^{-3} \sum_{n=-3}^{3} h(n)\, z^{-n}$$

$$= z^{-3}\left\{ h(-3)z^{3} + h(-2)z^{2} + h(-1)z^{+1} + h(0)z^{0} + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} \right\}$$

$$H(z) = z^{-3}\left\{h(3)z^3 + h(2)z^2 + h(1)z + h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3}\right\}$$

$\because$ Using symmetry condition, $h(-n) = h(n)$

$$= z^{-3}\left\{h(0) + h(1)[z+z^{-1}] + h(2)[z^2+z^{-2}] + h(3)[z^3+z^{-3}]\right\}$$

$$= h(0)z^{-3} + h(1)[z^{-2}+z^{-4}] + h(2)[z^{-1}+z^{-5}] + h(3)[z^0+z^{-6}]$$

$$= 0.4z^{-3} - 0.3027[z^{-2}+z^{-4}] + 0.0935[z^{-1}+z^{-5}] + 0.0623[1+z^{-6}]$$

Structure:

Let $H(z) = \dfrac{Y(z)}{X(z)} = 0.4z^{-3} - 0.3027[z^{-2}+z^{-4}] + 0.0935[z^{-1}+z^{-5}] + 0.0623[1+z^{-6}]$

$$Y(z) = 0.4z^{-3}X(z) - 0.3027[z^{-2}X(z)+z^{-4}X(z)] + 0.0935[z^{-1}X(z) + z^{-5}X(5)]$$
$$+ 0.0623[X(z) + z^{-6}X(z)]$$

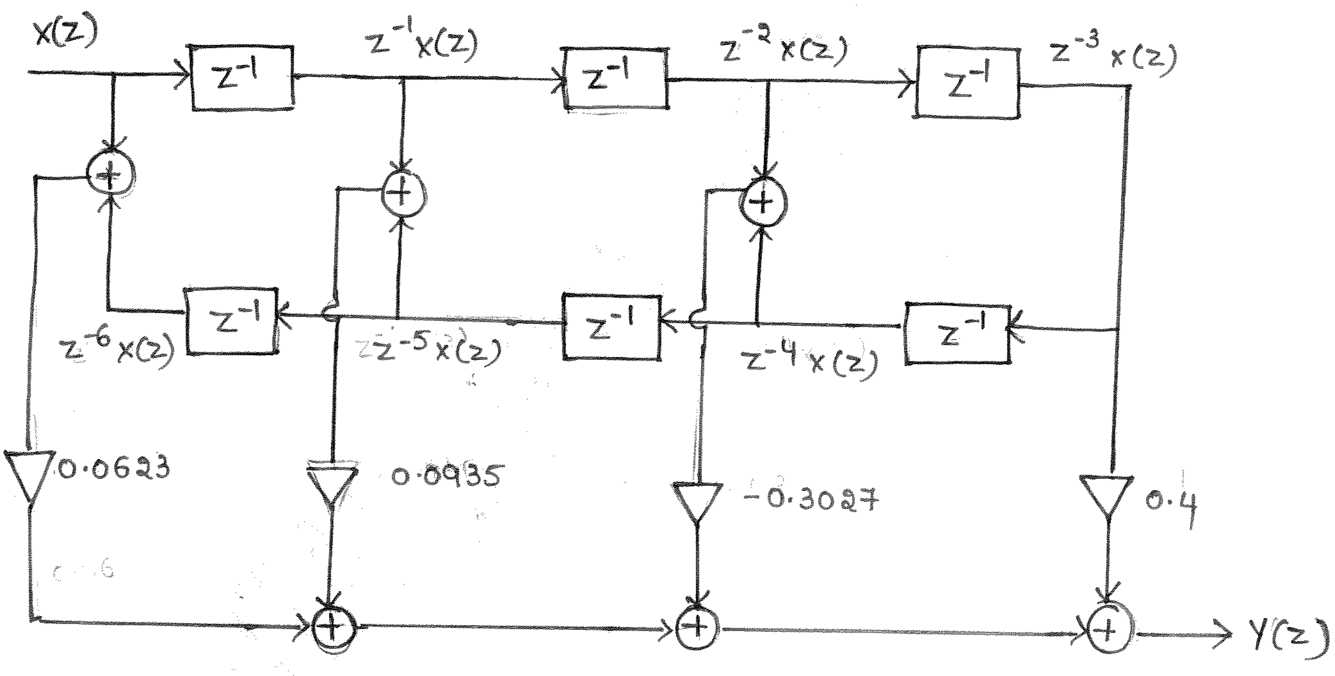


Fig: Linear phase structure of FIR high pass filter

Frequency response:

· When impulse response is symmetric & N is odd with centre of symmetry at $n=0$, the magnitude response $|H(e^{j\omega})|$ is given by $|A(\omega)|$,

where $A(\omega) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2h(n)\cos\omega n$

$$= h(0) + \sum_{n=1}^{3} 2h(n)\cos\omega n$$

$$A(\omega) = h(0) + 2h(1)\cos\omega + 2h(2)\cos 2\omega + 2h(3)\cos 3\omega$$

$$= 0.4 + 2\times -0.3027\cos\omega + 2\times 0.0935\cos 2\omega + 2\times 0.0623\cos 3\omega$$

$$= 0.4 - 0.6054\cos\omega + 0.187\cos 2\omega + 0.1246\cos 3\omega$$

Using above eq, the amplitude response $A(\omega)$ and magnitude function $|H(e^{j\omega})|$ are calculated for various values of ω

Table : $A(\omega)$ and $|H(e^{j\omega})|$ for various values of ω

ω	$A(\omega)$	$\lvert H(e^{j\omega})\rvert = \lvert A(\omega)\rvert$
$0\times\frac{\pi}{16}$	0.1062	0.1062
$1\times\frac{\pi}{16}$	0.0825	0.0826
$2\times\frac{\pi}{16}$	0.0205	0.0205
$3\times\frac{\pi}{16}$	−0.0561	0.0560
$4\times\frac{\pi}{16}$	−0.1161	0.1161
$5\times\frac{\pi}{16}$	−0.1301	0.1300
$6\times\frac{\pi}{16}$	−0.0790	0.0790
$7\times\frac{\pi}{16}$	0.0399	0.0399
$8\times\pi/16$	0.213	0.213

ω	$A(\omega)$	$\lvert H(e^{j\omega})\rvert = \lvert A(\omega)\rvert$
$9\times\pi/16$	0.4145	0.4145
$10\times\frac{\pi}{16}$	0.6145	0.6146
$11\times\frac{\pi}{16}$	0.7869	0.7869
$12\times\frac{\pi}{16}$	0.9161	0.9161
$13\times\frac{\pi}{16}$	0.9992	0.9991
$14\times\frac{\pi}{16}$	1.0438	1.0438
$15\times\frac{\pi}{16}$	1.0629	1.0629
$16\times\frac{\pi}{16}$	1.0678	1.0678

Alternate method for frequency response :

Frequency response $H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$

$$H(e^{j\omega}) = 0.4z^{-3} - 0.3027[z^{-2}+z^{-4}] + 0.0935[z^{-1}+z^{-5}] + 0.0623[1+z^{-6}]$$

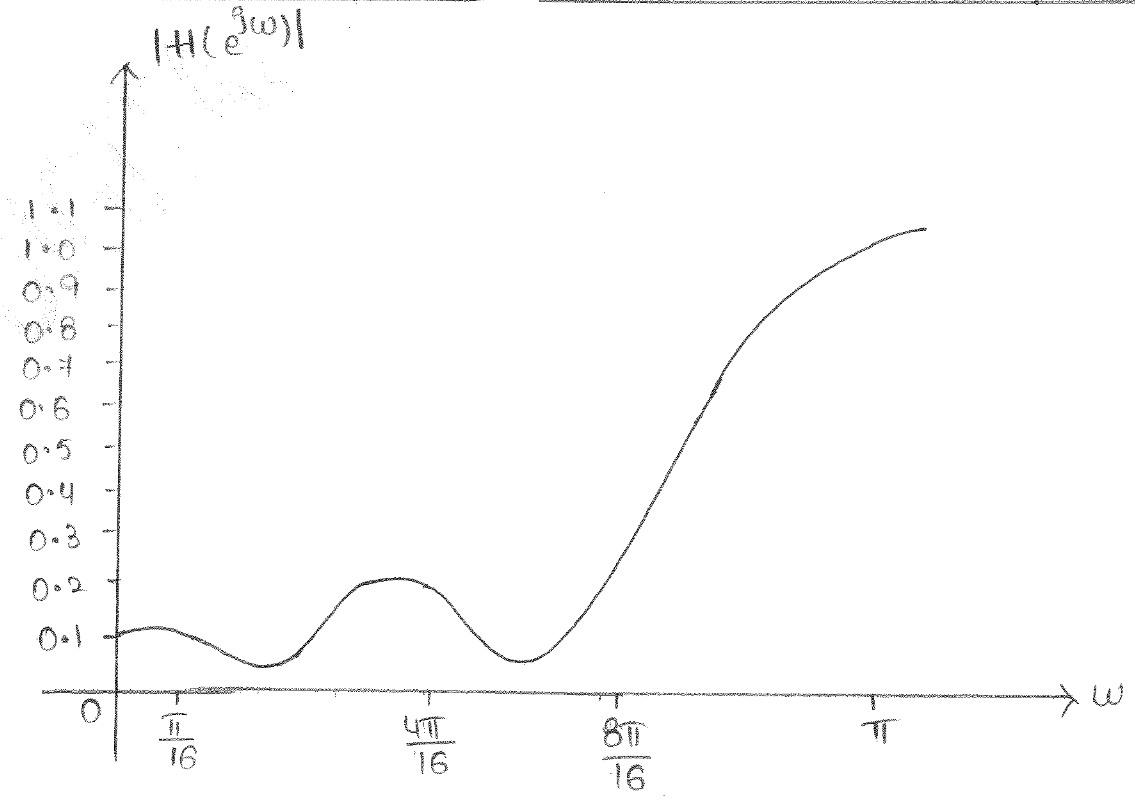
$$= 0.4e^{-j3\omega} - 0.3027[e^{-j2\omega}+e^{-j4\omega}] + 0.0935[e^{-j\omega}+e^{-j5\omega}] + 0.0623[1+e^{-j6\omega}]$$

$$= 0.4[\cos 3\omega - j\sin 3\omega] - 0.3027[\cos 2\omega - j\sin 2\omega + \cos 4\omega - j\sin 4\omega]$$
$$+ 0.0935[\cos\omega - j\sin\omega + \cos 5\omega - j\sin 5\omega] + 0.0623[1 + \cos 6\omega - j\sin 6\omega]$$

$H(e^{j\omega}) = [0.4\cos 3\omega - 0.3027\cos 2\omega - 0.3027\cos 4\omega + 0.0935\cos\omega +$
$0.0935\cos 5\omega + 0.0623\cos 6\omega + 0.0623] +$
$j[-0.4\sin 3\omega + 0.3027\sin 2\omega + 0.3027\sin 4\omega - 0.0935\sin\omega -$
$0.0935\sin 5\omega - 0.0623\sin 6\omega]$

ω	$H(e^{j\omega})$	$\lvert H(e^{j\omega})\rvert$
$0\times\frac{\pi}{16}$	$0.1062 + j0$	0.1062
$1\times\frac{\pi}{16}$	$0.0687 - j0.0459$	0.0826
$2\times\frac{\pi}{16}$	$0.0078 - j0.0190$	0.0205
$3\times\frac{\pi}{16}$	$0.0109 - j0.0550$	0.0560
$4\times\frac{\pi}{16}$	$0.0821 + j0.0821$	0.1161
$5\times\frac{\pi}{16}$	$0.1276 + j0.0253$	0.1300
$6\times\frac{\pi}{16}$	$0.0730 - j0.0302$	0.0790
$7\times\frac{\pi}{16}$	$-0.0221 + j0.0331$	0.0397
$8\times\frac{\pi}{16}$	$0 + j0.213$	0.213

ω	$H(e^{j\omega})$	$\lvert H(e^{j\omega})\rvert$
$9\times\pi/16$	$0.2303 + j0.3447$	0.4145
$10\times\frac{\pi}{16}$	$0.5678 + j0.2352$	0.6746
$11\times\frac{\pi}{16}$	$0.7718 - j0.1535$	0.7869
$12\times\frac{\pi}{16}$	$0.6478 - j0.6478$	0.9161
$13\times\frac{\pi}{16}$	$0.1949 - j0.9800$	0.9991
$14\times\frac{\pi}{16}$	$-0.3994 - j0.9644$	1.0438
$15\times\frac{\pi}{16}$	$-0.8838 + j0.5905$	1.0629
$16\times\frac{\pi}{16}$	$-1.0678 + j0$	1.0678



Design an FIR band pass filter to pass frequencies in the range 1.5 khz to 3 khz & sampling frequency of 8 khz with 7 samples using fourier series method. Determine frequency response & verify the design by sketching the magnitude response.

Given $f_{c1} = 1.5\,khz$; $f_{c2} = 3\,khz$; $f_s = 8\,khz$

$$\therefore \omega_{c1} = \Omega_{c1}T = \frac{\Omega_{c1}}{f_s} = \frac{2\pi f_{c1}}{f_s} = \frac{2\pi \times 1.5 \times 10^3}{8 \times 10^3} = 0.375\pi$$

$$\omega_{c2} = \Omega_{c2}T = \frac{\Omega_{c2}}{f_s} = \frac{2\pi f_{c2}}{f_s} = \frac{2\pi \times 3 \times 10^3}{8 \times 10^3} = 0.75\pi$$

The desired frequency response $H_d(e^{j\omega})$ of BPF is

$$H_d(e^{j\omega}) = 1 \; ; \text{ for } -\omega_{c2} \le \omega \le -\omega_{c1} \;\&\; \omega_{c1} \le \omega \le \omega_{c2}$$
$$= 0 \; ; \text{ otherwise}$$

The desired impulse response $h_d(n)$ of BPF is

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_{c2}}^{-\omega_{c1}} 1 \times e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_{c1}}^{\omega_{c2}} 1 \times e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_{c2}}^{-\omega_{c1}} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\omega_{c1}}^{\omega_{c2}}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega_{c1}n} - e^{-j\omega_{c2}n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\omega_{c2}n} - e^{j\omega_{c1}n}}{jn}\right]$$

$$= \frac{1}{\pi n}\left[\frac{e^{j\omega_{c2}n} - e^{-j\omega_{c2}n}}{2j} - \frac{e^{j\omega_{c1}n} - e^{-j\omega_{c1}n}}{2j}\right] \qquad \left\{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}\right\}$$

$$= \frac{\sin\omega_{c2}n - \sin\omega_{c1}n}{\pi n} \; ; \text{ for all } n, \text{ except } n=0$$

When $n=0$, $h_d(n)$ becomes $0/0$, which is indeterminate.

When $n=0$; $h_d(n) = h_d(0) = \underset{n\to 0}{Lt}\left[\dfrac{\sin\omega_{c2}n - \sin\omega_{c1}n}{\pi n}\right]$

$$= \frac{1}{\pi}\left[\underset{n\to 0}{Lt}\frac{\sin\omega_{c2}n}{n} - \underset{n\to 0}{Lt}\frac{\sin\omega_{c1}n}{n}\right]$$

$$= \frac{1}{\pi}\left[\omega_{c2} - \omega_{c1}\right]$$

Using L'Hospital rule, $\underset{\theta\to 0}{Lt}\dfrac{\sin A\theta}{\theta} = A$

The impulse response $h(n)$ of FIR filter is obtained by truncating $h_d(n)$ to 7 samples

$$\therefore h(n) = h_d(n) = \frac{\sin\omega_{c2}n - \sin\omega_{c1}n}{\pi n} \quad ; \text{ for } n = -\left(\frac{N-1}{2}\right) \text{ to } \left(\frac{N-1}{2}\right), \text{ except } n=0$$

$$= \frac{\omega_{c2} - \omega_{c1}}{\pi} \quad ; \text{ for } n=0$$

Here, $N=7$ $\therefore \dfrac{N-1}{2} = \dfrac{7-1}{2} = 3$

Hence calculate $h(n)$ for $n=-3$ to 3

Since, the impulse response $h(n)$ satisfies symmetry condition, $h(-n) = h(n)$, calculate $h(n)$ for $n=0$ to 3.

When $n=0$; $h(0) = \dfrac{\omega_{c2}-\omega_{c1}}{\pi} = \dfrac{0.75\pi - 0.375\pi}{\pi} = 0.375$

When $n=1$; $h(1) = \dfrac{\sin(0.75\pi\times 1) - \sin(0.375\pi\times 1)}{\pi\times 1} = -0.069$

When $n=2$; $h(2) = \dfrac{\sin(0.75\pi\times 2) - \sin(0.375\pi\times 2)}{\pi\times 2} = -0.2716$

When $n=3$; $h(3) = \dfrac{\sin(0.75\pi\times 3) - \sin(0.375\pi\times 3)}{\pi\times 3} = 0.1156$

When $n=-1$; $h(-1) = h(1) = -0.069$

$n=-2$; $h(-2) = h(2) = -0.2716$

$n=-3$; $h(-3) = h(3) = 0.1156$

The transfer function $H(z)$ of the digital FIR BPF is given by

$$H(z) = z^{-\left(\frac{N-1}{2}\right)} z\{h(n)\}$$

$$= z^{-\left(\frac{N-1}{2}\right)} \left[\sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} h(n)\, z^{-n} \right] = z^{-3} \sum_{n=-3}^{3} h(n) z^{-n}$$

$$= z^{-3}\left\{ h(-3)z^{3} + h(-2)z^{2} + h(-1)z^{1} + h(0)z^{0} + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} \right\}$$

$$= z^{-3}\left\{ h(3)z^{3} + h(2)z^{2} + h(1)z + h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} \right\} \quad \{\because h(-n) = h(n)\}$$

$$= z^{-3}\left\{ h(0) + h(1)[z + z^{-1}] + h(2)[z^{2} + z^{-2}] + h(3)[z^{3} + z^{-3}] \right\}$$

$$= h(0)z^{-3} + h(1)[z^{-2} + z^{-4}] + h(2)[z^{-1} + z^{-5}] + h(3)[z^{0} + z^{-6}]$$

$$= 0.375z^{-3} - 0.069[z^{-2} + z^{-4}] - 0.2716[z^{-1} + z^{-5}] + 0.1156[z^{0} + z^{-6}]$$

Structure :

Let $H(z) = \frac{Y(z)}{X(z)} = 0.375z^{-3} - 0.069(z^{-2} + z^{-4}) - 0.2716(z^{-1} + z^{-5}) + 0.1156(1 + z^{-6})$

$$Y(z) = 0.375z^{-3}X(z) - 0.069[z^{-2}X(z) + z^{-4}X(z)] - 0.2716[z^{-1}X(z) + z^{-5}X(z)] + 0.1156[X(z) + z^{-6}X(6)]$$

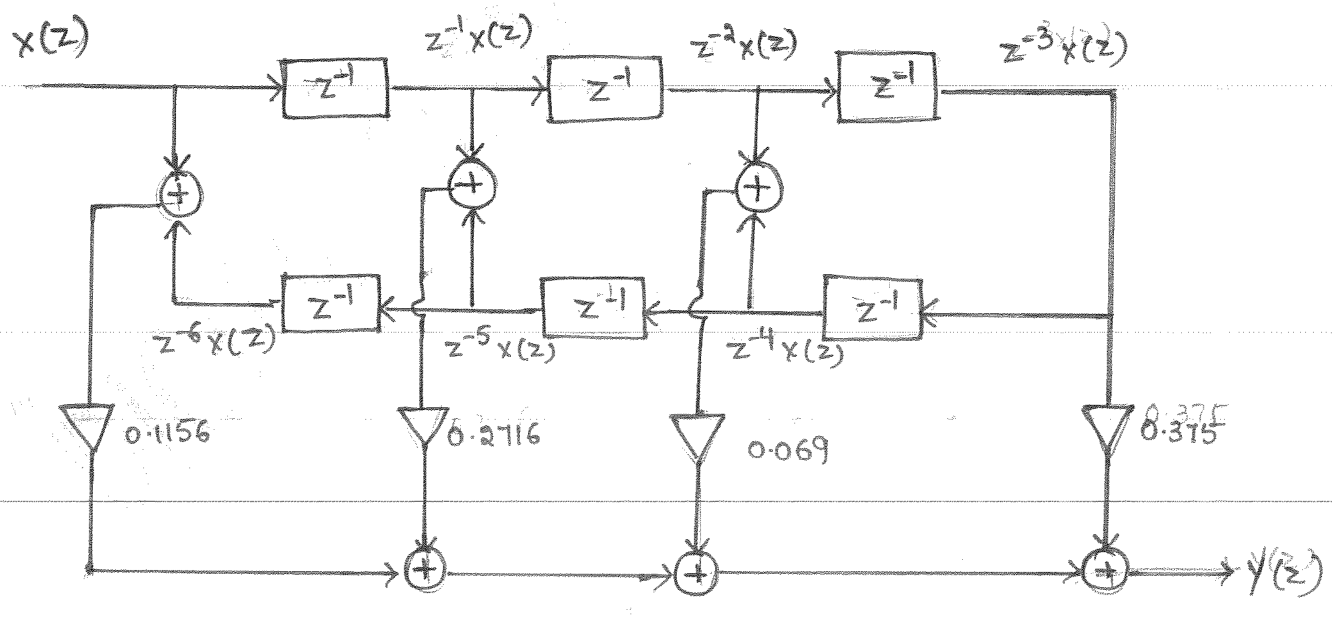


fig: Linear phase structure of FIR BPF.

Frequency response :

When impulse response is symmetric & N is odd with centre of symmetry at $n=0$, the magnitude response $|H(e^{j\omega})|$ is given by $|A(\omega)|$, where $A(\omega) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2h(n)\cos\omega n$

$$= h(0) + \sum_{n=1}^{3} 2h(n)\cos\omega n$$

$$= h(0) + 2h(1)\cos\omega + 2h(2)\cos\omega(2) + 2h(3)\cos 3\omega$$

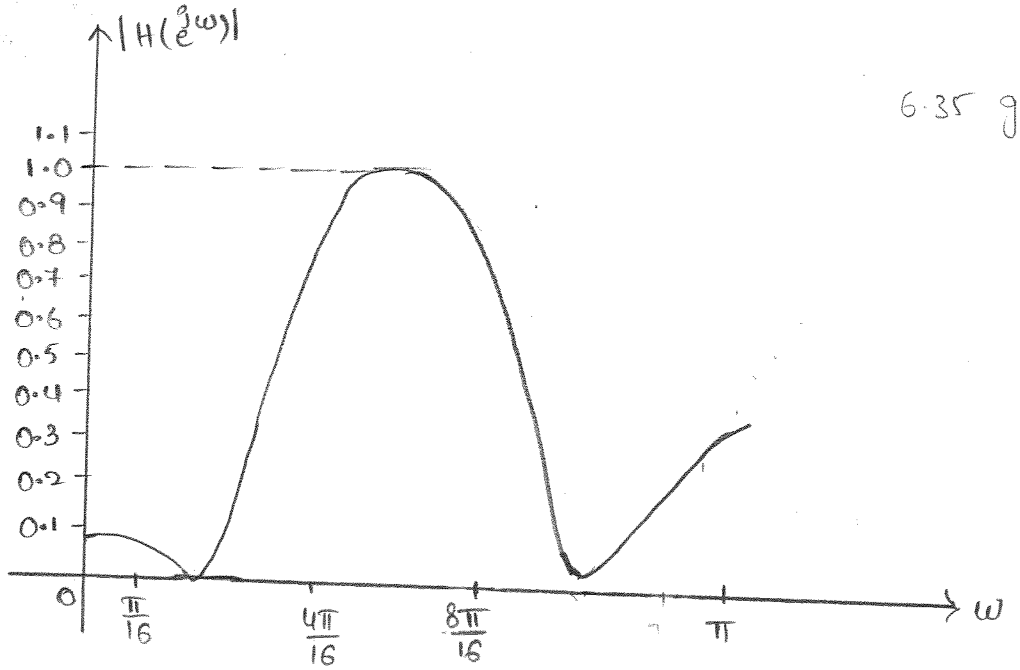
$$= 0.375 + 2\times(-0.069)\cos\omega + 2(-0.2716)\cos 2\omega + 2(0.1156)\cos 3\omega$$

$$= 0.375 - 0.138\cos\omega - 0.5432\cos 2\omega + 0.2312\cos 3\omega$$

Table : $A(\omega)$ & $|H(e^{j\omega})|$ for various values of ω

ω	$H(e^{j\omega})$	$\lvert H(e^{j\omega})\rvert = \lvert A(\omega)\rvert$
$0\times\pi/16$	-0.075	0.075
$1\times\pi/16$	-0.0699	0.0699
$2\times\pi/16$	-0.0481	0.0481
$3\times\pi/16$	0.0081	0.0081
$4\times\pi/16$	0.1139	0.1138
$5\times\pi/16$	0.2794	0.2793
$6\times\pi/16$	0.4926	0.4925
$7\times\pi/16$	0.7215	0.7215
$8\times\pi/16$	0.9182	0.9182

ω	$H(e^{j\omega})$	$\lvert H(e^{j\omega})\rvert = \lvert A(\omega)\rvert$
$9\times\pi/16$	1.0322	1.0321
$10\times\pi/16$	1.0255	1.0254
$11\times\pi/16$	0.8862	0.8862
$12\times\pi/16$	0.6230	0.6359
$13\times\pi/16$	0.3269	0.3269
$14\pi/16$	0.0299	0.0298
$15\times\pi/16$	-0.1837	0.1836
$16\times\pi/16$	-0.2614	0.2614



6.35 graph

Design a FIR band stop filter to reject frequencies in the range 1.5 khz to 3 khz & sampling frequency of 8 khz with 7 samples using fourier series method. Determine frequency response & verify the design by sketching magnitude response.

Given $f_{c1} = 1.5\text{ khz}$; $f_{c2} = 3\text{ khz}$; $f_s = 8\text{ khz}$

$$\therefore \omega_{c1} = \Omega_{c1}T = \frac{\Omega_{c1}}{f_s} = \frac{2\pi f_{c1}}{f_s} = \frac{2\pi \times 1.5\times 10^3}{8\times 10^3} = 0.375\pi$$

$$\omega_{c2} = \Omega_{c2}T = \frac{\Omega_{c2}}{f_s} = \frac{2\pi f_{c2}}{f_s} = \frac{2\pi \times 3\times 10^3}{8\times 10^3} = 0.75\pi$$

The desired frequency response $H_d(e^{j\omega})$ of bandstop filter is

$$H_d(e^{j\omega}) = 1 \;;\; -\pi \le \omega \le -\omega_{c2} \;\&\; -\omega_{c1} \le \omega \le \omega_{c1} \;\&\; \omega_{c2} \le \omega \le \pi$$
$$= 0 \;;\; \text{otherwise}$$

The desired impulse response $h_d(n)$ of bandstop filter is

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{-\omega_{c2}} 1\times e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{-\omega_{c1}}^{\omega_{c1}} 1\times e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_{c2}}^{\pi} 1\times e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-\omega_{c2}} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_{c1}}^{\omega_{c1}} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\omega_{c2}}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega_{c2}n}}{jn} - \frac{e^{-j\pi n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\omega_{c1}n} - e^{-j\omega_{c1}n}}{jn}\right] + \frac{1}{2\pi}\left[\frac{e^{j\pi n} - e^{j\omega_{c2}n}}{jn}\right]$$

$$= \frac{1}{\pi n}\left[\frac{e^{j\pi n} - e^{-j\pi n}}{2j} + \frac{e^{j\omega_{c1}n} + e^{-j\omega_{c1}n}}{2j} \mp \frac{e^{j\omega_{c2}n} - e^{-j\omega_{c2}n}}{2j}\right]$$

$$= \frac{\sin\pi n + \sin\omega_{c1}n + - \sin\omega_{c2}n}{\pi n}$$; for all n, except $n=0$

{ When $n=0$, $h_d(n)$ becomes 0 which is indeterminate }

When $n=0$; $h_d(n) = h_d(0) = \underset{n\to 0}{Lt}\left[\frac{\sin \pi n}{\pi n} + \frac{\sin \omega_{c1} n}{\pi n} - \frac{\sin \omega_{c2} n}{\pi n}\right]$

$$= \frac{1}{\pi}\underset{n\to 0}{Lt}\frac{\sin \pi n}{n} + \frac{1}{\pi}\underset{n\to 0}{Lt}\frac{\sin \omega_{c1} n}{n} - \frac{1}{\pi}\underset{n\to 0}{Lt}\frac{\sin \omega_{c2} n}{n}$$

$$= \frac{1}{\pi}\times \pi + \frac{1}{\pi}\times \omega_{c1} - \frac{1}{\pi}\times \omega_{c2}$$

Using L'Hospital rule $\underset{\theta\to 0}{Lt}\frac{\sin A\theta}{\theta} = A$

$$= 1 - \left(\frac{\omega_{c2} - \omega_{c1}}{\pi}\right)$$

The impulse response $h(n)$ of FIR filter is obtained by truncating $h_d(n)$ to 7 samples

$$h(n) = h_d(n) = \frac{\sin \omega_{c1} n - \sin \omega_{c2} n}{\pi n} \;;\; \text{for } n = -\left(\frac{N-1}{2}\right) \text{ to } \left(\frac{N-1}{2}\right),$$

except $n=0$

$$= 1 - \left(\frac{\omega_{c2} - \omega_{c1}}{\pi}\right) \;;\; \text{for } n=0$$

Here $N=7$, $\therefore \frac{N-1}{2} = \frac{7-1}{2} = 3$

Hence calculate $h(n)$ for $n = -3$ to 3

Since, the impulse response $h(n)$ satisfies symmetry condition, $h(n) = h(-n)$, calculate $h(n)$ for $n=3$

When $n=0$; $h(0) = 1 - \left(\frac{\omega_{c2} - \omega_{c1}}{\pi}\right) = 1 - \left(\frac{0.75\pi - 0.375\pi}{\pi}\right) = 0.625$

$$n=1 \;;\; h(1) = \frac{\sin(0.375\pi\times 1) - \sin(0.75\pi\times 1)}{\pi\times 1} = 0.069$$

$$n=2 \;;\; h(2) = \frac{\sin(0.375\pi\times 2) - \sin(0.75\pi\times 2)}{\pi\times 2} = 0.2716$$

$$n=3 \;;\; h(3) = \frac{\sin(0.375\pi\times 3) - \sin(0.75\pi\times 3)}{\pi\times 3} = -0.1156$$

When $n=-1$; $h(-1) = h(1) = 0.069$

$n=-2$; $h(-2) = h(2) = 0.2716$

$n=3$; $h(-3) = h(3) = -0.1156$

The transfer function $H(z)$ of digital band stop filter is

$$H(z) = z^{\frac{N-1}{2}} \, Z\{h(n)\}$$

$$= z^{\frac{N-1}{2}} \left[\sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} h(n) z^{-n} \right] = z^{-3} \sum_{n=-3}^{3} h(n) z^{-n}$$

$$= z^{-3} \left\{ h(-3)z^{3} + h(-2)z^{2} + h(-1)z^{1} + h(0)z^{0} + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} \right\}$$

$$= z^{-3} \left[h(3)z^{3} + h(2)z^{2} + h(1)z + h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} \right\}$$

$$\left\{ \because h(-n) = h(n) \right\}$$

$$= z^{-3} \left\{ h(0) + h(1)[z + z^{-1}] + h(2)[z^{2} + z^{-2}] + h(3)[z^{3} + z^{-3}] \right\}$$

$$= h(0)z^{-3} + h(1)[z^{-2} + z^{-4}] + h(2)[z^{-1} + z^{-5}] + h(3)[z^{0} + z^{-6}]$$

$$= 0.625z^{-3} + 0.069[z^{-2} + z^{-4}] + 0.2716[z^{-1} + z^{-5}] - 0.1156[1 + z^{-6}]$$

Structure:

Let $H(z) = \dfrac{Y(z)}{X(z)} = 0.625z^{-3} + 0.069[z^{-2} + z^{-4}] + 0.2716[z^{-1} + z^{-5}] - 0.1156[1 + z^{-6}]$

$$Y(z) = 0.625z^{-3}X(z) + 0.069[z^{-2}X(z) + z^{-4}X(z)] + 0.2716[z^{-1}X(z) + z^{-5}X(z)] - 0.1156[X(z) + z^{-6}X(z)]$$

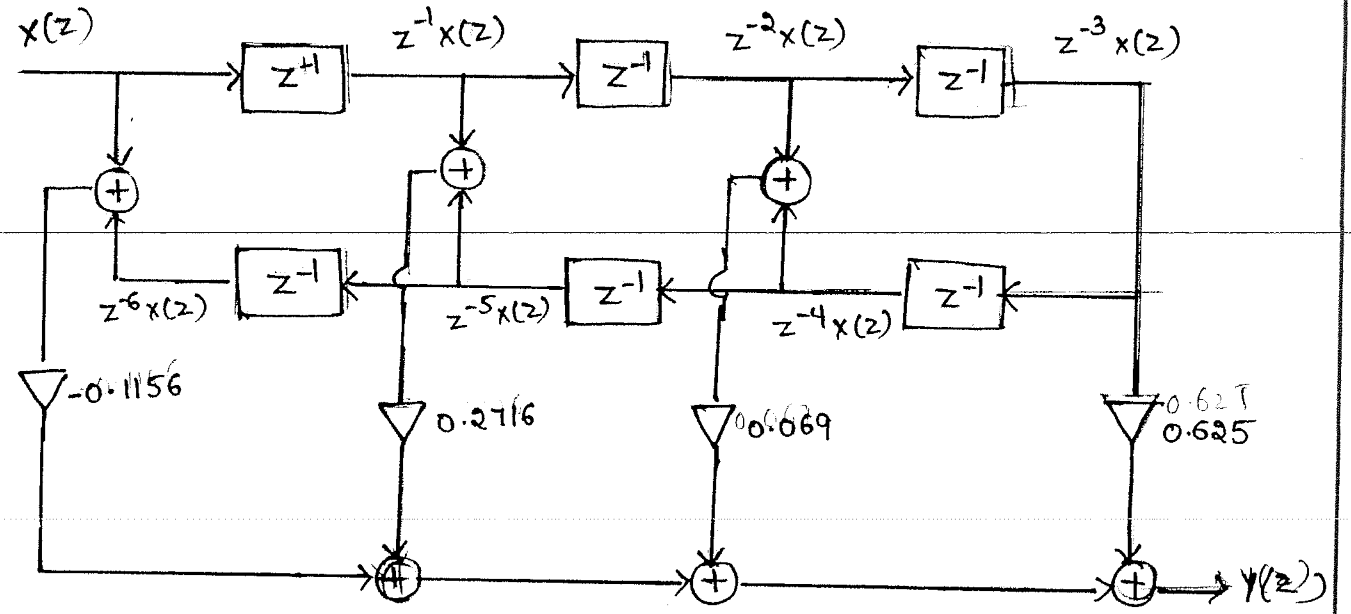


fig: Linear phase structure of FIR bandstop filter

Frequency response :

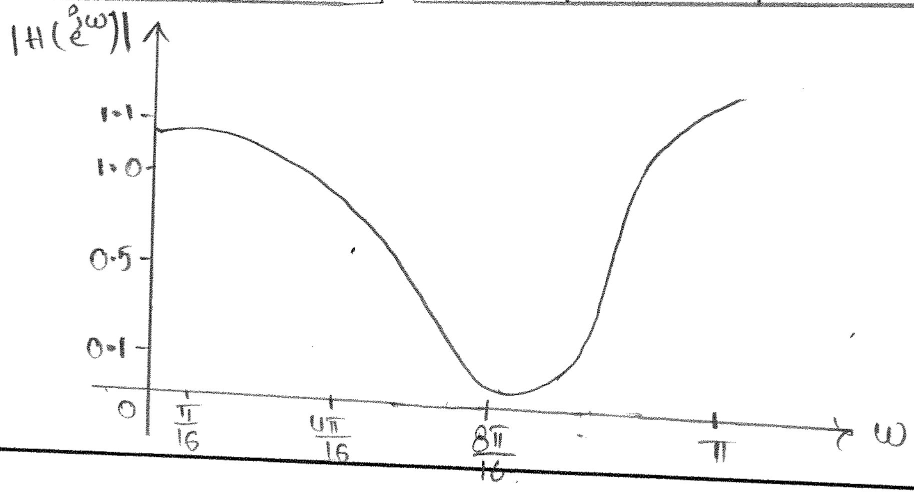
When impulse response is symmetric & N is odd with centre of symmetry at $n=0$, the magnitude function $|H(e^{j\omega})|$ is given by $|A(\omega)|$

where,
$$A(\omega) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2h(n)\cos\omega n$$
$$= h(0) + \sum_{n=1}^{3} 2h(n)\cos\omega n$$
$$= h(0) + 2h(1)\cos\omega + 2h(2)\cos 2\omega + 2h(3)\cos 3\omega$$
$$= 0.625 + 2\times 0.069\cos\omega + 2\times 0.2716\cos 2\omega + 2\times -0.1156\cos 3\omega$$
$$= 0.625 + 0.138\cos\omega + 0.5432\cos 2\omega - 0.2312\cos 3\omega$$

Table : $A(\omega)$ & $|H(e^{j\omega})|$ for various values of ω

ω	$H(e^{j\omega})$	$\lvert H(e^{j\omega})\rvert = \lvert A(\omega)\rvert$
$0\times\pi/16$	1.075	1.075
$1\times\pi/16$	1.0699	1.0699
$2\times\pi/16$	1.0481	1.0481
$3\times\pi/16$	0.9927	0.9926
$4\times\pi/16$	0.8860	0.8860
$5\times\pi/16$	0.7205	0.7205
$6\times\pi/16$	0.5073	0.5073
$7\times\pi/16$	0.2785	0.2785
$8\times\pi/16$	0.0818	0.0818

ω	$H(e^{j\omega})$	$\lvert H(e^{j\omega})\rvert = \lvert A(\omega)\rvert$
$9\times\pi/16$	-0.0322	0.0322
$10\times\pi/16$	-0.0255	0.0255
$11\times\pi/16$	0.1137	0.1137
$12\times\pi/16$	0.3639	0.3639
$13\times\pi/16$	0.6730	0.6730
$14\times\pi/16$	0.9700	0.9700
$15\times\pi/16$	1.1837	1.1837
$16\times\pi/16$	1.2614	1.2614



Prob: Design a Linear phase FIR Lowpass filter using the rectangular window by taking 7 samples of the window sequence, with cut off frequency $\omega_c = 0.2\pi$ rad/sample.

Sol: Let us choose symmetric impulse response with the symmetry condition $h(N-1-n) = h(n)$. Therefore the desired ideal frequency response $H_d(e^{j\omega})$ for FIR LPF is

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} & ; -\omega_c \le \omega \le +\omega_c \\ 0 & ; \text{otherwise} \end{cases}$$

The desired impulse response $h_d(n)$ is obtained by taking I.F.T of $H_d(e^{j\omega})$

$$h_d(n) = IFT[H_d(e^{j\omega})] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}\,d\omega.$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\alpha}\cdot e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-\alpha)}\,d\omega.$$

$$\Rightarrow \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\omega_c}^{\omega_c} \Rightarrow \frac{1}{2\pi}\left[\frac{e^{j\omega_c(n-\alpha)}}{j(n-\alpha)} - \frac{e^{-j\omega_c(n-\alpha)}}{j(n-\alpha)}\right]$$

$$\therefore h_d(n) = \frac{1}{\pi(n-\alpha)}\left[\frac{e^{j\omega_c(n-\alpha)} - e^{-j\omega_c(n-\alpha)}}{2j}\right]$$

$$\left\{ \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}. \right\}$$

$$h_d(n) = \frac{\sin\omega_c(n-\alpha)}{\pi(n-\alpha)} \text{ for all } n \text{ except } n=\alpha$$

$\because$ When $n=\alpha$ { $h_d(n)$ becomes $0/0$ }

Using L'hospital rule, At $n=\alpha$

$$h_d(n) = \lim_{(n-\alpha)\to 0}\frac{\sin\omega_c(n-\alpha)}{\pi(n-\alpha)} = \frac{1}{\pi}\lim_{(n-\alpha)\to 0}\frac{\sin\omega_c(n-\alpha)}{(n-\alpha)}$$

$$h_d(n) = \frac{1}{\pi}\times\omega_c = \frac{\omega_c}{\pi}$$

The impulse response $h(n)$ of FIR filter is obtained by multiplying $h_d(n)$ by window sequence.

Rectangular window sequence, $W_R(n) = \begin{cases} 1 & \text{for } n=0 \text{ to } (N-1) \\ 0 & ; \text{Otherwise} \end{cases}$

Impulse response $h(n) = h_d(n) \times W_R(n)$

$= h_d(n)$ for $n=0$ to $(N-1)$

Here $\underline{N=7}$; $\omega_c = 0.2\pi$ rad/sample; $\alpha = \left(\frac{N-1}{2}\right) = \frac{7-1}{2} = 3$; $N-1=6$

Since, Impulse response $h(n)$ satisfies the symmetry condition $\underline{h(N-1-n) = h(n)}$, calculate $h(n)$ for $\underline{n=0 \text{ to } 3}$

When $n=0 \Rightarrow h(0) = \dfrac{\sin(0.2\pi \times (0-3))}{\pi \times (0-3)} = 0.1009$

$n=1 \Rightarrow h(1) = \dfrac{\sin(0.2\pi \times (1-3))}{\pi \times (1-3)} = 0.1514$

$n=2 \Rightarrow h(2) = \dfrac{\sin(0.2\pi \times (2-3))}{\pi \times (2-3)} = 0.1871$

$n=3 \Rightarrow h(3) = \dfrac{0.2\pi}{\pi} = 0.2$

$h(4) = h(2) = 1.871$

$h(5) = h(1) = 0.1514$ $\boxed{h(N-1-n) = h(n)}$

$h(6) = h(0) = 0.1009$

The Transfer function $H(z)$ for FIR LPF is given as

$$H(z) = Z\{h(n)\} = \sum_{n=0}^{N-1} h(n) z^{-n} = \sum_{n=0}^{6} h(n) z^{-n}.$$

$$\Rightarrow h(0) + h(1) z^{-1} + h(2) z^{-2} + h(3) z^{-3} + h(4) z^{-4} + h(5) z^{-5} + h(6) z^{-6}$$

$$\Rightarrow h(0)\left[1+z^{-6}\right] + h(1)\left[z^{-1}+z^{-5}\right] + h(2)\left[z^{-2}+z^{-4}\right] + h(3) z^{-3}$$

$$H(z) = 0.1009\left[1+z^{-6}\right] + 0.1514\left[z^{-1}+z^{-5}\right] + 1.871\left[z^{-4}+z^{-2}\right] + 0.2z^{-3}$$

Structure

$$H(z) = \frac{Y(z)}{X(z)} = 0.1009\left[1+z^{-6}\right] + 0.1514\left[z^{-1}+z^{-5}\right] + 1.871\left[z^{-2}+z^{-4}\right] + 0.2z^{-3}$$

$$Y(z) = 0.1009\left[X(z) + z^{-6}X(z)\right] + 0.1514\left(X(z)z^{-1} + X(z)z^{-5}\right] + 1.871\left(X(z)z^{-2} + X(z)z^{-4}\right] + 0.2X(z)z^{-3}$$

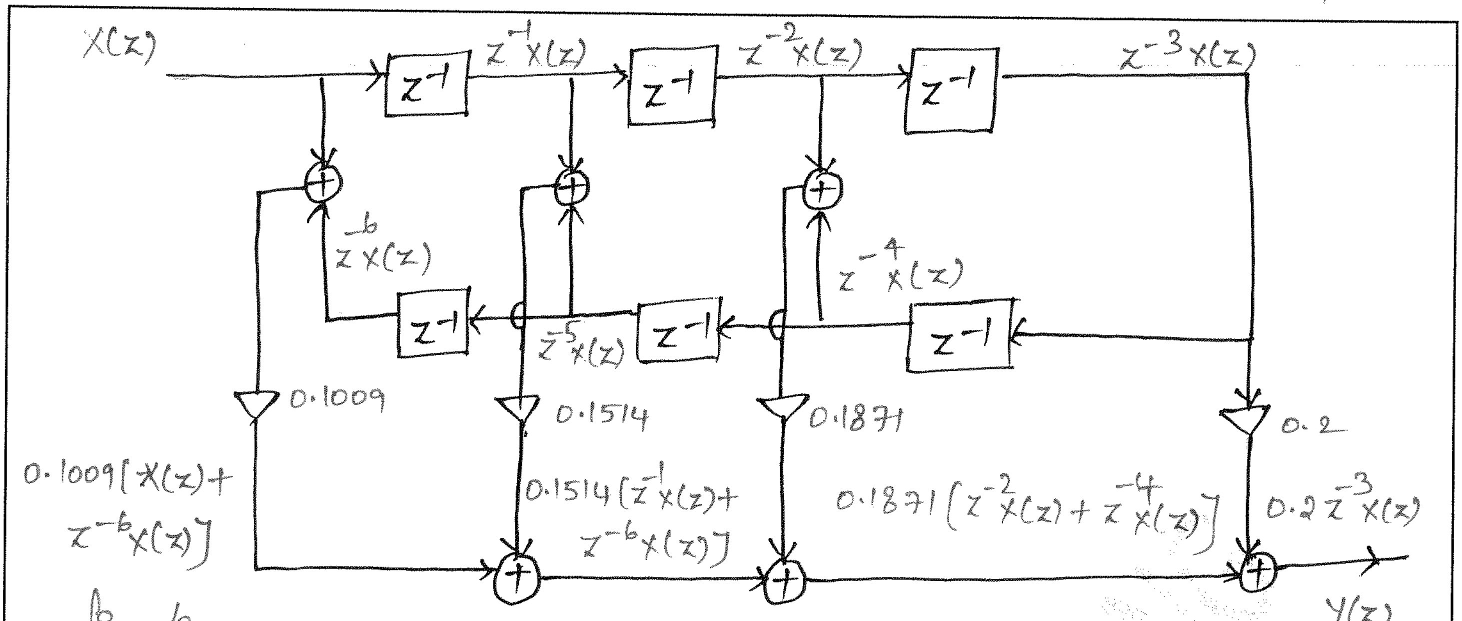


fig: Linear phase Structure of FIR LPF.

FREQUENCY RESPONSE: When Impulse response is symmetric and N is odd with centre of symmetry $\left(\frac{N-1}{2}\right)$, the Magnitude response $|H(e^{j\omega})|$ is given by $|A(\omega)|$

$$A(\omega) = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{N-1/2} 2h\left(\frac{N-1}{2}-n\right)\cos\omega n$$

$$A(\omega) = h(3) + \sum_{n=1}^{3} 2h(3-n)\cos\omega n$$

$$\Rightarrow h(3) + 2h(2)\cos\omega + 2h(1)\cos 2\omega + 2h(0)\cos 3\omega.$$

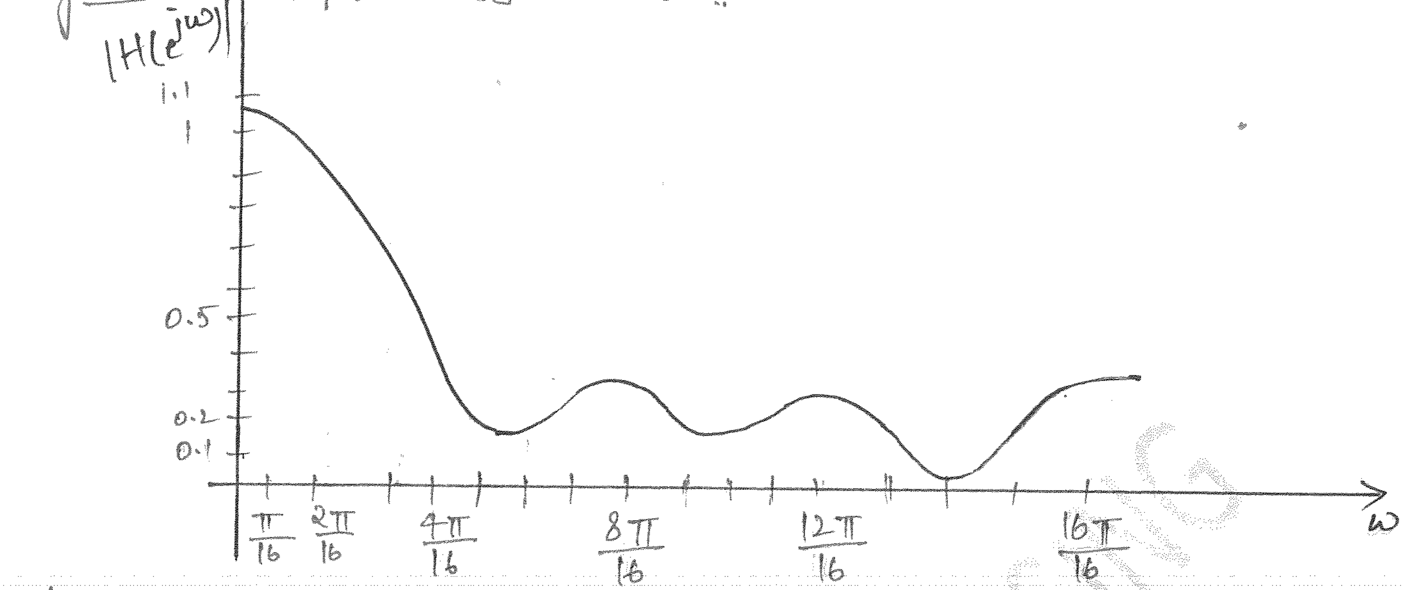
$$A(\omega) \Rightarrow 0.2 + 2(0.1871)\cos\omega + 2\times 0.1514\cos 2\omega + 2\times 0.1009\cos 3\omega$$

$$A(\omega) = 0.2 + 0.3742\cos\omega + 0.3028\cos 2\omega + 0.2018\cos 3\omega.$$

ω	$0\times\pi/16$	$\pi/16$	$2\pi/16$	$4\pi/16$	$5\pi/16$	$6\pi/16$	$7\pi/16$	$8\pi/16$	$9\pi/16$	$10\pi/16$	$11\pi/16$
$A(\omega)$	1.0788	1.014	0.8370	0.5876	0.3219	-0.0573	-0.1188	-0.1028	-0.0406	0.0291	0.0741
$\lvert H(e^{j\omega})\rvert = \lvert A(\omega)\rvert$	1.0788	1.0145	0.8370	0.5876	0.3219	0.0573	0.1188	0.1028	0.0406	0.0291	0.0741

$12\pi/16$	$13\pi/16$	$14\pi/16$	$15\pi/16$	$16\pi/16$.
0.0780	0.0441	-0.0088	-0.0550	-0.0732
0.0780	0.0441	0.0088	0.055	0.0732

Magnitude response of FIR LPF:



Alternate method for Filter Design:

Let the symmetry condition $h(-n) = h(n)$. Therefore desired Ideal frequency response $H_d(e^{j\omega})$ for FIR LPF is

$$H_d(e^{j\omega}) = \begin{cases} 1 \; ; & -\omega_c \le \omega \le \omega_c \\ 0 \; ; & \text{otherwise} \end{cases}$$

The desired impulse response $h_d(n)$ is obtained by taking IFT of $H_d(e^{j\omega})$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega. = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1 \times e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi}\left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2jn}\right]$$

$$\Rightarrow \frac{1}{\pi n}\left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right] = \frac{\sin\omega_c n}{\pi n} \cdot \text{ for all } n \text{ except at } n=0.$$

At $n=0$; $h_d(n) = \lim_{n\to 0} \frac{\sin n\omega_c}{\pi n}$ By using L'Hospital rule

$$\Rightarrow \frac{1}{\pi} \lim_{n\to 0} \frac{\sin \omega_c n}{n} = \frac{\omega_c}{\pi}$$

The Impulse response $h(n)$ is obtained by multiplying $h_d(n)$ by window sequence. Rectangular window sequence.

$$W_R(n) = \begin{cases} 1 \text{ for } & |n| \le \frac{N-1}{2} \\ 0 \text{ for others} \end{cases}$$

Impulse response $h(n) = h_d(n) \times W_R(n)$

$= h_d(n)$ for $|n| \le \frac{N-1}{2}$

Here $N = 7$, $\omega_c = 0.2\pi$ rad/sample ; $\frac{N-1}{2} = \frac{7-1}{2} = 3$

Calculate $h(n)$ for -3 to $+3$

$n=0 \Rightarrow h(0) = \frac{\omega_c}{\pi} = \frac{0.2\pi}{\pi} = 0.2$

$n=1 \Rightarrow h(1) = \frac{\sin(0.2\pi \times 1)}{\pi \times 1} = 0.1871$

$n=2 \Rightarrow h(2) = \frac{\sin(0.2\pi \times 2)}{\pi \times 2} = 0.1514$

$n=3 \Rightarrow h(3) = \frac{\sin(0.2 \times 3)}{\pi \times 3} = 0.1009$

$h(n)$ Satisfies symmetry Condition $\underline{h(-n) = h(n)}$

$n=-1 \Rightarrow h(-1) = h(1) = 0.1871$

$n=-2 \Rightarrow h(-2) = h(2) = 0.1514$

$n=-3 \Rightarrow h(-3) = h(3) = 0.1009$

The T/F of FIR LPF is

$$H(z) = z^{-\left(\frac{N-1}{2}\right)} Z\{h(n)\} = z^{-\left(\frac{N-1}{2}\right)} \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} h(n) z^{-n}$$

$$= z^{-3} \sum_{n=-3}^{3} h(n) z^{-n}$$

$$\Rightarrow z^{-3}\left[h(-3)z^{3} + h(-2)z^{2} + h(-1)z + h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3}\right]$$

$$\Rightarrow z^{-3}\left[h(3)\left[z^{3} + z^{-3}\right] + h(2)\left[z^{2} + z^{-2}\right] + h(1)\left[z + z^{-1}\right] + h(0)\right]$$

$$\Rightarrow h(3)\left[z^{0} + z^{-6}\right] + h(2)\left[z^{-1} + z^{-5}\right] + h(1)\left[z^{-2} + z^{-4}\right] + h(0)z^{-3}$$

$$= 0.1009\left[1 + z^{-6}\right] + 0.1514\left[z^{-1} + z^{-5}\right] + 0.1871\left[z^{-2} + z^{-4}\right] + 0.2z^{-3}$$

The T/F obtained is same in both methods.

Alternate method for frequency response.

Frequency response $H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$

$$H(e^{j\omega}) = 0.1009\left[1+z^{-6}\right] + 0.1514\left[z^{-1}+z^{-5}\right] + 0.1871\left[z^{-2}+z^{-4}\right] + 0.2z^{-3}\Big|_{z=e^{j\omega}}$$

$$= 0.1009\left[1+e^{-j6\omega}\right] + 0.1514\left[e^{-j\omega}+e^{-j5\omega}\right] + 0.1871\left[e^{-j2\omega}+e^{-j4\omega}\right] + 0.2e^{-3j\omega}$$

$$\Rightarrow 0.1009\left[1+\cos 6\omega - j\sin 6\omega\right] + 0.1514\left[\cos\omega - j\sin\omega + \cos 5\omega - j\sin 5\omega\right]$$
$$+0.1871\left[\cos 2\omega - j\sin 2\omega + \cos 4\omega - j\sin 4\omega\right] + 0.2\left[\cos 3\omega - j\sin 3\omega\right]$$

$$\Rightarrow \left[0.1009 + 0.1009\cos 6\omega + 0.1514\cos\omega + 0.1514\cos 5\omega + 0.1871\cos 2\omega\right.$$
$$\left. + 0.1871\cos 4\omega + 0.2\cos 3\omega\right] + j\left[-0.1009\sin 6\omega - 0.1514\sin\omega - \right.$$
$$\left. 0.1514\sin 5\omega - 0.1871\sin 2\omega - 0.1871\sin 4\omega - 0.2\sin 3\omega\right]$$

Calculate $H(e^{j\omega})$ and $|H(e^{j\omega})|$ for various values of ω.

ω	$H(e^{j\omega})$	$\lvert H(e^{j\omega})\rvert$
$0\times\pi/16$	$1.0788 + j.0$	1.0788
$2\pi/16$	$0.8435 - j0.563$	1.0141
$2\pi/16$	$0.3203 - j0.7733$	0.8370
$3\pi/16$	$-0.1146 - j0.576$	0.5872
$4\pi/16$	$-0.2276 - j0.2276$	0.3218
$5\pi/16$	$-0.0922 - j0.0183$	0.0940
$6\pi/16$	$0.0529 - j0.0129$	0.0572
$7\pi/16$	$0.0660 - j0.0988$	0.1188
$8\pi/16$	$0 - j0.1028$	0.1028

ω	$H(e^{j\omega})$	$\lvert H(e^{j\omega})\rvert$
$9\pi/16$	$-0.0225 - j0.0337$	0.0405
$10\pi/16$	$0.0269 + j0.0114$	0.0292
$11\pi/16$	$0.0727 - j0.0144$	0.0741
$12\pi/16$	$0.0552 - j0.0552$	0.0738
$13\pi/16$	$0.0086 - j0.0432$	0.0440
$14\pi/16$	$0.0033 + j0.0081$	0.0087
$15\pi/16$	$0.0457 + j0.0305$	0.0549
$16\pi/16$	$0.0732 - j.0$	0.0732

Prob: Design Linear phase FIR high pass filter using hamming window, with a cutoff frequency $\omega_c = 0.8\pi$ rad/sample and $N=7$

Sol: Let us choose symmetric impulse response with the symmetry condition $h(N-1-n) = h(n)$. Therefore, desired Ideal frequency response $H_d(e^{j\omega})$ for FIR HPF is

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha} & ; \quad -\pi \le \omega \le -\omega_c \text{ and } +\omega_c \le \omega \le \pi \\ 0 & ; \quad \text{otherwise.} \end{cases}$$

$$h_d(n) = IFT[H_d(e^{j\omega})]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\left\{\int_{-\pi}^{-\omega_c} e^{-j\omega\alpha}\cdot e^{j\omega n}\, d\omega + \int_{\omega_c}^{\pi} e^{-j\omega\alpha}\cdot e^{j\omega n}\, d\omega\right\}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{j\omega(n-\alpha)}\, d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{j\omega(n-\alpha)}\, d\omega.$$

$$h_d(n) = \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\pi}^{-\omega_c} + \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{\omega_c}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega_c(n-\alpha)}}{j(n-\alpha)} - \frac{e^{-j\pi(n-\alpha)}}{j(n-\alpha)}\right] + \frac{1}{2\pi}\left[\frac{e^{j\pi(n-\alpha)}}{j(n-\alpha)} - \frac{e^{j\omega_c(n-\alpha)}}{j(n-\alpha)}\right]$$

$$= \frac{1}{\pi(n-\alpha)}\left[\frac{e^{j\pi(n-\alpha)} - e^{-j\pi(n-\alpha)}}{2j} - \left[\frac{e^{j\omega_c(n-\alpha)} - e^{j\omega_c(n-\alpha)}}{2j}\right]\right]$$

$$h_d(n) = \frac{1}{\pi(n-\alpha)}\left[\sin\pi(n-\alpha) - \sin\omega_c(n-\alpha)\right] \text{ ; for all } n, \text{ except at } n=\alpha.$$

When $n=\alpha$, $h_d(n)$ becomes $0/0$, which is indeterminate

$$n=\alpha \Rightarrow h_d(n) = \lim_{(n-\alpha)\to 0} \frac{\sin\pi(n-\alpha) - \sin\omega_c(n-\alpha)}{\pi(n-\alpha)}$$

$$= \frac{1}{\pi}\left[\underset{(n-\alpha)\to 0}{\mathrm{Lt}} \frac{\sin\pi(n-\alpha)}{(n-\alpha)} - \underset{(n-\alpha)\to 0}{\mathrm{Lt}} \frac{\sin\omega_c(n-\alpha)}{(n-\alpha)}\right]$$

$$= \frac{1}{\pi}(\pi-\omega_c) = 1-\frac{\omega_c}{\pi}$$

The impulse response $h(n)$ of FIR filter is obtained by multiplying $h_d(n)$ by window sequence.

Hamming window sequence $W_H(n)$ is

$$W_H(n) = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right) & \text{for } n = 0 \text{ to } (N-1) \\ 0 ; & \text{otherwise.} \end{cases}$$

$$h(n) = h_d(n)\, \omega_H(n)$$

$$= \frac{\sin\pi(n-\alpha) - \sin\omega_c(n-\alpha)}{\pi(n-\alpha)}\left[0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)\right] \text{ for } n\neq\alpha$$

$$h(n) = \left(1-\frac{\omega_c}{\pi}\right)\left[0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)\right] \text{ for } n=\alpha$$

Given that, $N=7$, $\omega_c = 0.8\pi$ rad/sample

$$\because \alpha = \frac{N-1}{2} = 3, \quad (N-1) = 6$$

$h(n)$ satisfies symmetry condition $h(N-1-n) = h(n)$

$$h(n) = \frac{-\sin\omega_c(n-3)}{\pi(n-3)}\left[0.54 - 0.46\cos\left(\frac{n\pi}{3}\right)\right] \text{ for } n\neq 3$$

$$= \left(1-\frac{\omega_c}{\pi}\right)\left[0.54 - 0.46\cos\left(\frac{n\pi}{3}\right)\right] \text{ for } n=3$$

$$n=0 \Rightarrow h(0) = \frac{-\sin(0.8\pi(0-3))\left[0.54-0.46\cos 0\right]}{\pi\times(0-3)} = -0.0081$$

$$n=1 \Rightarrow h(1) = \frac{-\sin(0.8\pi(1-3))\left[0.54-0.46\cos\frac{1\times\pi}{3}\right]}{\pi\times(1-3)} = 0.0469$$

$$n=2 \Rightarrow h(2) = \frac{-\sin(0.8\pi(2-3))\left[0.54-0.46\cos\frac{2\pi}{3}\right]}{\pi(2-3)} = -0.1441$$

$$n=3 \Rightarrow h(3) = \left(1-\frac{0.8\pi}{\pi}\right)\left[0.54-0.46\cos\frac{3\pi}{3}\right] = 0.2$$

$n=4;\ h(4) = h(2) = -0.1441$

$n=5;\ h(5) = h(1) = 0.0469$

$n=6;\ h(6) = h(0) = -0.0081$

$\left\{ \because h(n) = h(N-1-n) = h(6-n) \right\}$

The T/F $H(z)$ of $(H(z))$ of FIR HPF is given by

$$H(z) = Z.T\{h(n)\} = \sum_{n=0}^{N-1} h(n) z^{-n} = \sum_{n=0}^{6} h(n) z^{-n}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6}$$

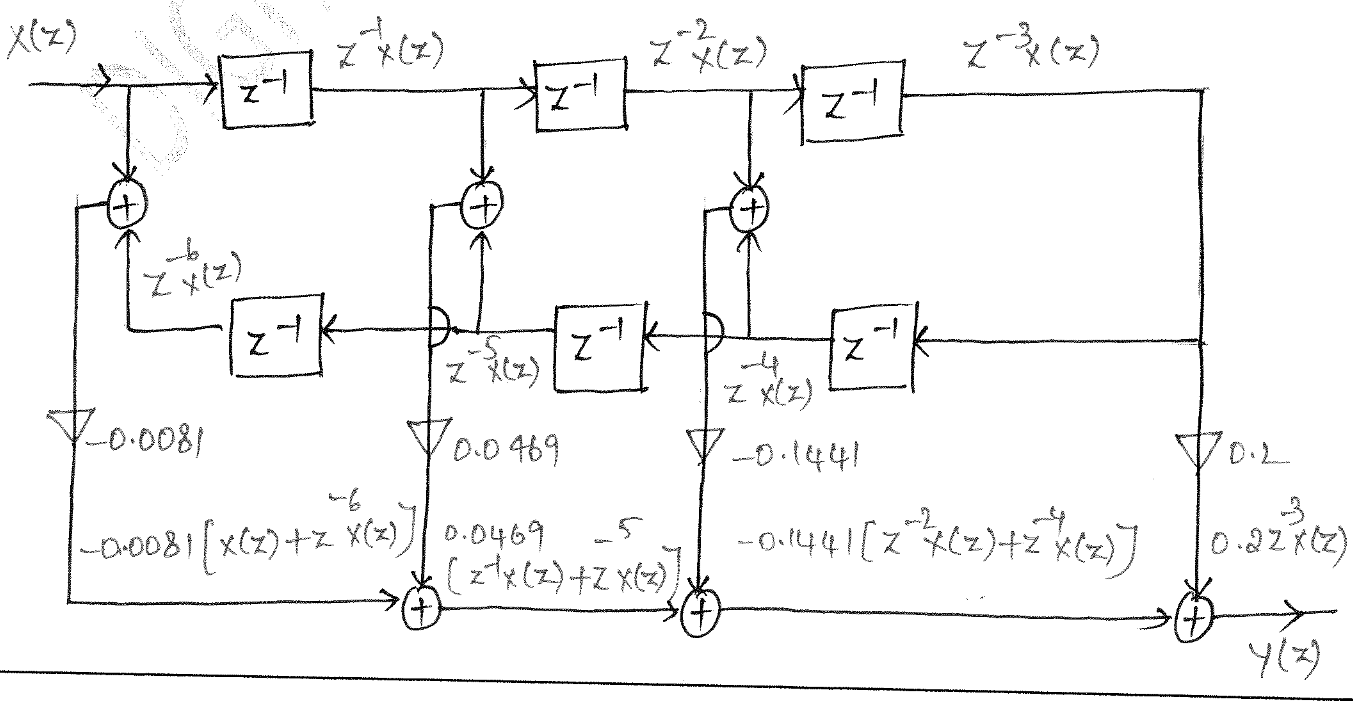
$$= h(0)\left[1+z^{-6}\right] + h(1)\left[z^{-1}+z^{-5}\right] + h(2)\left[z^{-2}+z^{-4}\right] + h(3)z^{-3}$$

$$= -0.0081\left[1+z^{-6}\right] + 0.0469\left[z^{-1}+z^{-5}\right] - 0.1441\left[z^{-2}+z^{-4}\right] + 0.2z^{-3}$$

Structure:

$$H(z) = \frac{Y(z)}{X(z)} = -0.0081\left(1+z^{-6}\right) + 0.0469\left[z^{-1}+z^{-5}\right] - 0.1441\left[z^{-2}+z^{-4}\right] + 0.2z^{-3}$$

$$Y(z) = -0.081\left[X(z)+X(z)z^{-6}\right] + 0.0469\left[X(z)z^{-1}+X(z)z^{-5}\right] - 0.1441\left[X(z)z^{-2}+X(z)z^{-4}\right] + 0.2X(z)z^{-3}.$$



Frequency response: When Impulse response is symmetric and N is odd with centre of symmetry at $\left(\frac{N-1}{2}\right)$. Magnitude response $|H(e^{j\omega})|$ is given by $|A(\omega)|$

$$A(\omega) = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{N-1/2} 2h\left(\frac{N-1}{2} - n\right) \cos \omega n$$

$$= h(3) + \sum_{n=1}^{3} 2h(3-n) \cos \omega n$$

$$= h(3) + 2h(2)\cos\omega + 2h(1)\cos 2\omega + 2h(0)\cos 3\omega$$

$$= 0.2 + 2\times -0.1441 \cos\omega + 2\times 0.0469 \cos 2\omega + 2\times(-0.0081)\cos 3\omega$$

$$= 0.2 - 0.2882\cos\omega + 0.0938\cos 2\omega - 0.0162\cos 3\omega.$$

Using above eqn the Amplitude func. $A(\omega)$ & magnitude function $|H(e^{j\omega})|$ are calculated for various values of ω.

ω	$A(\omega)$	$\lvert H(e^{j\omega})\rvert = \lvert A(\omega)\rvert$
$0\times\pi/16$	-0.0106	0.0106
$\pi/16$	-0.0094	0.0094
$\frac{2\pi}{16}$	-0.0061	0.0061
$3\pi/16$	-0.0005	0.0005
$4\pi/16$	0.0076	0.0076
$5\pi/16$	0.0198	0.0198
$6\pi/16$	0.0383	0.0383
$7\pi/16$	0.0661	0.0661
$8\pi/16$	0.1062	0.1062
$9\pi/16$	0.1605	0.1605
$10\pi/16$	0.2289	0.2289
$11\pi/16$	0.3083	0.3083
$12\pi/16$	0.3923	0.3923
$13\pi/16$	0.4723	0.4723
$\frac{14\pi}{16}$	0.5387	0.5387
$15\pi/16$	0.5827	0.5827
$16\pi/16$	0.5982	0.5982.

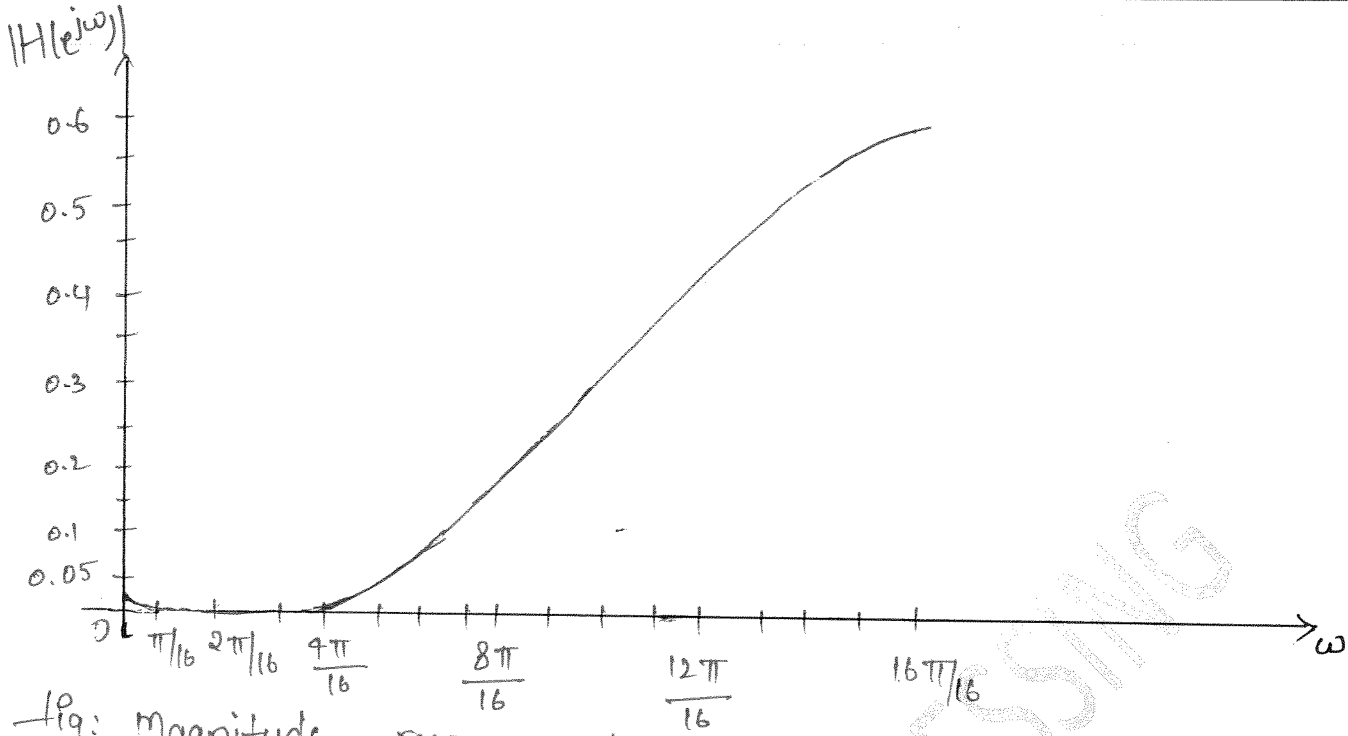


Fig: Magnitude response of FIR High pass filter.

1. Design an ideal high pass filter with a frequency response $H_d(e^{j\omega}) = 1$ for $\frac{\pi}{4} \le |\omega| \le \pi$
$= 0$ for $|\omega| \le \frac{\pi}{4}$
using Hamming window with 11 samples and plot the response.

Given

$$H_d(e^{j\omega}) = 1, \quad \frac{\pi}{4} \le |\omega| \le \pi$$
$$0, \quad |\omega| \le \frac{\pi}{4}$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-\pi/4} 1 \cdot e^{j\omega n}\, d\omega + \int_{\pi/4}^{\pi} e^{j\omega n}\, d\omega\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{e^{j\omega n}}{jn}\right)_{-\pi}^{-\frac{\pi}{4}} + \left(\frac{e^{j\omega n}}{jn}\right)_{\pi/4}^{\pi}\right]$$

$$= \frac{1}{2\pi(jn)}\left[e^{-j\frac{\pi n}{4}} - e^{-j\pi n} + e^{j\pi n} - e^{j\pi/4 n}\right]$$

$$= \frac{1}{\pi n}\left[\cancel{\sin \pi n} \cdot \frac{e^{j\pi n} - e^{-j\pi n}}{2j} - \left(\frac{e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}}{2j}\right)\right]$$

$$h_d(n) = \frac{1}{\pi n}\left[\sin \pi n - \sin \frac{\pi}{4} n\right] \quad \forall\, n \text{ expt at } n = 0$$

$$w_H(n) = \begin{cases} 0.54 + 0.46 \cos \frac{2\pi n}{N-1}, & |n| \leq \left(\frac{N-1}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

The window sequence for $N=11$ is given by.

$$w_H(n) = 0.54 + 0.46 \cos \frac{\pi n}{5} \quad \text{for} \quad -5 \leq n \leq 5$$

$$0 \qquad \text{otherwise}$$

$$w_H(0) = 0.54 + 0.46 = 1$$

$$w_H(-1) = w_H(1) = 0.54 + 0.46 \cos \frac{\pi}{5} = 0.912$$

$$w_H(-2) = w_H(2) = 0.54 + 0.46 \cos \frac{2\pi}{5} = 0.682$$

$$w_H(-3) = w_H(3) = 0.54 + 0.46 \cos \frac{3\pi}{5} = 0.398$$

$$w_H(-4) = w_H(4) = 0.54 + 0.46 \cos \frac{4\pi}{5} = 0.1678$$

$$w_H(-5) = w_H(5) = 0.54 + 0.46 \cos \pi = 0.08$$

$$h_d(n) = \frac{\sin \pi n - \sin \frac{\pi}{4} n}{n\pi}$$

$$h_d(0) = \lim_{n \to 0} \left[\frac{\sin \pi n - \sin \frac{\pi}{4} n}{n\pi}\right] = \lim_{n \to 0} \frac{\sin \pi n}{\pi n} - \lim_{n \to 0} \frac{\sin \frac{\pi}{4} n}{n\pi}$$

$$\left[\Rightarrow \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \; ; \; \lim_{\theta \to 0} \frac{\sin n\theta}{\theta} = n\right] \quad (\because \text{L H' Rule})$$

$$h_d(0) = 1 - \frac{1}{4} = 0.75$$

$$h_d(-1) = h_d(1) = \frac{\sin \pi - \sin \frac{\pi}{4}}{\pi} = -0.225$$

$$h_d(-2) = h_d(2) = \frac{\sin 2\pi - \sin \frac{\pi}{2}}{2\pi} = -0.159$$

$$h_d(-3) = h_d(3) = \frac{\sin 3\pi - \sin \frac{3\pi}{4}}{3\pi} = -0.075$$

$$h_d(-4) = h_d(4) = \frac{\sin 4\pi - \sin \pi}{4\pi} = 0$$

$$h_d(-5) = h_d(5) = \frac{\sin 5\pi - \sin \frac{5\pi}{4}}{5\pi} = 0.045$$

The filter coefficients using Hamming window sequence are.

$$h(n) = h_d(n) w_H(n) \quad \text{for } -5 \le n \le 5$$
$$0 \quad \text{for otherwise}$$

$h(0) = h_d(0) w_{Hn}(0) = (1)(0.75) = 0.75$

$h(-1) = h(1) = h_d(1) w_{Hn}(1) = (-0.225)(0.912) = -0.2052$

$h(-2) = h(2) = h_d(2) w_{Hn}(2) = (-0.159)(0.682) = -0.1084$

$h(-3) = h(3) = h_d(3) w_{Hn}(3) = (-0.075)(0.398) = -0.03$

$h(-4) = h(4) = h_d(4) w_{Hn}(4) = (0)(0.1678) = 0$

$h(-5) = h(5) = h_d(5) w_{Hn}(5) = (-0.045)(0.08) = 0.0036$

The transfer function of the filter is given by.

$$H(z) = h(0) + \sum_{n=1}^{5} \left[h(n)\left(z^{-n} + z^{n}\right)\right]$$

$$= 0.75 - 0.2052(z^{-1} + z) - 0.1084(z^{-2} + z^{2}) - 0.03(z^{-3} + z^{3})$$
$$+ 0 + 0.0036(z^{-5} + z^{+5})$$

$H(z)$ is non causal FIR filter TF and it can not be realised

convert $H(z)$ into causal FIR filter TF $H'(z)$

$$H'(z) = z^{-\left(\frac{N-1}{2}\right)} H(z)$$

$$H'(z) = z^{-5} H(z) \qquad (\because N = 11)$$

$$H'(z) = 0.0036 - 0.03z^{-2} - 0.1084z^{-3} - 0.2052z^{-4} + 0.75z^{-5}$$
$$- 0.2052z^{-6} - 0.1084z^{-7} - 0.03z^{-8} + 0.0036z^{-10} \rightarrow (a)$$

The filter coefficients of causal filter are

$h(0) = h(10) = 0.0036$; $h(1) = h(9) = 0$; $h(2) = h(8) = -0.03$

$h(3) = h(7) = -0.1084$; $h(4) = h(6) = -0.2052$

$h(5) = 0.75$

$$\bar{H}(e^{j\omega}) = \sum_{n=0}^{\frac{N-1}{2}} \alpha(n) \cos\omega n$$

$$\alpha(0) = h\left(\frac{N-1}{2}\right) = h(5) = 0.75$$

$$\alpha(n) = 2h\left[\frac{N-1}{2} - n\right]$$

$$\alpha(1) = 2h(5-1) = 2(h(4)) = -0.4104$$

$$\alpha(2) = 2h(5-2) = 2h(3) = -0.2168$$

$$\alpha(3) = 2h(5-3) = 2h(2) = -0.06$$

$$\alpha(4) = 2h(5-4) = 2h(1) = 0$$

$$\alpha(5) = 2h(5-5) = 2h(0) = 0.0072$$

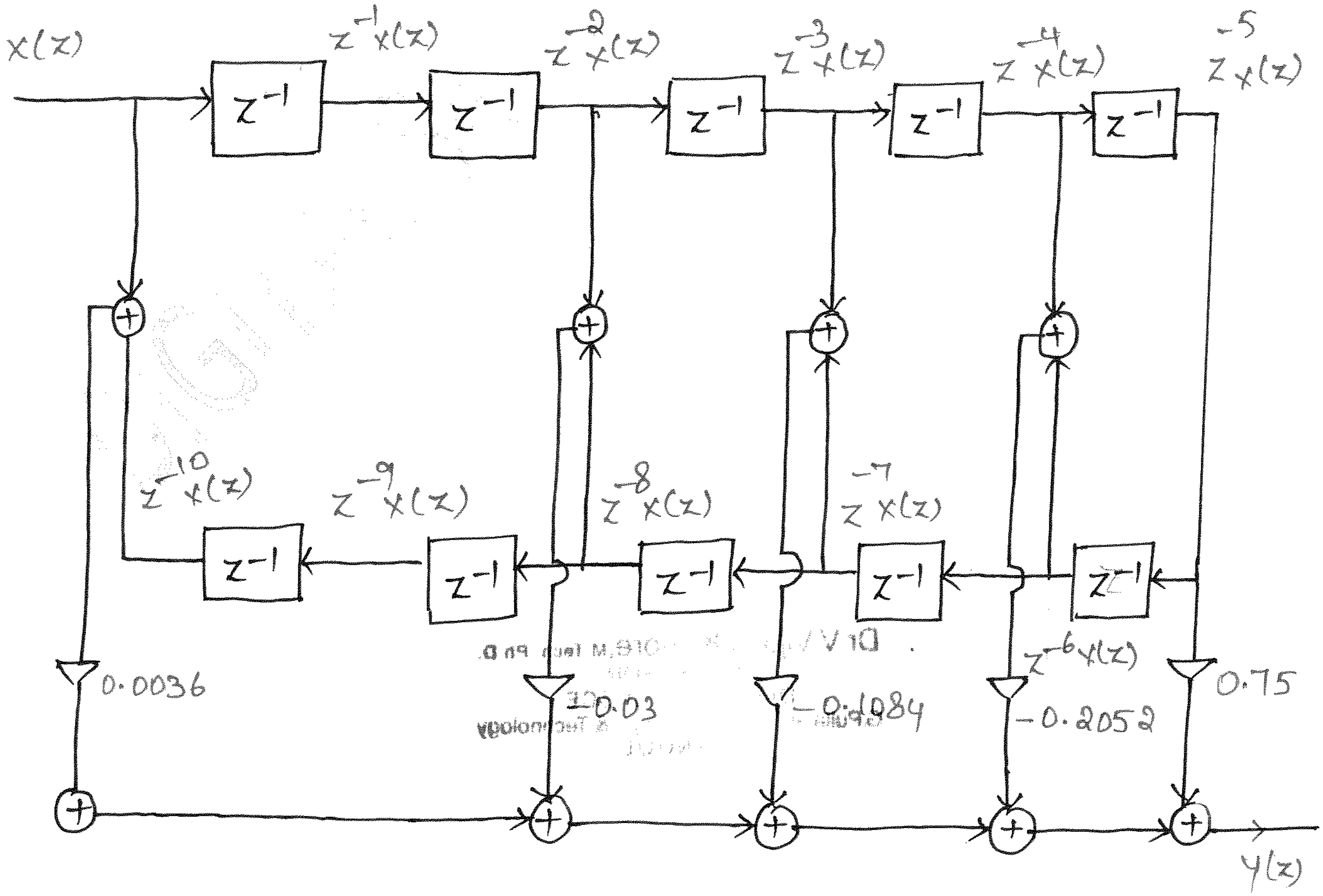
$$\bar{H}(e^{j\omega}) = 0.75 - 0.4104\cos\omega - 0.2168\cos 2\omega - 0.06\cos 3\omega$$
$$+ 0.0072\cos 5\omega$$

Structure:

from eqn (a) $H'(z) = 0.0036 - 0.03z^{-2} - 0.1084z^{-3} - 0.2052z^{-4} + 0.75z^{-5} - 0.2052z^{-6} - 0.1084z^{-7} - 0.03z^{-8} + 0.0036z^{-10}$.

$$H'(z) = \frac{Y(z)}{X(z)} = 0.0036(1+z^{-10}) - 0.03(z^{-2}+z^{-8}) - 0.1084(z^{-3}+z^{-7}) - 0.2052(z^{-4}+z^{-6}) + 0.75z^{-5}.$$

$$Y(z) = 0.0036X(z) + 0.0036X(z)z^{-10} - 0.03X(z)z^{-2} - 0.03X(z)z^{-8} - 0.1084X(z)z^{-3} - 0.1084X(z)z^{-7} - 0.2052X(z)z^{-4} - 0.2052X(z)z^{-6} + 0.75X(z)z^{-5}$$



ω (in degrees)	0	15	30	45	60	75
$H(e^{j\omega})$	0.07	0.125	0.28	0.497	0.7168	0.88
$\lvert H(e^{j\omega})\rvert_{dB}$	-23.1	-18	-11	-6.07	-2.89	-1.1

90	105	120	135	150	165	180
0.9668	0.9945	1	1.0026	1.003	1	1.0108
-0.29	-0.0478	0	0.0229	0.028	0	0.093

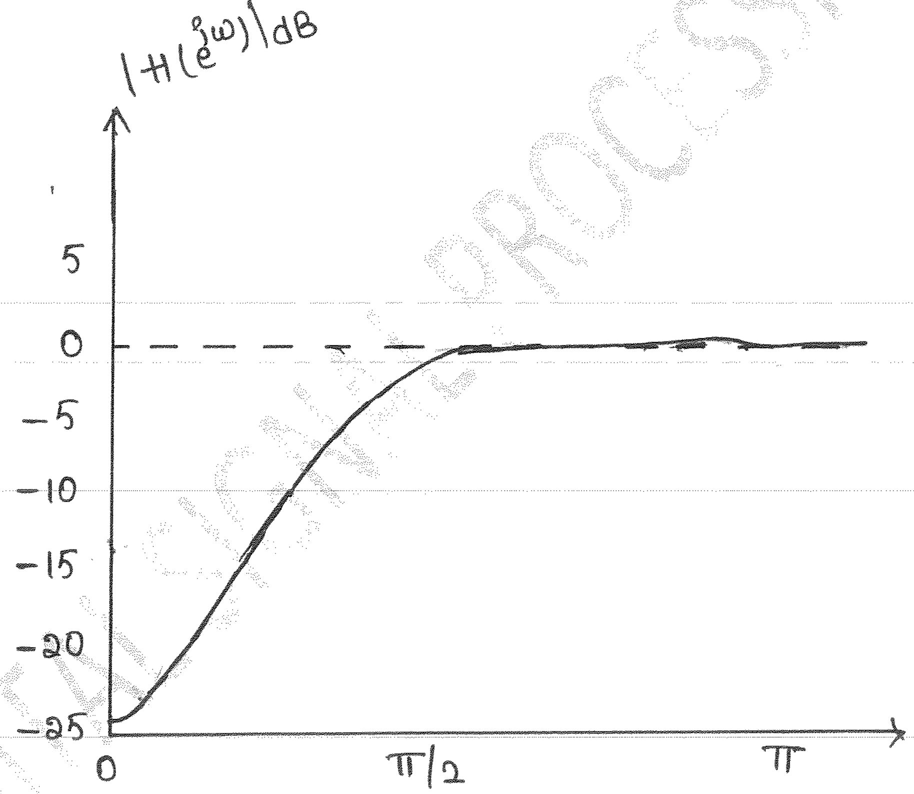


fig: Log Magnitude response using Hanning Window

Prob: Determine the coefficients of a linear-phase FIR filter of $N=15$ which has a symmetric ~~circuit~~ unit sample response and a frequency response that satisfies the conditions

$$H\left(\frac{2\pi k}{15}\right) = \begin{cases} 1 \;; & \text{for } k=0,1,2,3 \\ 0.4 \;; & \text{for } k=4 \\ 0 \;; & \text{for } k=5,6,7. \end{cases}$$

Sol: For linear phase FIR filter the phase function $\theta(\omega) = -\alpha\omega$ where $\alpha = \frac{N-1}{2}$

Here $N=15$, $\alpha = \frac{N-1}{2} = \frac{15-1}{2} = 7$

Also, here $\omega = \omega_k = \frac{2\pi k}{N} = \frac{2\pi k}{15}$; Hence we can go for type-1 design. In this problem the samples of the magnitude response of the ideal filter are directly given for the values of k.

$$\because H(k) = H_d(\omega)\Big|_{\omega=\omega_k} = 1 \cdot e^{-j\alpha\omega_k} = e^{-j7\times\frac{2\pi k}{15}} \;;\; k=0,1,2,3$$

$$= 0.4 \cdot e^{-j\alpha\omega_k} = 0.4 \cdot e^{-j7\cdot\frac{2\pi k}{15}} \;;\; k=4$$

$$= 0 \;;\; k=5,6,7$$

The samples of the Impulse response $h(n)$ are given by.

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} \mathrm{Re}\left\{H(k)\cdot e^{\frac{j2\pi nk}{N}}\right\}\right]$$

$$= \frac{1}{15}\left[H(0) + 2\sum_{k=1}^{7} \mathrm{Re}\left[H(k)\cdot e^{\frac{+j2\pi nk}{15}}\right]\right]$$

$$= \frac{1}{15}\left[H(0) + 2\sum_{k=1}^{3} \mathrm{Re}\left[H(k)\cdot e^{\frac{j2\pi nk}{15}}\right] + 2\,\mathrm{Re}\left[H(4)\cdot e^{\frac{j2\pi n\times 4}{15}}\right]\right]$$

$$h(n) = \frac{1}{15}\left[1 + 2\sum_{k=1}^{3} \mathrm{Re}\left\{e^{\frac{-j7\times 2\pi k}{15}} \times e^{\frac{j2\pi nk}{15}}\right\} + 2\mathrm{Re}\left\{0.4 e^{\frac{-j7\times 2\pi\times 4}{15}} \cdot e^{\frac{j8\pi n}{15}}\right\}\right]$$

$$h(n) = \frac{1}{15}\left[1 + 2\sum_{K=1}^{3} Re\left[e^{\frac{j2\pi k}{15}(n-7)}\right] + 2Re\left[0.4 \cdot e^{\frac{j8\pi}{15}(n-7)}\right]\right]$$

$$h(n) = \frac{1}{15}\left[1 + 2\sum_{K=1}^{3} \cos\frac{2\pi k}{15}(n-7) + 0.8\cos\frac{8\pi}{15}(n-7)\right]$$

$$= \frac{1}{15}\left[1 + 2\cos\frac{2\pi(n-7)}{15} + 2\cos\frac{4\pi(n-7)}{15} + 2\cos\frac{6\pi(n-7)}{15} + 0.8\cos\left(\frac{8\pi(n-7)}{15}\right)\right]$$

Here $N=15$, $N-1=14$, $\frac{N-1}{2}=7$

Calculate $h(n)$ for $n = 0$ to 14

$h(n)$ satisfies symmetry condition $\underline{\underline{h(n) = h(N-1-n)}}$ with centre of symmetry at $\left(\frac{N-1}{2}\right)$. Calculate $h(n)$ for $n=0$ to 7

When $n=0 \Rightarrow h(0) = \frac{1}{15}\left[1 + 2\cos\frac{2\pi(0-7)}{15} + 2\cos\frac{4\pi(0-7)}{15} + 2\cos\frac{6\pi(0-7)}{15} + 0.8\cos\frac{8\pi(0-7)}{15}\right]$

$h(0) = -0.0141$

$n=1 \Rightarrow h(1) = \frac{1}{15}\left[1 + 2\cos\frac{2\pi(1-7)}{15} + 2\cos\frac{4\pi(1-7)}{15} + 2\cos\frac{6\pi(1-7)}{15} + 0.8\cos\frac{8\pi(1-7)}{15}\right]$

$h(1) = -0.0019$

$n=2$.

$h(2) = \frac{1}{15}\left[1 + 2\cos\frac{2\pi(2-7)}{15} + 2\cos\frac{4\pi(2-7)}{15} + 2\cos\frac{6\pi(2-7)}{15} + 0.8\cos\frac{8\pi(2-7)}{15}\right]$

$h(2) = 0.04$

$n=3$

$h(3) = \frac{1}{15}\left[1 + 2\cos\frac{2\pi(3-7)}{15} + 2\cos\frac{4\pi(3-7)}{15} + 2\cos\frac{6\pi(3-7)}{15} + 0.8\cos\frac{8\pi(3-7)}{15}\right]$

$h(3) = 0.0122.$

$n=4$

$$h(4) = \frac{1}{15}\left[1+2\cos\frac{2\pi(4-7)}{15} + 2\cos\frac{4\pi(4-7)}{15} + 2\cos\frac{6\pi(4-7)}{15} + 0.8\cos\frac{8\pi(4-7)}{15}\right]$$

$h(4) = -0.0914$

$n=5$.

$$h(5) = \frac{1}{15}\left[1+2\cos\frac{2\pi(5-7)}{15} + 2\cos\frac{4\pi(5-7)}{15} + 2\cos\frac{6\pi(5-7)}{15} + 0.8\cos\frac{8\pi(5-7)}{15}\right]$$

$h(5) = -0.0181$

$n=6$

$$h(6) = \frac{1}{15}\left[1+2\cos\frac{2\pi(6-7)}{15} + 2\cos\frac{4\pi(6-7)}{15} + 2\cos\frac{6\pi(6-7)}{15} + 0.8\cos\frac{8\pi(6-7)}{15}\right]$$

$h(6) = 0.3130$

$n=7$

$$h(7) = \frac{1}{15}\left[1+2\cos\frac{2\pi(7-7)}{15} + 2\cos\frac{4\pi(7-7)}{15} + 2\cos\frac{6\pi(7-7)}{15} + 0.8\cos\frac{8\pi(7-7)}{15}\right]$$

$h(7) = 0.52$

$n=8 \Rightarrow h(8) = h(6) = 0.3130$

$n=9 \Rightarrow h(9) = h(5) = -0.0181$

$n=10 \Rightarrow h(10) = h(4) = -0.0914$

$n=11 \Rightarrow h(11) = h(3) = 0.0122$

$n=12 \Rightarrow h(12) = h(2) = 0.04$ $\quad \because h(n) = h(N-1-n)$

$n=13 \Rightarrow h(13) = h(1) = -0.0019$

$n=14 \Rightarrow h(14) = h(0) = -0.0141$

The transfer function $H(z)$ of filter is given by Z-Transform of $h(n)$

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n} = \sum_{n=0}^{14} h(n)\cdot z^{-n}$$

$$\Rightarrow h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7} + h(8)z^{-8} + h(9)z^{-9} + h(10)z^{-10} + h(11)z^{-11} + h(12)z^{-12} + h(13)z^{-13} + h(14)z^{-14}$$

$h(n) = h(N-1-n)$

$$H(z) = h(0)[1+z^{-14}] + h(1)[z^{-1}+z^{-13}] + h(2)[z^{-2}+z^{-12}] + h(3)[z^{-3}+z^{-11}] + h(4)[z^{-4}+z^{-10}] + h(5)[z^{-5}+z^{-9}] + h(6)[z^{-6}+z^{-8}] + h(7)z^{-7}.$$

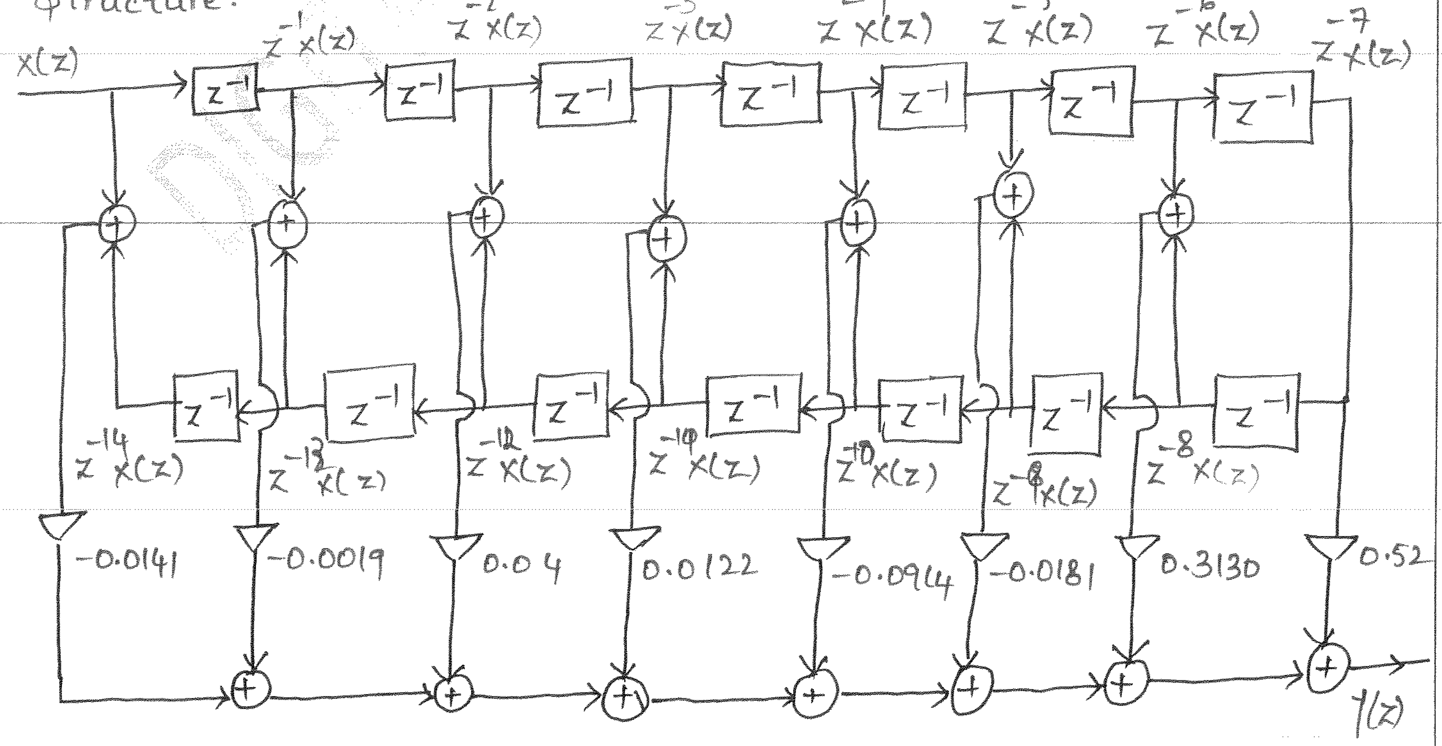
$$\Rightarrow -0.0141[1+z^{-14}] - 0.0019[z^{-1}+z^{-13}] + 0.04[z^{-2}+z^{-12}] + 0.0122[z^{-3}+z^{-11}] + (-0.0914)(z^{-4}+z^{-10}) - 0.0181[z^{-5}+z^{-9}) + 0.3130[z^{-6}+z^{-8}] + 0.52z^{-7}$$

Structure:

$$H(z) = \frac{Y(z)}{X(z)} = -0.0141[1+z^{-14}] - 0.0019[z^{-1}+z^{-13}] + 0.04[z^{-2}+z^{-12}] + 0.0122[z^{-3}+z^{-11}] - 0.0914[z^{-4}+z^{-10}] - 0.0181[z^{-5}+z^{-9}] + 0.3130[z^{-6}+z^{-8}] + 0.52z^{-7}$$

$$Y(z) = -0.0141[X(z)+X(z)z^{-14}] - 0.0019[X(z)z^{-1}+X(z)z^{-13}] + 0.04[X(z)z^{-2}+X(z)z^{-12}] + 0.0122[X(z)z^{-3}+X(z)z^{-11}] - 0.0914[X(z)z^{-4} + X(z)z^{-10}] - 0.0181[X(z)z^{-5}+X(z)z^{-9}] + 0.3130[X(z)z^{-6}+X(z)z^{-8}] + 0.52X(z)z^{-7}$$

Structure:



Frequency Response: When Impulse response is symmetric and N is odd with centre of symmetry at $\left(\frac{N-1}{2}\right)$ the magnitude response is $|H(e^{j\omega})|$ given by $|A(\omega)|$

$$A(\omega) = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right)\cos\omega n$$

$$= h(7) + \sum_{n=1}^{7} 2h(7-n)\cos\omega n$$

$$= 0.52 + 2h(6)\cos\omega + 2h(5)\cos 2\omega + 2h(4)\cos 3\omega + 2h(3)\cos 4\omega$$
$$+ 2h(2)\cos 5\omega + 2h(1)\cos 6\omega + 2h(0)\cos 7\omega$$

$$A(\omega) = 0.52 + 0.626\cos\omega + (-0.0362)\cos 2\omega - 0.1828\cos 3\omega +$$
$$0.0244\cos 4\omega + 0.08\cos 5\omega - 0.0038\cos 6\omega - 0.0282\cos 7\omega.$$

$A(\omega)$ and $|H(e^{j\omega})|$ for various values of ω.

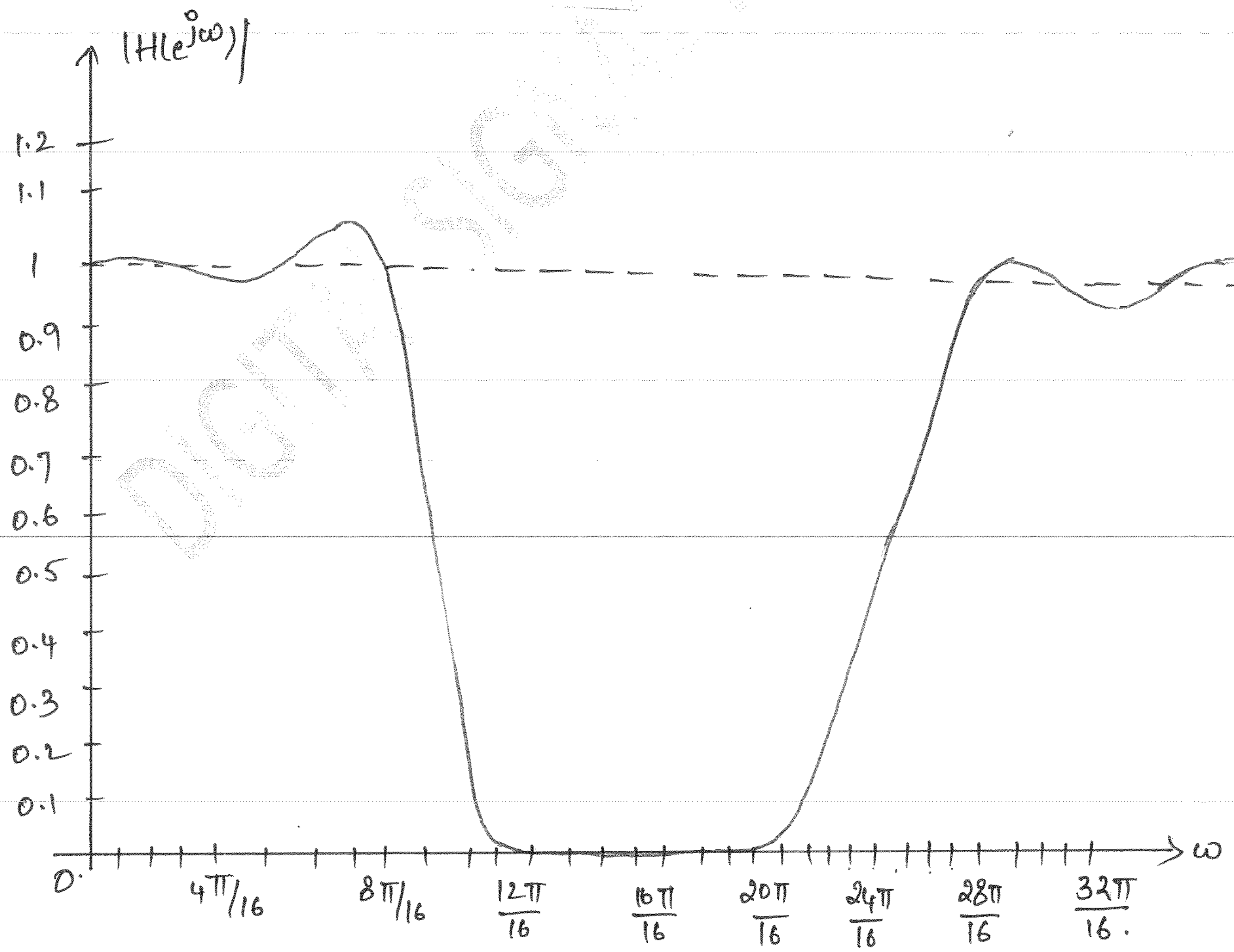
ω	$A(\omega)$	$\lvert H(e^{j\omega})\rvert = A(\omega)$
0	0.9994	0.9994
$\pi/16$	1.0032	1.0032
$2\pi/16$	1.0009	1.0009
$3\pi/16$	0.9856	0.9856
$4\pi/16$	0.9909	0.9909
$5\pi/16$	1.0323	1.0323
$6\pi/16$	1.0360	1.0360
$7\pi/16$	0.8900	0.8900
$8\pi/16$	0.5844	0.5844
$9\pi/16$	0.2542	0.2542
$10\pi/16$	0.0497	0.0497
$11\pi/16$	-0.0061	0.0061
$12\pi/16$	0.0020	0.0020

$\omega =$	$A(\omega)$	$\lvert H(e^{j\omega})\rvert = A(\omega)$
$13\pi/16$	-0.0009	0.0009
$14\pi/16$	-0.0067	0.0067
$15\pi/16$	-0.0014	0.0014
$16\pi/16$	0.0094	0.0094
$17\pi/16$	0.0014	0.0014
$18\pi/16$	-0.0067	0.0067
$19\pi/16$	-0.0009	0.0009
$20\pi/16$	0.0020	0.0020
$21\pi/16$	-0.0061	0.0061
$22\pi/16$	0.0497	0.0497
$23\pi/16$	0.2542	0.2542
$24\pi/16$	0.5844	0.5844

ω	$A(\omega)$	$\lvert H(e^{j\omega})\rvert = A(\omega)$
$25\pi/16$	0.8900	0.8900
$26\pi/16$	1.0360	1.0360
$27\pi/16$	1.0323	1.0323
$28\pi/16$	0.9909	0.9909
$29\pi/16$	0.9857	0.9857
$30\pi/16$	1.0009	1.0009
$31\pi/16$	1.0032	1.0032
$32\pi/16$	0.9994	0.9994

∵ Alternate method for freq response is

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$$



Magnitude response of FIR

prob: Design a Linear phase FIR Low pass filter with a cutoff frequency of 0.5π rad/sample by taking 11 samples of Ideal frequency response.

Sol: The magnitude response of Ideal Lowpass filter is shown in fig 1. The desired frequency response $H_d(e^{j\omega})$ of Linear phase FIR Lowpass filter with cut off frequency of 0.5π rad/sample is given

$$H_d(e^{j\omega}) = \begin{cases} e^{-j2\omega} & ; \ 0 \le \omega \le 0.5\pi \text{ and } \\ & \ \ \ 1.5\pi \le \omega \le 2\pi \\ 0 & ; \ 0.5\pi \le \omega < 1.5\pi \end{cases}$$

The DFT sequence $H(k)$ is obtained by sampling $H_d(e^{j\omega})$ at 11 equidistant frequency points in a period of 2π. The 11 freq. for type-1 design are given by.

$$\omega_k = \frac{2\pi k}{N} = \frac{2\pi k}{11} ; \ k = 0 \text{ to } 11$$

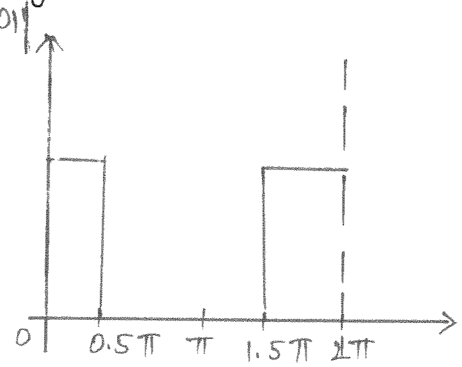


fig 1: Ideal Magnitude response of FIR LPF

$k=0; \ \omega_k = \frac{2\pi \times 0}{11} = 0$

$k=1; \ \omega_k = \frac{2\pi \times 1}{11} = 0.18\pi$

$k=2; \ \omega_k = \frac{2\pi \times 2}{11} = 0.36\pi$

$k=3; \ \omega_k = \frac{2\pi \times 3}{11} = 0.55\pi$

$k=4; \ \omega_k = \frac{2\pi \times 4}{11} = 0.73\pi$

$k=5; \ \omega_k = \frac{2\pi \times 5}{11} = 0.91\pi$

$k=6; \ \omega_k = \frac{2\pi \times 6}{11} = 1.09\pi$

$k=7; \ \omega_k = \frac{2\pi \times 7}{11} = 1.27\pi$

$k=8;\quad \omega_k = \frac{2\pi \times 8}{11} = 1.45\pi$

$k=9;\quad \omega_k = \frac{2\pi \times 9}{11} = 1.64\pi$

$k=10;\quad \omega_k = \frac{2\pi \times 10}{11} = 1.82\pi$ from these calculations the Observations are made as follows

For $k=0$ to 2 samples lie in range $0 \le \omega \le 0.5\pi$

$k=3$ to 8 samples lie in range $0.5\pi < \omega < 1.5\pi$

$k=9$ to 10 samples lie in range $1.5\pi \le \omega < 2\pi$

The Sampling points on ideal frequency response are shown below. The Magnitude of samples of $H(k)$ (Magnitude spectrum)

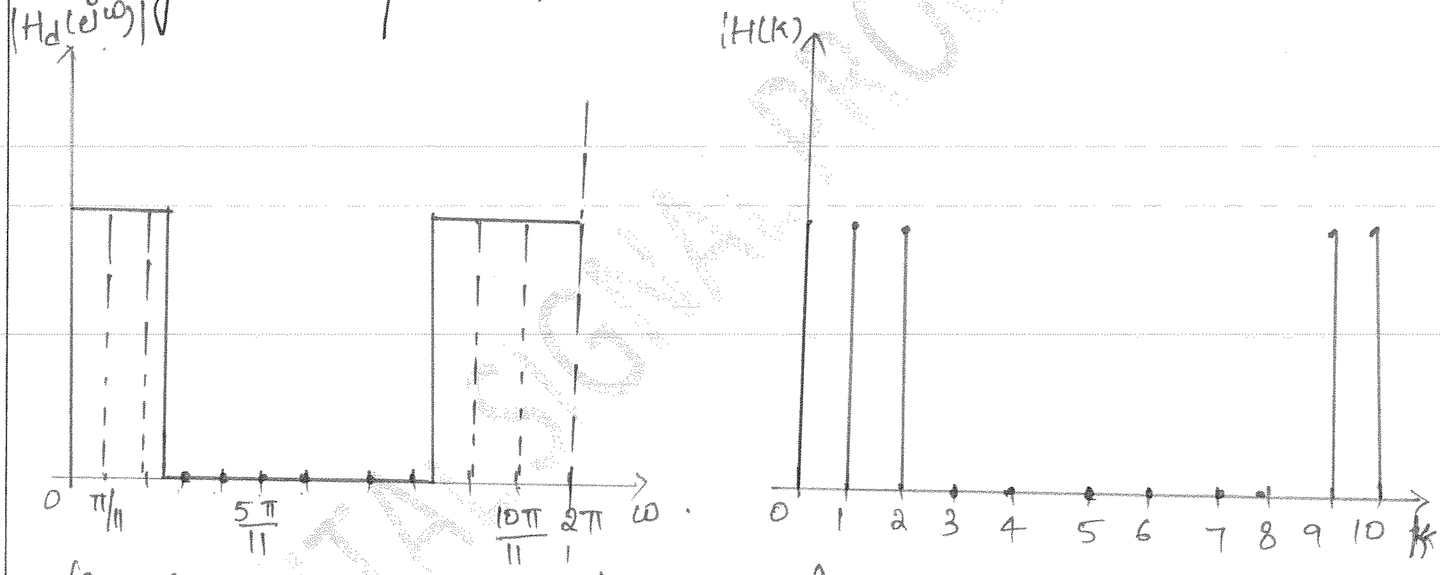


fig: Sampling points of $H_d(e^{j\omega})$

fig: Magnitude spectrum of $H(k)$

Based on above discussions, the eqn of DFT coe. $H(k)$ can be written as

$$H(k) = H_d(e^{j\omega})\Big|_{\omega=\omega_k} = e^{-j\,2\omega_k} = \begin{cases} e^{-j\frac{5\times 2\pi k}{11}}; & k=0,1,2. \\ 0; & k=3 \text{ to } 8 \\ e^{-j\frac{5\times 2\pi k}{11}}; & k=9,10 \end{cases}$$

The samples of Impulse response, $h(n)$ are given by

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{K=1}^{N-1/2} Re\left[H(k)\cdot e^{\frac{j2\pi nk}{N}}\right]\right]$$

$$= \frac{1}{11}\left[1 + 2\sum_{K=1}^{5} Re\left[H(k)\cdot e^{\frac{j2\pi nk}{N}}\right]\right]$$

$$= \frac{1}{11}\left[1 + 2\sum_{K=1}^{2} Re\left[e^{-j5\cdot\frac{2\pi k}{11}}\cdot e^{\frac{j2\pi nk}{11}}\right]\right] \qquad \because H(0)=1$$

$$= \frac{1}{11}\left[1 + 2Re\left[e^{j\frac{2\pi}{11}(n-5)}\right] + 2Re\left[e^{j\frac{4\pi}{11}(n-5)}\right]\right]$$

$$= \frac{1}{11}\left[1 + 2\cos\frac{2\pi(n-5)}{11} + 2\cos\frac{4\pi(n-5)}{11}\right]$$

for $N=1$, $N-1=10$, $\frac{N-1}{2}=5$

Calculate $h(n)$ for $n=0$ to 10. $h(n)$ satisfies the symmetry condition $h(N-1-n)=h(n)$ with centre of symmetry $\left(\frac{N-1}{2}\right)$

Calculate $h(n)$ for $n=0$ to 5

$n=0 \Rightarrow h(0) = \frac{1}{11}\left[1+2\cos\frac{2\pi(0-5)}{11} + 2\cos\frac{4\pi(0-5)}{11}\right] = 0.0694$

$n=1 \Rightarrow h(1) = \frac{1}{11}\left[1+2\cos\frac{2\pi(1-5)}{11} + 2\cos\frac{4\pi(1-5)}{11}\right] = -0.054$

$n=2 \Rightarrow h(2) = \frac{1}{11}\left[1+2\cos\frac{2\pi(2-5)}{11} + 2\cos\frac{4\pi(2-5)}{11}\right] = -0.1094$

$n=3 \Rightarrow h(3) = \frac{1}{11}\left[1+2\cos\frac{2\pi(3-5)}{11} + 2\cos\frac{4\pi(3-5)}{11}\right] = 0.0474$

$n=4 \Rightarrow h(4) = \frac{1}{11}\left[1+2\cos\frac{2\pi(4-5)}{11} + 2\cos\frac{4\pi(4-5)}{11}\right] = 0.3194$

$n=5 \Rightarrow h(5) = \frac{1}{11}\left[1+2\cos\frac{2\pi(5-5)}{11} + 2\cos\frac{4\pi(5-5)}{11}\right] = 0.4545$

$n=6 \Rightarrow h(6) = h(4) = 0.3194$

$n=7 \Rightarrow h(7) = h(3) = 0.0474$

$n=8 \Rightarrow h(8) = h(2) = -0.1094$

$n=9 \Rightarrow h(9) = h(1) = -0.054$ $\quad \because h(n) = h(N-1-n)$

$n=10 \Rightarrow h(10) = h(0) = 0.0694$

The transfer function $H(z)$ of filter is given by Z-T of $h(n)$

$$H(z) = \sum_{n=0}^{N-1} h(n) \cdot z^{-n} = \sum_{n=0}^{10} h(n) \cdot z^{-n}$$

$$= h(0) + h(1) z^{-1} + h(2) z^{-2} + h(3) z^{-3} + h(4) z^{-4} + h(5) z^{-5} + h(6) z^{-6} + h(7) z^{-7} + h(8) z^{-8} + h(9) z^{-9} + h(10) \cdot z^{-10}$$

$\because h(n) = h(N-1-n)$

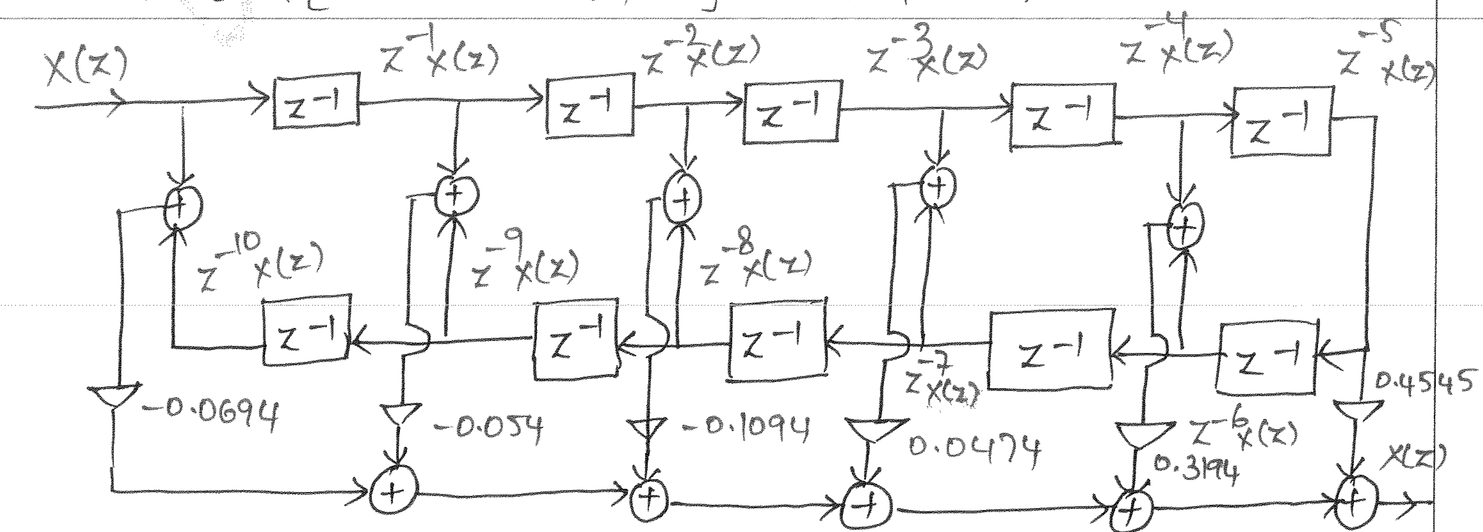
$$= h(0)\left[1+z^{-10}\right] + h(1)\left[z^{-1}+z^{-9}\right] + h(2)\left[z^{-2}+z^{-8}\right] + h(3)\left[z^{-3}+z^{-7}\right] + h(4)\left[z^{-4}+z^{-6}\right] + h(5) z^{-5}$$

$$= 0.0694\left[1+z^{-10}\right] - 0.054\left[z^{-1}+z^{-9}\right] - 0.1094\left[z^{-2}+z^{-8}\right] + 0.0474\left[z^{-3}+z^{-7}\right] + 0.3194\left[z^{-4}+z^{-6}\right] + 0.4545 z^{-5}$$

Structure:

$$H(z) = \frac{Y(z)}{X(z)} = 0.0694\left[1+z^{-10}\right] - 0.054\left[z^{-1}+z^{-9}\right] - 0.1094\left[z^{-2}+z^{-8}\right] + 0.0474\left[z^{-3}+z^{-7}\right] + 0.3194\left[z^{-4}+z^{-6}\right] + 0.4545 z^{-5}$$

$$Y(z) = 0.0694\left[X(z) + X(z) z^{-10}\right] - 0.054\left[X(z) z^{-1} + X(z) z^{-9}\right] - 0.1094\left[X(z) z^{-2} + X(z) z^{-8}\right] + 0.0474\left[X(z) z^{-3} + X(z) z^{-7}\right] + 0.3194\left[X(z) z^{-4} + X(z) z^{-6}\right] + 0.4545 X(z) z^{-5}$$



Frequency response:

When Impulse response is symmetric and N is odd with centre of symmetry at $\left(\frac{N-1}{2}\right)$ the magnitude response $|H(e^{j\omega})|$ is given by $|A(\omega)|$

$$A(\omega) = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{N-1/2} 2h\left(\frac{N-1}{2}-n\right)\cos \omega n \, .$$

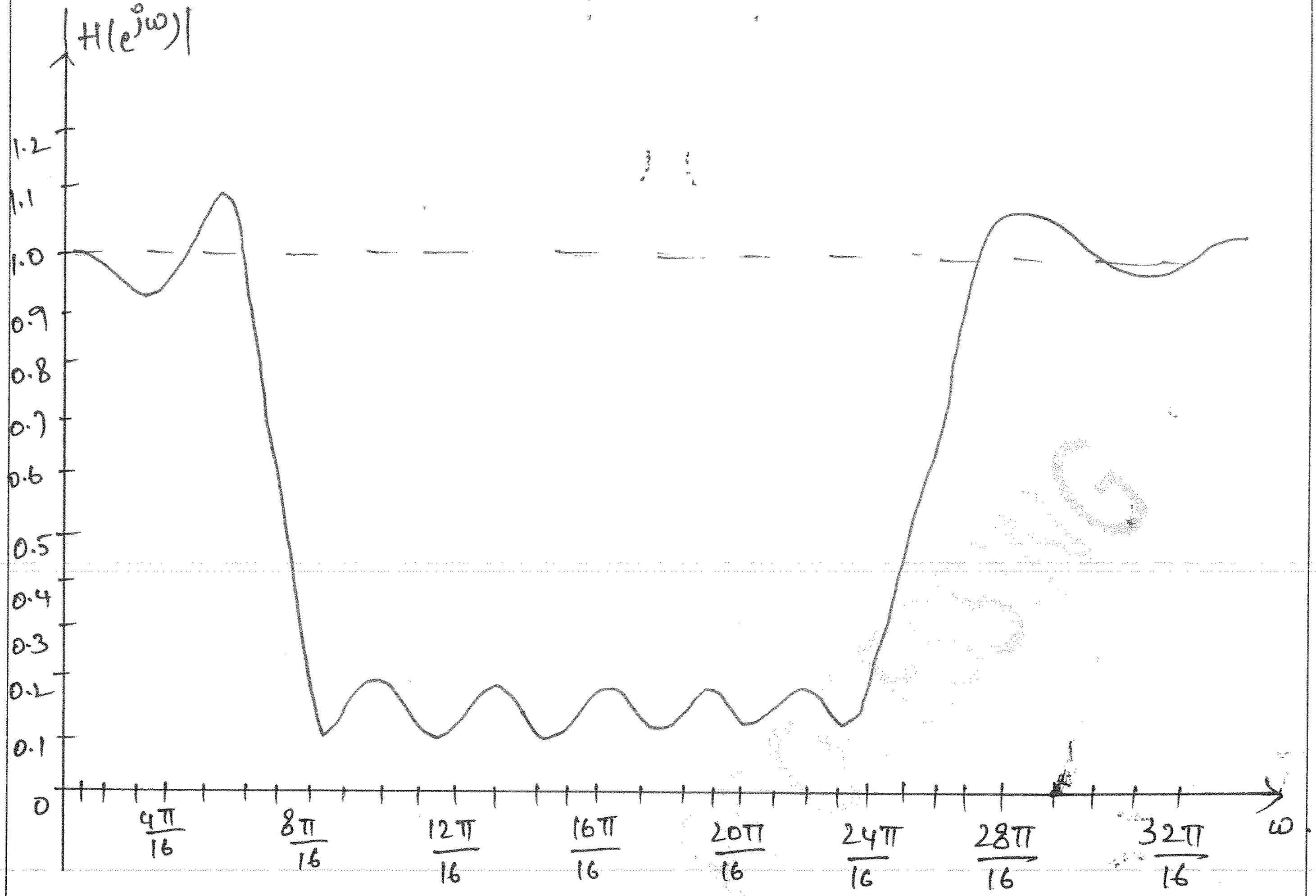
$$A(\omega) = h(5) + \sum_{n=1}^{5} 2h(5-n) \cos \omega n$$

$$= h(5) + 2h(4)\cos\omega + 2h(3)\cos 2\omega + 2h(2)\cos 3\omega + 2h(1)\cos 4\omega + 2h(0)\cos 5\omega \, .$$

$$= 0.4545 + 0.6388\cos\omega + 0.0948\cos 2\omega + 0.2188\cos 3\omega \pm 0.108\cos 4\omega + 0.1388\cos 5\omega \, .$$

$A(\omega)$ and $|H(e^{j\omega})|$ for various values of ω.

ω	$A(\omega)$	$\|H(e^{j\omega})\| = A(\omega)$
$0 \cdot \pi/16$	1.0001	1.0001
$\pi/16$	0.9874	0.9874
$2\pi/16$	0.9748	0.9748
$3\pi/16$	1.0048	1.0048
$4\pi/16$	1.0707	1.0707
$5\pi/16$	1.0911	1.0911
$6\pi/16$	0.9623	0.9623
$7\pi/16$	0.6521	0.6521
$8\pi/16$	0.2517	0.2517
$9\pi/16$	−0.0710	0.0710
$10\pi/16$	−0.1873	0.1873
$11\pi/16$	−0.1019	0.1019
$12\pi/16$	0.0542	0.0542
$13\pi/16$	0.1294	0.1294
$14\pi/16$	0.0682	0.0682
$15\pi/16$	−0.0559	0.0559
$16\pi/16$	−0.1175	0.1175
$17\pi/16$	−0.0559	0.0559
$18\pi/16$	0.0682	0.0682
$19\pi/16$	0.1294	0.1294
$20\pi/16$	0.0542	0.0542
$21\pi/16$	−0.1019	0.1019
$22\pi/16$	−0.1873	0.1873
$23\pi/16$	−0.0710	0.0710
$24\pi/16$	0.2517	0.2517
$25\pi/16$	0.6521	0.6521
$26\pi/16$	0.9623	0.9623
$27\pi/16$	1.0911	1.0911
$28\pi/16$	1.0707	1.0707
$29\pi/16$	1.0048	1.0048
$30\pi/16$	0.9748	0.9748
$31\pi/16$	0.9874	0.9874
$32\pi/16$	1.0001	1.0001



$|H(e^{j\omega})|$
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
$\frac{4\pi}{16}$
$\frac{8\pi}{16}$
$\frac{12\pi}{16}$
$\frac{16\pi}{16}$
$\frac{20\pi}{16}$
$\frac{24\pi}{16}$
$\frac{28\pi}{16}$
$\frac{32\pi}{16}$
ω

Infinite Impulse Response Filters

Basically a digital filter is a linear time invariant discrete time system. The terms FIR & IIR are used to distinguish filter types. The FIR filters are of non-recursive type, whereby the present output sample depends on the present input sample & previous input samples, whereas the IIR filters are of recursive type, whereby the present output sample depends on present input, past input samples & output samples

The impulse response $h(n)$ for a realizable filter is

$$h(n) = 0 \quad \text{for} \quad n \leq 0$$

& for stability it must satisfy the condition

$$\sum_{n=0}^{\infty} |h(n)| < \infty$$

IIR digital filters have the transfer function of form

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

The design of an IIR filter for given specifications is to find filter coefficients a_k's & b_k's of above eq

Frequency selective filters :

A filter is one, which rejects unwanted frequencies from the input signal & allow the desired frequencies of signal that are passed through the filter. The range of frequencies of signal that are passed through the filter is called Pass band & those frequencies that are blocked is called stop band.

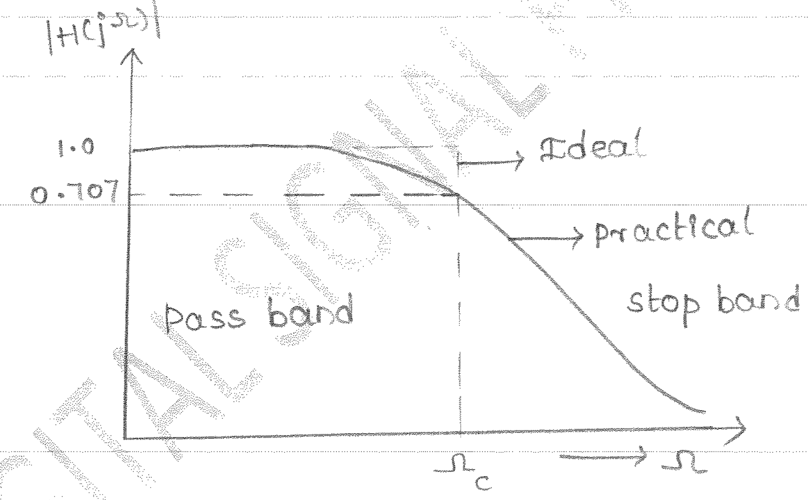
The filters are of different types

1. Low pass filter
2. High pass filter
3. Band pass filter
4. Band reject filter

1. Low pass filter :

The magnitude response of an ideal LPF allows low frequencies in the pass band $0 < \Omega < \Omega_c$ to pass, whereas the higher frequencies in the stop band $\Omega > \Omega_c$ are blocked. The frequency Ω_c between the two bands is the cutoff frequency, where the magnitude $|H(j\Omega)| = 1/\sqrt{2}$

In practice it is impossible to obtain the ideal response. The practical response of a LPF is shown in solid line



(a) Magnitude response of LPF

2. High pass filter :

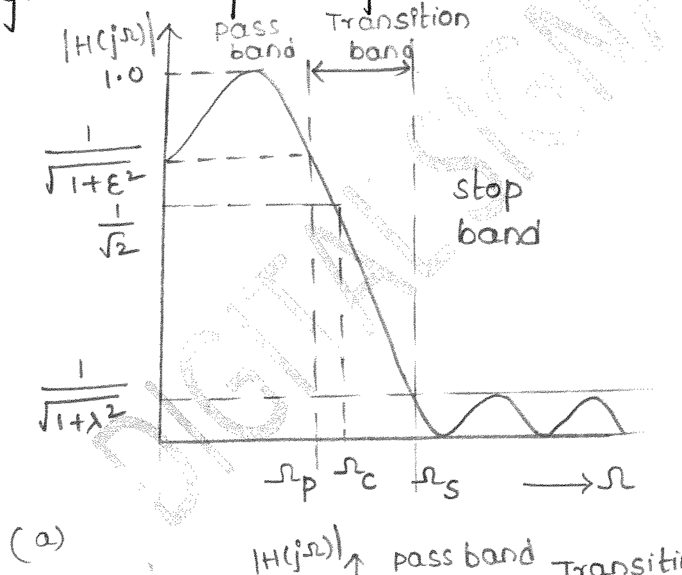
The HPF allows high frequencies above $\Omega > \Omega_c$ & rejects the frequencies between $\Omega = 0$ and $\Omega = \Omega_c$. The magnitude response of an ideal & practical HPF is shown in fig.

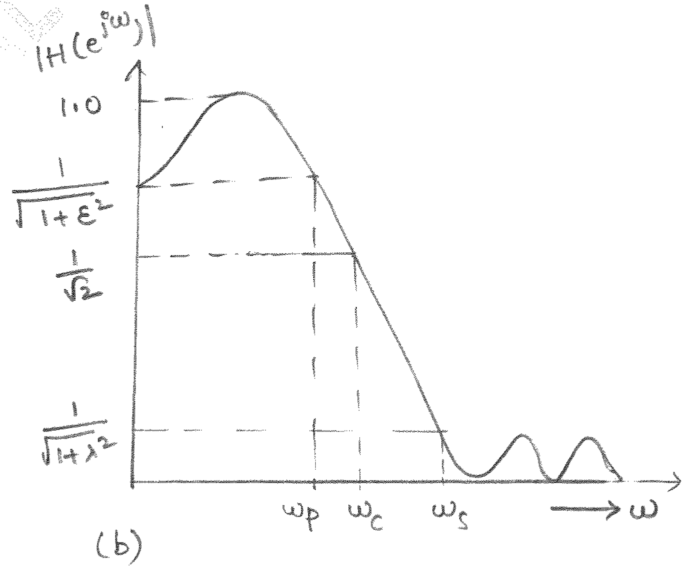
Design of Digital filters from Analog filters:

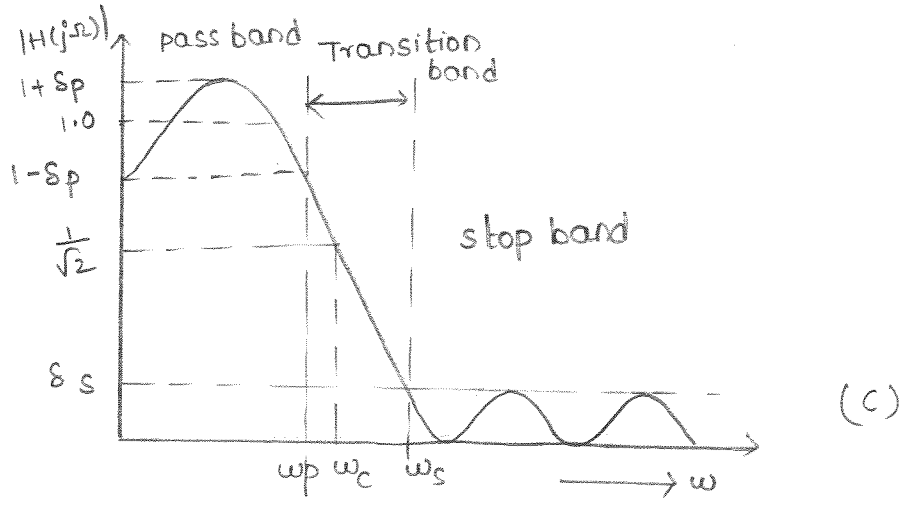
The most common technique used for designing IIR digital filters known as indirect method, involves first designing an analog prototype filter & then transforming the prototype to a digital filter. For the given specifications of a digital filter, the derivation of a digital filter transfer function requires three steps.

1. Map the desired digital filter specifications in to those for an equivalent analog filter
2. Derive the analog transfer function for the analog prototype.
3. Transform the transfer function of the analog prototype into an equivalent digital filter transfer function

The below fig shows the magnitude response of a digital low pass filter







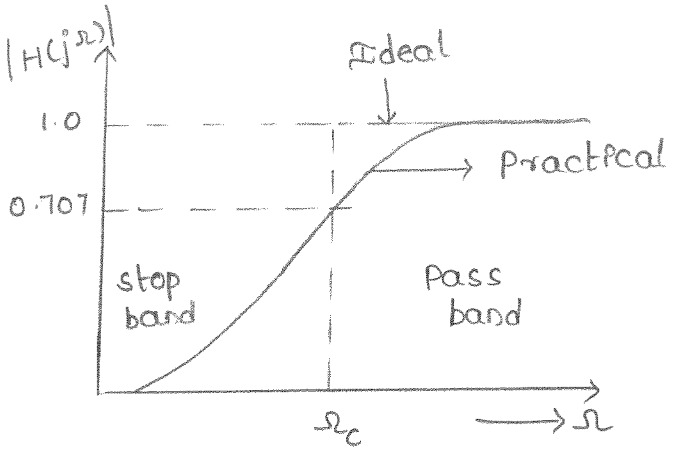


fig: Magnitude response of HPF

3. BandPass filter :

It allows only a band of frequencies Ω_1 to Ω_2 to pass & stops all other frequencies. The ideal & practical response of BPF are shown in below fig

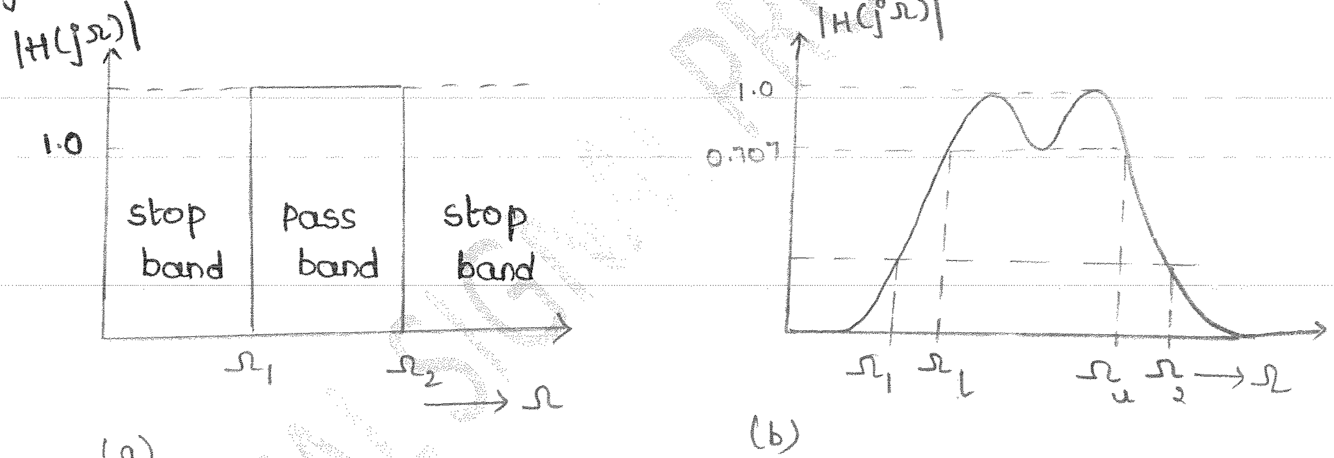


fig: Magnitude response of BPF (a) ideal (b) Practical

4. Band reject filter : It rejects all the frequencies between Ω_1 and Ω_2 & allows remaining frequencies.

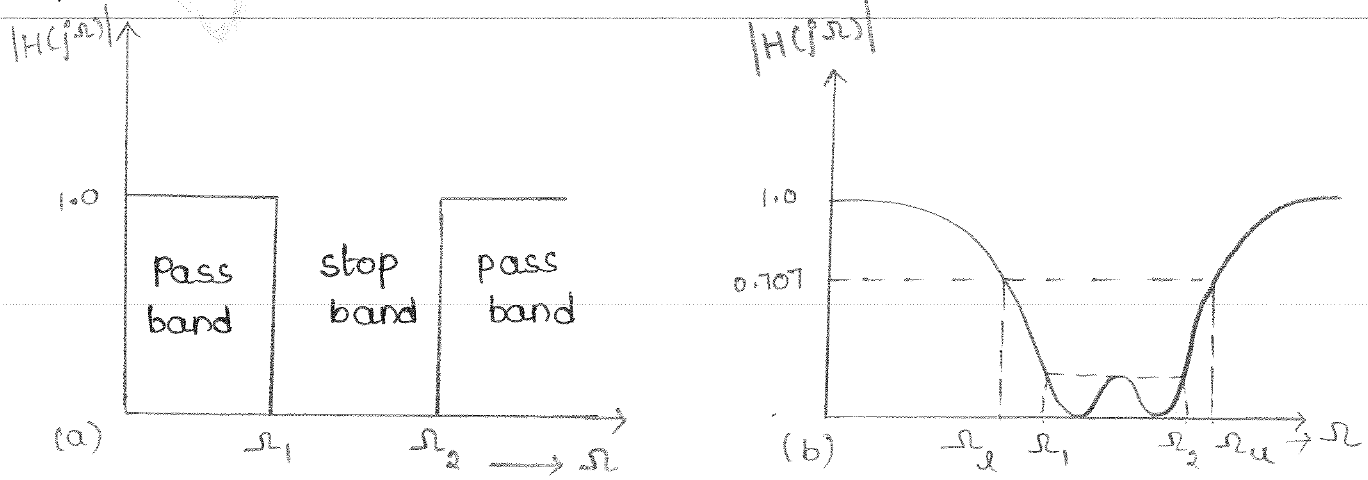


fig: Magnitude response of BRF a) Ideal b) Practical

Specifications for the magnitude response of lowpass filter

(a) analog

(b) digital

(c) Alternate specifications of magnitude response of LPF

The various parameters in the fig are

ω_p = Passband frequency in radians

ω_s = stop band frequency in radians

ω_c = 3-dB cutoff frequency in radians

ε = Parameter specifying allowable passband

λ = Parameter specifying allowable stopband

fig (b) can be modified to apply to analog low pass filter as shown in fig (a). Here the digital frequencies ω_p, ω_s & ω_c are replaced by analog frequencies Ω_p, Ω_s & Ω_c whose units are in radians/sec.

δ_p represents the passband error tolerance &

δ_s represents the maximum allowable magnitude in the stop band. The relation between the parameters is shown in fig (b), (c). & are given by

$$\boxed{\epsilon = 2\frac{\sqrt{\delta_p}}{1-\delta_p} \quad \text{and} \quad \lambda = \frac{\sqrt{(1+\delta_p)^2-\delta_s^2}}{\delta_s}}$$

Using the analog filter specifications the transfer function of analog LPF is designed & it is transformed to digital filter using suitable transformation methods. Among the transformation methods available, the bilinear transformation is usually the method of choice for the design of filters like LPF, HPF, BPF, BRF.

Digital versus Analog filters :

Analog filter	Digital filter
1. Analog filters processes analog inputs & generates analog outputs	1. A digital filter processes & generates digital data
2. Analog filters are constructed from active or passive electronic components	2. A digital filter consists of elements like adder, multiplier & delay unit
3. Analog filter is described by a differential equation	3. Digital filter is described by a difference equation
4. The frequency response of an analog filter can be modified by changing the components	4. The frequency response can be changed by changing the filter coefficients

Advantages of digital filters :

1. Unlike analog filter, the digital filter performance is not influenced by component ageing, temperature & power supply variations
2. A digital filter is highly immune to noise & possesses considerable parameter stability
3. Digital filters afford a wide variety of shapes for the amplitude & phase responses
4. Digital filters can be operated over a wide range of frequencies
5. There are no problems of input or output impedance matching with digital filters
6. The coefficients of digital filter can be programmed & altered any time to obtain the desired characteristics.

7. Multiple filtering is possible only in digital filter.

Disadvantage of digital filter:

The quantization error arises due to finite word length in the representation of signals & parameters.

Butterworth versus Chebyshev (Type-I) filter

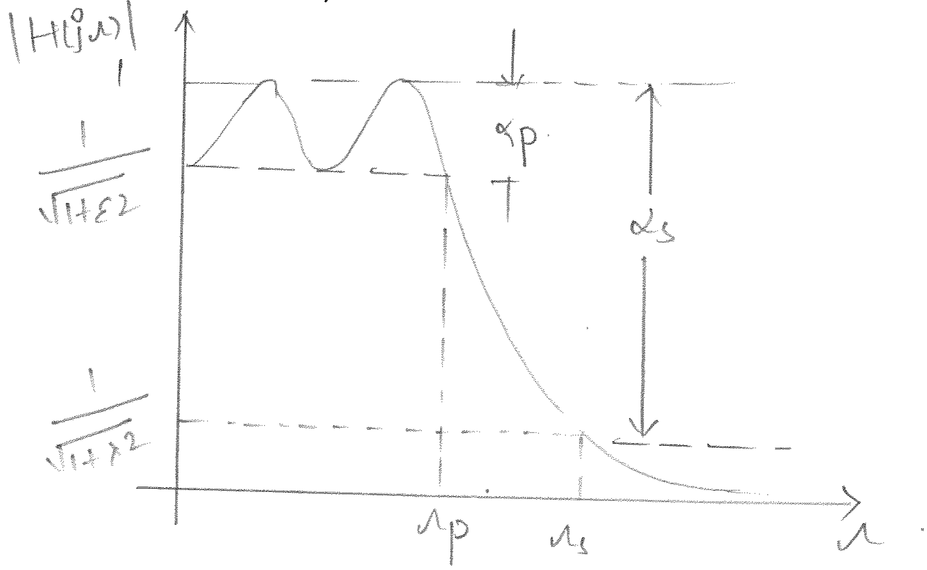
Butterworth filter	Chebyshev filter
1. The magnitude response of Butterworth filter decreases monotonically as frequency Ω increases from 0 to ∞	1. The magnitude response of Chebyshev filter exhibits ripple in pass band & monotonically decreasing in stop band
2. The transition band is more in Butterworth filter compared to Chebyshev	2. The transition band is less in chebyshev compared to Butterworth
3 Poles lie on a circle	3. Poles lie on an ellipse
4. for the same specifications the no. of poles in Butterworth filter are more compared to chebyshev	4. for the same specifications the no. of poles in chebyshev are less compared to Butterworth i.e order of Chebyshev filter is less than that of Butterworth

FIR versus IIR :

FIR	IIR
1. FIR is a non recursive system	1. IIR is a recursive system
2. The current o/p y(n) is a function of only past & present inputs	2. The o/p y(n) is a function of past outputs, past & present inputs
3. FIR has linear phase	3 IIR filters donot have linear phase
4. h(n) is finite	4. The impulse response h(n) is infinite

Algorithm for Design of Chebyshev Filter:

1. Given Specifications
2. Draw the Low pass filter characteristic.



3. For High pass filter

The pass band frequency ω_{ph} and Stop band frequency ω_s are exchanged (with Lowpass filter)

$$\Omega_p = \Omega_s \text{ and } \Omega_s = \Omega_p$$

4. Select the transformation techniques:

a) For Bilinear Transformation technique:

$$\Omega = \frac{2}{T} \tan\left(\frac{\omega}{2}\right)$$

b) For impulse Invarient technique

$$\Omega = \frac{\omega}{T}$$

5. For order of the filter

a) For Low pass and high pass filters

$$N \geq \frac{\cosh^{-1}(\lambda/\varepsilon)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

b) For Band pass and Band elimination filters

$$N \geq \frac{\cosh^{-1}(\lambda/\varepsilon)}{\cosh^{-1}(\Omega_p)}$$

where $\Omega_p = \min\{|A|, |B|\}$

i. For Band pass filter:

$$A = \frac{-\Omega_1^2 + \Omega_l \Omega_u}{\Omega_l(\Omega_u - \Omega_l)} \qquad B = \frac{\Omega_2^2 - \Omega_l \Omega_u}{\Omega_2(\Omega_u - \Omega_l)}$$

ii. For Band Elimination filter:

$$A = \frac{\Omega_1 + (\Omega_u - \Omega_l)}{-\Omega_1^2 + \Omega_u \Omega_l} \; ; \quad B = \frac{\Omega_2(\Omega_u - \Omega_l)}{\Omega_2^2 - \Omega_l \Omega_u}.$$

6. Chebyshev polynominal can be obtained by

$$S_K = a\cos\phi_K + jb\sin\phi_K$$

$$a = \Omega_p\left(\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right) \; ; \; b = \Omega_p\left(\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right)$$

$\phi_K =$

7. The Transfer function of chebyshev filter is given by

$$H(j\Omega) = \frac{\text{Numerator}}{\text{chebyshev polynomial}}$$

8. Find values of numerator term of the Transfer function.

a) For odd value of filter order, substitute $s=0$ in chebyshev polynomial

b) Fore even value of filter order, Substitute $s=0$ and divide by $\sqrt{1+\varepsilon^2}$ in chebyshev polynomial.

9. Filter Conversion:

a) For Lowpass to High pass
s must be replaced by $\frac{\Omega_p}{s}$

b) For Low pass to Band pass
s must be replaced by $\frac{s^2+\Omega_l\Omega_u}{s(\Omega_u-\Omega_l)}$

c. For Low pass to Band Elimination.
s must be replaced by. $\frac{s(\Omega_u-\Omega_l)}{s^2+\Omega_l\Omega_u}$.

10. Apply Transformation technique to convert analog filter into respective Digital filter

a) For Bilinear Transformation technique
Replace s by $\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$

b) For Impulse Invarient technique.
Replace $\frac{1}{s-P_k}$ by $\frac{1}{1-e^{P_kT}z^{-1}}$

11. Final Transfer function must be presented in following format.

$$H(z) = \frac{b_0+b_1z^{-1}+b_2z^{-2}+---}{1+a_1z^{-1}+a_2z^{-2}+---}$$

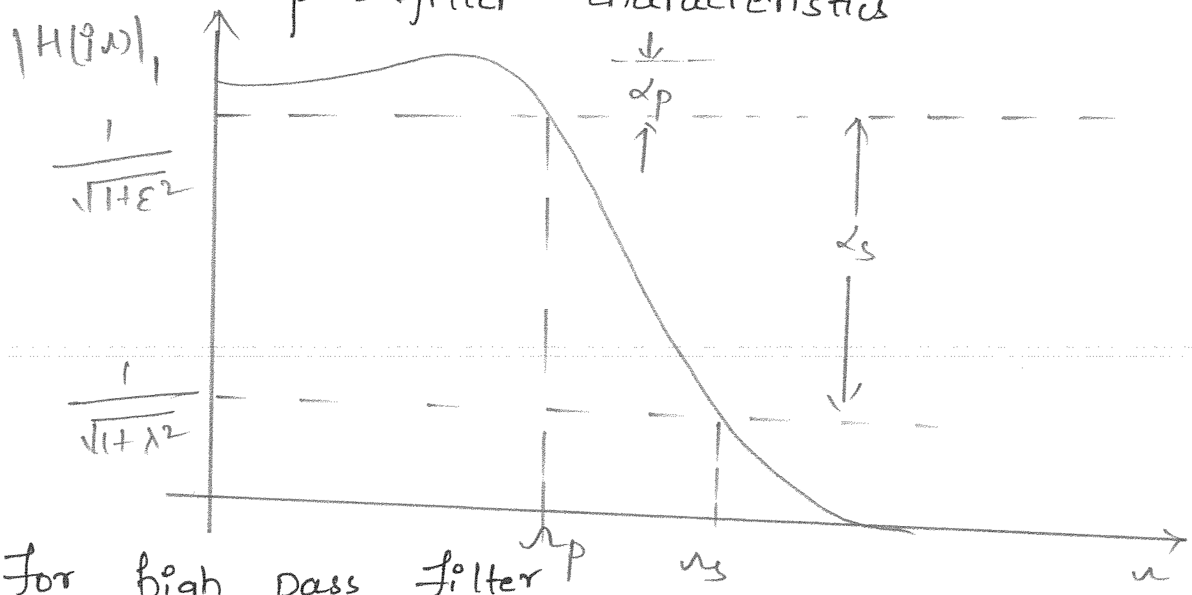
Note: a_0 is always unity

Algorithm for Design of Butterworth filter:

1. Given specifications

Note: If freq is in Hertz convert into rad/sec $\omega_s = \frac{2\pi f_0}{F_s}$

2. Draw Low pass filter characteristics



3. For high pass filter

The pass band frequency Ω_p and stop band freq Ω_s

$$\Omega_p = \Omega_s \text{ and } \Omega_s = \Omega_p$$

4. Select Transformation Technique

a) $\Omega = \frac{2}{T}\tan(\omega/2)$; Bilinear Transformation

b) $\Omega = \frac{\omega}{T}$; Impulse Invariant technique.

5. Find order of filter

a) For Lowpass and high pass filters

$$N \geq \frac{\log(\lambda/\varepsilon)}{\log(\Omega_s/\Omega_p)}$$

b) For Bandpass and Band elimination filters:

$$N \geq \frac{\log(\lambda/\varepsilon)}{\log(\Omega_p)}$$

where $\Omega_p = \min\{|A|, |B|\}$

i) For Band pass filter $A = \dfrac{\Omega_1^2 + \Omega_l\Omega_u}{\Omega_1(\Omega_u - \Omega_l)}$ $\quad B = \dfrac{\Omega_2^2 - \Omega_l - \Omega_u}{\Omega(\Omega_u - \Omega_l)}$

ii) For Band Elimination filter. $A = \dfrac{\Omega_1 + \Omega_u\Omega_l}{-\Omega_1^2 + \Omega_u\Omega_l}$; $B = \dfrac{\Omega_2 - (\Omega_u - \Omega_l)}{\Omega_2^2 - \Omega_u\Omega_l}$

Find poles $S_K = e^{j\phi_K}$; $\phi_K = \dfrac{\pi}{2} + \dfrac{(2K-1)\pi}{2N}$ $\quad K = 1, 2 --- N$

6. The Normalised Butterworth polynomials for various values of N is given as . Find the Transfer function

$$H(S) = \prod_{K=1}^{N} \frac{1}{(S - P_K)}$$

N	polynomial
1	$s+1$
2	$s^2 + \sqrt{2}s + 1$
3	$(s+1)(s^2+s+1)$
4	$(s^2 + 0.7654s + 1)(s^2 + 1.8477s + 1)$
5	$(s+1)(s^2 + 0.618s + 1)(s^2 + 1.618s + 1)$
6	$(s^2 + 1.932s + 1)(s^2 + \sqrt{2}s + 1)(s^2 + 0.5176s + 1)$

7. De-normalise the Transfer function.

a. For Low pass to Low pass

S must be replaced by $\dfrac{s}{\Omega_c}$

where $\Omega_c = \dfrac{\Omega_p}{\varepsilon^{1/N}}$

b. For Lowpass to High pass

S must be replaced by $\dfrac{\Omega_c}{s}$

c. For Lowpass to Band pass

S must be replaced by $\dfrac{s^2 + \Omega_l\Omega_u}{s(\Omega_u - \Omega_l)}$

d. For Lowpass to Band Elimination

S must be replaced by $\dfrac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l\Omega_u}$

8. Apply Transformation technique to convert Analog filters to Digital filter.

a. By Bilinear Transformation technique

Replace $\underline{s}$ by $\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$

b. For Time Impulse- Invarient Technique

Replace $\frac{1}{s-P_K}$ by $\frac{1}{1-e^{+P_K T}z^{-1}}$

9. The final T/F must be presented in following format

$$H(z) = \frac{b_0+b_1z^{-1}+b_2z^{-2}+\text{------}}{1+a_1z^{-1}+a_2z^{-2}+\text{------}}$$

Note: a_0 must be unity

The pass band and stop band cut off frequencies are 350 hz and 1000 hz respectively. The attenuation at pass band & stop band are -3dB and -10dB respectively. The sampling frequency is 5000hz. Design a digital low pass Butterworth filter using bilinear transformation.

The given specifications of the filter are:

$\alpha_p = -3dB$

$\alpha_s = -10dB$

$f_p = 350hz$

$f_s = 1000hz$

$F_s = 5000hz$

The corresponding analog frequencies are:

$$\omega_p = \frac{2\pi f_p}{F_s} = \frac{2\pi \times 350}{5000} = 0.14\pi \text{ rad}|$$

$$\omega_s = \frac{2\pi f_s}{F_s} = \frac{2\pi \times 1000}{5000} = 0.4\pi \text{ rad}|$$

Apply bilinear transformation,

$$\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$$

$$\Omega_p = \frac{2}{2\times 10^{-4}}\tan\left(\frac{0.14\pi}{2}\right) = 2235.27 \text{ rad}|\text{sec}$$

$$\Omega_s = \frac{2}{2\times 10^{-4}}\tan\left(\frac{0.4\pi}{2}\right) = 7265.43 \text{ rad}|\text{sec}$$

$$\left\{ F = \frac{1}{T} \quad T = \frac{1}{F_s} = \frac{1}{5000} = 2\times 10^{-4} \right\}$$

The order of low pass filter is

$$N \geq \frac{\log\left(\frac{\lambda}{\varepsilon}\right)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$\lambda = \sqrt{10^{0.1\alpha_s} - 1} = \sqrt{10^{0.1\times 10} - 1} = 3$$

$$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1} = \sqrt{10^{0.1\times 3} - 1} = 0.999$$

$$N \geq \frac{\log\left(\frac{3}{0.999}\right)}{\log\left(\frac{7265.43}{2235.27}\right)} = 0.933 \approx 1$$

$S_K = e^{j\phi_K}$ $\quad k = 1$ to N

$\phi_K = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}$

$= \frac{\pi}{2} + \frac{(1)\pi}{2} = \pi$

$S_1 = e^{j\pi} = -1$

The Order of LPF for the given magnitude response is 1

The transfer function for I order normalized LPF is given by

$H(s) = \prod_{K=1}^{N} \frac{1}{s - s_K} = \frac{1}{s - s_1} = \frac{1}{s+1}$

$$H(s) = \frac{1}{s+1}$$

The denormalized filter transfer function can be obtained by replacing $s \longrightarrow \frac{s}{\Omega_c}$

$$\Omega_c = \frac{\Omega p}{\varepsilon^{1/N}} = \frac{2235.27}{0.999} = 2237.8 \text{ rad/sec}$$

Replace $s \longrightarrow \frac{s}{2237.5}$ in $H(s)$

$$H_a(s) = \frac{1}{\left(\frac{s}{2237.5}\right) + 1}$$

$$= \frac{2237.5}{s + 2237.5}$$

To convert analog LPF to digital LPF using bilinear transformation, substitute

$$s \longrightarrow \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

$$H_a(s) = \frac{2237.5}{\frac{2}{2\times10^{-4}}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 2237.5}$$

$$H(z) = \frac{2257.5 + 2257.5 z^{-1}}{12237.5 - 7762.5 z^{-1}}$$

$$H(z) = \frac{0.1845 + 0.1845 z^{-1}}{1 - 0.6343 z^{-1}}$$

Design a Butterworth digital high pass filter with the following specifications

$$|H(e^{j\omega})| \le 0.2 \,, \quad 0 \le \omega \le 0.2\pi$$

$$0.8 \le |H(e^{j\omega})| \le 1 \,, \quad 0.6\pi \le \omega \le \pi$$

Using i) Impulse invariance transformation
ii) Bilinear transformation.

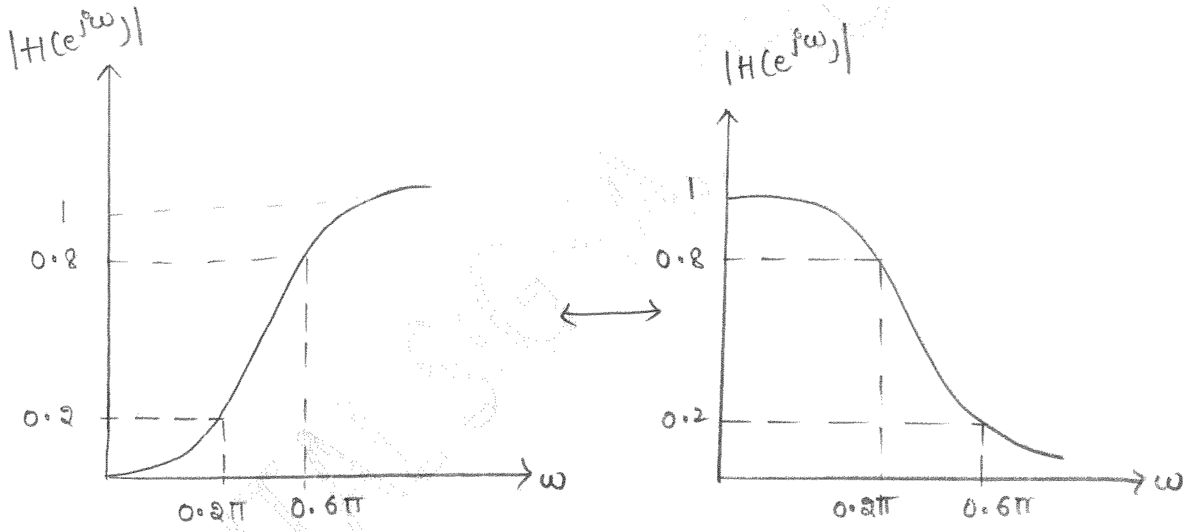


Fig: HPF to LPF transformation

The given specifications of high pass filter are:

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.2 \,; \quad \frac{1}{\sqrt{1+\varepsilon^2}} = 0.8$$

$$\omega_p = 0.6\pi \,; \quad \omega_s = 0.2\pi$$

For low pass filter these specifications are to be taken as $\omega_p = 0.2\pi$; $\omega_s = 0.6\pi$ refer fig shown above

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.2 \Rightarrow \lambda = 4.89$$

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.8 \Rightarrow \varepsilon = 0.75$$

i) Impulse - invariant transformation :

Apply impulse - invariant transformation, $\omega = \Omega T$

Since sampling period T is not given, assume T = 1 second

$$\therefore \Omega_p = \omega_p = 0.2\pi \text{ rad/sec}$$

$$\Omega_s = \omega_s = 0.6\pi \text{ rad/sec}$$

The Order of LPF is

$$N \geq \frac{\log\left(\frac{\lambda}{\varepsilon}\right)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$N \geq \frac{\log\left(\frac{4.89}{0.75}\right)}{\log\left(\frac{0.6\pi}{0.2\pi}\right)} = 1.706 = 2$$

The order of LPF for the given magnitude response is 2.

The transfer function for II order normalized LPF is given by

$$H(s) = \frac{1}{s^2+\sqrt{2}s+1}$$

The denormalized HPF transfer function can be obtained by replacing s by Ω_c/s.

$$\text{where } \Omega_c = \frac{\Omega_p}{\varepsilon^{1/N}} = \frac{\Omega_p}{\varepsilon^{1/2}} = \frac{0.2\pi}{(0.75)^{1/2}}$$

$$= 0.7255 \text{ rad/sec}$$

$$H_a(s) = \frac{1}{\left(\frac{0.7255}{s}\right)^2 + \sqrt{2}\left(\frac{0.7255}{s}\right) + 1}$$

$$= \frac{s^2}{s^2 + 1.026s + 0.5264}$$

$$= 1 - \frac{1.026s + 0.5264}{s^2 + 1.026s + 0.5264} \longrightarrow (1)$$

Consider 2nd term of eq.(1)

$$\frac{1.026s + 0.5264}{s^2 + 1.026s + 0.5264} = \frac{1.026s + 0.5264}{(s + 0.5131 + j0.5131)(s + 0.5131 - j0.5131)}$$

$$= \frac{A}{s + 0.5131 + j0.5131} + \frac{B}{s + 0.5131 - j0.5131}$$

$$A = \frac{1.026s + 0.5264}{\cancel{(s + 0.5131 + j0.5131)}(s + 0.5131 - j0.5131)} \times \cancel{(s + 0.5131 + j0.5131)}\Bigg|_{s = -0.5131 - j0.5131}$$

$$= \frac{1.026(-0.5131 - j0.5131) + 0.5264}{-0.\cancel{5}131 - j0.5131 + 0.\cancel{5}131 - j0.5131}$$

$$= 0.5129 \qquad \text{(Magnitude)}$$

$$B = 0.5129$$

$$\therefore H_a(s) = 1 - \left[\frac{0.5129}{s + 0.5131 + j0.5131} + \frac{0.5129}{s + 0.5131 - j0.5131}\right] \longrightarrow (2)$$

To convert analog HPF to digital HPF using IIT

$$\frac{1}{s - P_k} \longrightarrow \frac{1}{1 - e^{P_k T} z^{-1}}$$

From eq.(2)

$P_1 = -0.5131 - j0.5131$

$P_2 = -0.5131 + j0.5131$

$$\therefore \quad H(z) = 1 - \left[\frac{0.5129}{1 - e^{(-0.5131 - j0.5131)T} z^{-1}} + \frac{0.5129}{1 - e^{(-0.5131 + j0.5131)T} z^{-1}} \right]$$

since $T = 1\,sec$

$$H(z) = 1 - \left[\frac{0.5129}{1 - e^{(-0.5131 - j0.5131)} z^{-1}} + \frac{0.5129}{1 - e^{(-0.5131 + j0.5131)} z^{-1}} \right]$$

$$H(z) = 1 - \left[\frac{0.5129}{1 - 0.5986 e^{-j0.5131} z^{-1}} + \frac{0.5129}{1 - 0.5986 e^{j0.5131} z^{-1}} \right] \quad \left\{ \because e^{-0.5131} = 0.5986 \right\}$$

On simplification

$$H(z) = 1 - \left\{ \frac{0.5129\left(2 - 0.5986\left(e^{j0.5131} + e^{-j0.5131}\right)z^{-1}\right)}{1 - 0.5986\left(e^{j0.5136} + e^{-j0.5136}\right)z^{-1} + (0.5986)^2 z^{-2}} \right\}$$

$$= 1 - \frac{1.0258 - 0.535z^{-1}}{1 - 1.043z^{-1} + 0.3583z^{-2}}$$

$2\cos(0.5136) \times 0.5986 = 1.04$

$$= \frac{-0.025 - 0.508z^{-1} + 0.3583z^{-2}}{1 - 1.043z^{-1} + 0.3583z^{-2}}$$

ii) Bilinear transformation:

Prewarping in Bilinear transformation

$$\Omega = \frac{2}{T} \tan\left(\frac{\omega}{2}\right)$$

$$\Omega_p = 2\tan\left(\frac{0.2\pi}{2}\right) = 0.6498 \text{ rad/sec}$$

$$\Omega_s = 2\tan\left(\frac{0.6\pi}{2}\right) = 2.7528 \text{ rad/sec}$$

On simplification,

$$H(z) = \frac{0.598 - 1.196z^{-1} + 0.598z^{-2}}{1 - 1.028z^{-1} + 0.365z^{-2}}$$

Design a Chebyshev LPF whose transfer function

$$0.8 \le |H(e^{j\omega})| \le 1, \quad 0 \le \omega \le 0.2\pi$$

$$|H(e^{j\omega})| \le 0.2, \quad 0.6\pi \le \omega \le \pi$$

Using i) Impulse invariant transformation

ii) Bilinear transformation

Given $\frac{1}{\sqrt{1+\varepsilon^2}} = 0.8$

$$\sqrt{1+\varepsilon^2} = \frac{1}{0.8}$$

$$1+\varepsilon^2 = \frac{1}{0.8^2}$$

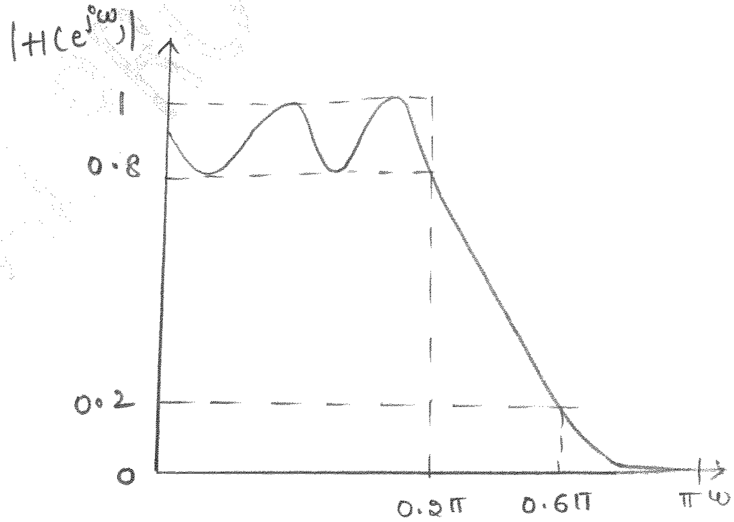
$$\varepsilon^2 = \frac{1}{0.8^2} - 1$$

$$\varepsilon = \sqrt{\frac{1}{0.8^2} - 1}$$

$$\varepsilon = 0.75$$

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.2$$

$$\lambda = 4.89$$



i) IIT:

Apply IIT, $\omega = \Omega T$.

since sampling period T is not given, assume T = 1 sec

$$\therefore \quad \Omega_p = \omega_p = 0.2\pi \text{ rad/sec}$$

$$\Omega_s = \omega_s = 0.6\pi \text{ rad/sec}$$

The order of LPF is

$$N \ge \frac{\cosh^{-1}\left(\frac{\lambda}{\varepsilon}\right)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

The order of LPF is

$$N \geq \frac{\log\left(\frac{\lambda}{\varepsilon}\right)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$N \geq \frac{\log\left(\frac{4.89}{0.75}\right)}{\log\left(\frac{2.7529}{0.6498}\right)} = 1.2986 \approx 2$$

The order of LPF for given magnitude response is 2. The transfer function for II order normalized LPF is

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

The denormalized HPF transfer function can be obtained by replacing s by $\frac{\Omega_c}{s}$

where $\Omega_c = \frac{\Omega_p}{\varepsilon^{1/N}} = \frac{0.6498}{(0.75)^{1/2}} = 0.7503$ rad/sec

$$H(s) = \frac{1}{\left(\frac{0.7503}{s}\right)^2 + \sqrt{2}\left(\frac{0.7503}{s}\right) + 1}$$

$$= \frac{s^2}{s^2 + 1.061s + 0.562}$$

To convert analog HPF to digital HPF using bilinear transformation

$$s \longrightarrow \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$H(z) = \frac{4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2}{4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 1.061 \times 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.562}$$

$$N \geq \frac{\cosh^{-1}\left(\frac{4.89}{0.75}\right)}{\cosh^{-1}\left(\frac{0.6\pi}{0.2\pi}\right)} = 1.45 \approx 2$$

$\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2-1}\right)$

The polynomial equation of chebyshev filter is

$$s_k = a\cos\phi_k + jb\sin\phi_k$$

$$= \sigma_k + j\Omega_k$$

$k = 1$ to N

where $\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}$ $\quad k = 1$ to N $= 1, 2$

$$\phi_1 = \frac{\pi}{2} + \frac{(2-1)\pi}{2\times 2} = \frac{3\pi}{4}$$

$$\phi_2 = \frac{\pi}{2} + \frac{(2\times 2-1)}{2\times 2} = \frac{5\pi}{4}$$

$$D(s) = \prod_{k=1}^{N} (s - s_k)$$

$$= \prod_{k=1}^{2} (s - s_k)$$

$$= (s-s_1)(s-s_2) \longrightarrow (1)$$

$s_1 = a\cos\phi_1 + jb\sin\phi_1$ $\qquad$ $s_2 = a\cos\phi_2 + jb\sin\phi_2$

$$a = \Omega_p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right]$$

$$b = \Omega_p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right]$$

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}}$$

$$= (0.75)^{-1} + \sqrt{1+(0.75)^{-2}}$$

$$\mu = 3$$

Now $a = 0.2\pi\left[\frac{3^{1/N} - 3^{-1/N}}{2}\right] = 0.3627$ $\qquad \because N = 2$

$$b = 0.2\pi\left[\frac{3^{1/N} + 3^{-1/N}}{2}\right] = 0.7255$$

$\phi_1 = \frac{3\pi}{4}$, $\phi_2 = \frac{5\pi}{4}$

$s_k = a\cos\phi_k + jb\sin\phi_k$

$s_1 = a\cos\phi_1 + jb\sin\phi_1$

$= (0.3627)\cos\left(\frac{3\pi}{4}\right) + j(0.7255)\sin\left(\frac{3\pi}{4}\right)$

$= -0.2564 + 0.5130j$

$s_2 = a\cos\phi_2 + jb\sin\phi_2$

$= (0.3627)\cos\left(\frac{5\pi}{4}\right) + j(0.7255)\left(\sin\left(\frac{5\pi}{4}\right)\right)$

$= -0.2564 - 0.5130j$

Now eq. (1) $\Rightarrow$

$D(s) = (s-s_1)(s-s_2)$

$= [s-(-0.2564+0.5130j)]\,[s-(-0.2564-0.5130j)]$

N(S) :

$N = 2 \longrightarrow$ Even

put $s=0$ in $D(s)$ & divide by $\sqrt{1+\varepsilon^2}$

$s=0 \Rightarrow (0.2564-0.5130j)(0.2564+0.5130j)$

$= (0.564)^2 - (0.5130j)^2$

$\div$ by $\sqrt{1+\varepsilon^2} = \sqrt{1+0.75^2} = 1.25$

$\therefore\ N(s) = \frac{(0.2564)^2 + (0.5130)^2}{1.25} = 0.2631$

$$H(s) = \frac{N(s)}{D(s)} = \frac{0.2631}{[s-(-0.2564+0.5130j)]\,[s-(-0.2564-0.5130j)]}$$

$$H_a(s) = H(s)\Big|_{s=\frac{s}{\Omega_c}}$$

Now by partial fractions

$$H(s) = \frac{A}{[s-(-0.2564+0.5130j)]} + \frac{B}{[s-(-0.2564-0.5130j)]}$$

$$A = \frac{0.2631}{[s-(-0.2564+0.5130j)][s-(-0.2564-0.5130j)]} \times [s-(-0.2564+0.5130j)] \Bigg|_{s=-0.2564+0.5130j}$$

$$= \frac{0.2631}{2\times 0.5130j}$$

$$= -0.2564j$$

$$B = 0.2564j$$

$$\therefore H(s) = \frac{-0.2564j}{[s-(-0.2564+0.5130j)]} + \frac{0.2564j}{[s-(-0.2564-0.5130j)]}$$

s_1 or $P_1 = -0.2564+0.5130j$

s_2 or $P_2 = -0.2564-0.5130j$

$$\frac{1}{s-P_K} \longrightarrow \frac{1}{1-e^{P_K T} z^{-1}} \qquad \{T = 1\,\text{sec}\}$$

$$\therefore H(z) = \frac{-0.2564j}{1-e^{-(0.2564+0.5130j)} z^{-1}} + \frac{0.2564j}{1-e^{-(-0.2564-0.5130j)} z^{-1}}$$

$$= \frac{-0.2564j}{1-\overset{1.29}{0.77}\, e^{-0.5130j} z^{-1}} + \frac{0.2564j}{1-1.29 e^{0.5130j} z^{-1}}$$

$$= 0.2564j \left\{ \frac{-1+1.29 e^{0.5130j} z^{-1} + 1 - 1.29 e^{-0.5130j} z^{-1}}{1-1.29(e^{0.5130j}+e^{-0.5130j}) z^{-1} + (1.29)^2 z^{-2}} \right\}$$

$$H(z) = 0.2564j \left\{ \frac{1.29\left(e^{0.5130j} - e^{-0.5130j}\right) z^{-1}}{1 - 1.29\left(e^{0.5130j} + e^{-0.5130j}\right) z^{-1} + (1.29)^2 z^{-2}} \right\}$$

$$= 0.2564j \left\{ \frac{1.29\,(2j \sin(0.5130))\, z^{-1}}{1 - 1.29\,(2\cos(0.5130))\, z^{-1} + (1.29)^2 z^{-2}} \right\}$$

$$= \frac{0.2564j \times 1.26j\, z^{-1}}{1 - 2.24 z^{-1} + 1.66 z^{-2}}$$

$$= \frac{-0.32 z^{-1}}{1 - 2.24 z^{-1} + 1.66 z^{-2}}$$

ii) BIT:

$\omega_p = 0.2\pi$ rad

$\omega_s = 0.6\pi$ rad

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.8 \qquad \frac{1}{\sqrt{1+\lambda^2}} = 0.2$$

$$\Rightarrow \varepsilon = 0.75 \qquad \lambda = 4.89$$

By prewarping,

$$\Omega = \frac{2}{T} \tan\left(\frac{\omega}{2}\right)$$

$T = 1$ sec

$$\Omega_p = 2 \tan\left(\frac{\omega_p}{2}\right)$$

$= 0.6498$ rad|sec

$$\Omega_s = 2 \tan\left(\frac{\omega_s}{2}\right)$$

$= 2.7527$ rad|sec

$$N \geq \frac{\cosh^{-1}\left(\frac{\lambda}{\varepsilon}\right)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)} = \frac{\cosh^{-1}\left(\frac{4.89}{0.75}\right)}{\cosh^{-1}\left(\frac{2.7527}{0.6498}\right)}$$

$$N = \frac{\cosh^{-1}(6.52)}{\cosh^{-1}(4.23)} \qquad \cosh^{-1}x = \ln\left(x+\sqrt{x^2-1}\right)$$

$$N \geq \frac{\ln\left(6.52+\sqrt{6.52^2-1}\right)}{\ln\left(4.23+\sqrt{4.23^2-1}\right)}$$

$$\geq 1.207$$

$$N \approx 2$$

Poles

$$s_k = a\cos\phi_k + jb\sin\phi_k$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad k = 1 \text{ to } N = 1,2$$

$$\mu = \epsilon^{-1} + \sqrt{1+\epsilon^{-2}}$$

$$= (0.75)^{-1} + \sqrt{1+(0.75)^{-2}} = 3$$

$$a = \Omega_p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] = 0.6498\left[\frac{3^{1/2} - 3^{-1/2}}{2}\right] = 0.3751$$

$$b = \Omega_p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 0.7503$$

$$\phi_1 = \frac{\pi}{2} + \frac{(2-1)\pi}{4} = \frac{3\pi}{4}$$

$$\phi_2 = \frac{\pi}{2} + \frac{(4-1)\pi}{4} = \frac{5\pi}{4}$$

$$s_1 = 0.3751\cos\phi_1 + j(0.7503)\sin\phi_1$$

$$= 0.3751\cos\left(\frac{3\pi}{4}\right) + j(0.7503)\sin\left(\frac{3\pi}{4}\right)$$

$$= -0.2652 + 0.5305j$$

$$s_2 = 0.3751\cos\left(\frac{5\pi}{4}\right) + j(0.7503)\sin\left(\frac{5\pi}{4}\right)$$

$$= -0.2652 - 0.5305j$$

Now $D(s) = \prod_{k=1}^{N}(s-s_k) = \prod_{k=1}^{2}(s-s_k)$

$$H(z) = D(s) = (s-s_1)(s-s_2)$$

$$D(s) = \left[s-(-0.2652+0.5305j)\right]\left[s-(-0.2652-0.5305j)\right]$$

N(S):

$N = 2 \longrightarrow$ Even

put $s=0$ in $D(s)$ & $\div$ by $\sqrt{1+\varepsilon^2}$

$$\sqrt{1+\varepsilon^2} = \sqrt{1+0.75^2} = 1.25$$

$$s=0 \Rightarrow (0.2652-0.5305j)(0.2652+0.5305j)$$

$$N(s) = \frac{(0.2652)^2+(0.5305)^2}{1.25} = 0.2814$$

$$\therefore H(s) = \frac{N(s)}{D(s)}$$

$$= \frac{0.2814}{\left[s-(-0.2652+0.5305j)\right]\left[s-(-0.2652-0.5305j)\right]}$$

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \qquad T = 1\,\text{sec}$$

$$= \frac{0.2814}{\left\{\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.2652\right]-0.5305j\right\}\left\{\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.2652\right]+0.5305j\right\}}$$

$$= \frac{0.2814}{\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.2652\right\}^2+(0.5305)^2}$$

$$= \frac{0.2814}{4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2+(0.2652)^2+4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)(0.2652)+(0.5305)^2}$$

$$= \frac{0.2814 \times (1+z^{-1})^2}{4(1-z^{-1})^2 + 0.07(1+z^{-1})^2 + 4(1-z^{-1})(1+z^{-1})(0.2652) + (0.2814)(1+z^{-1})^2}$$

$$= \frac{0.2814\,(1+z^{-2}+2z^{-1})}{4(1+z^{-2}-2z^{-1}) + 0.07(1+z^{-2}+2z^{-1}) + 4(1-z^{-2})(0.2652) + (0.2814)(1+z^{-2}+2z^{-1})}$$

$$H(z) = \frac{0.2814\,(1+z^{-2}+2z^{-1})}{z^{-2}(3.2906) + z^{-1}(-7.29) + 5.4122}$$

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prob: design bandstop Butterworth and Chebyshev type filter to meet following specifications

a) Stop band 100 to 600 Hz

b) 20dB attenuation at 200 & 400Hz

c) The gain at $\omega = 0$ is unity

d) The pass band ripple for Chebyshev filter is 1.1dB.

e) The pass band attenuation for Butterworth filter is 3dB.

Sol: Given $f_l = 100Hz$, $f_1 = 200Hz$, $f_2 = 400Hz$, $f_u = 600Hz$

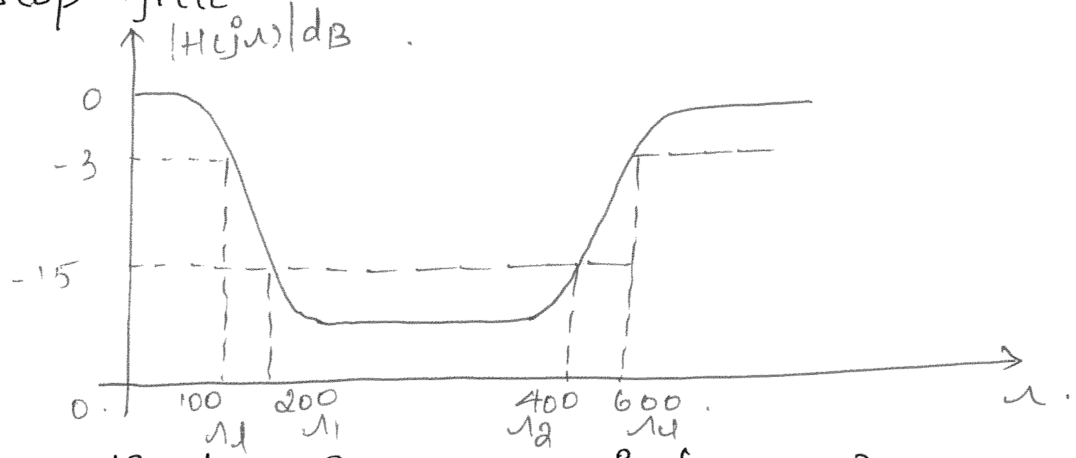
$$\Omega_l = 2 \times \pi \times 100 = 200\pi \text{ rad/sec}$$

$$\Omega_1 = 2\pi \times 200 = 400\pi \text{ rad/sec}$$

$$\Omega_2 = 2\pi \times 400 = 800\pi \text{ rad/sec}$$

$$\Omega_u = 2 \times \pi \times 600 = 1200\pi \text{ rad/sec}$$

First we design prototype normalised LPF and then use suitable transformation to obtain the T/F of Bandstop filter.



For normalised LPF $\Omega_r = \min\{|A|, |B|\}$

$$A = \frac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^2 + \Omega_l\Omega_u} = \frac{400\pi(1200\pi - 200\pi)}{-(400\pi)^2 + (200\pi)(1200\pi)}$$

$$= 5$$

$$B = \frac{\Omega_2(\Omega_l - \Omega_u)}{-\Omega_2^2 + \Omega_l\Omega_u} = \frac{800\pi\,(1200\pi - 200\pi)}{-(800\pi)^2 + (200\pi)(1200\pi)} = -2$$

$\Omega_r = \min\{|5|, |-2|\} = 2$.

a) Butterforth filter:

The order of normalised Butterworth filter is

$$N = \frac{\log\sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\dfrac{\Omega_s}{\Omega_p}}$$

Given

$\alpha_p = 3dB$, $\alpha_s = 20dB$

$\frac{\Omega_s}{\Omega_p} = \Omega_r = 2$.

$$= \frac{\log\sqrt{\dfrac{10^2-1}{10^{0.3}-1}}}{\log 2} = \frac{0.9978}{0.3010} = 3.32. \simeq 4$$

Take $N=4$

The transfer function of Butterworth filter is

$$H(s) = \frac{1}{(s^2+0.765s+1)(s^2+1.848s+1)}$$

To get the T/F of Bandstop filter, use the transformation

$$s \longrightarrow \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u\Omega_l}$$

$$s \longrightarrow \frac{s(1000\pi)}{s^2 + 24 * 10^4\pi^2}$$

$$H_a(s) = H(s)\Big|_{s \to \frac{1000\pi s}{s^2+24\times10^4\pi^2}}$$

$$= \frac{(s^2+24\times10^4\pi^2)^4}{\left[(1000\pi)^2s^2+(0.765\times1000\pi s)(s^2+24\times10^4\pi^2)+(s^2+24\times10^4\pi^2)^2\right]\left[(1000\pi)^2s^2+1.848\times1000\pi s\,(s^2+24\times10^4\pi^2)+(s^2+24\times10^4\pi^2)^2\right]}$$

b. Chebyshev filter:

The order of chebyshev filter

$$N = \frac{\cosh^{-1}\sqrt{\dfrac{10^2-1}{10^{0.3}-1}}}{\cosh^{-1}(2)} = 2.75 \simeq 3$$

Take $N=3$

$$\varepsilon = (10^{0.1\alpha_p}-1)^{0.5} = (10^{0.11}-1)^{0.5} = 0.5368.$$

$$\lambda = (10^{0.1\alpha_s}-1)^{0.5} = (10^2-1)^{0.5} = 9.95$$

$$\mu = \varepsilon^{-1}+\sqrt{1+\varepsilon^{-2}} = 3.97$$

$$a = \Omega_p\left[\frac{\mu^{1/N}-\mu^{-1/N}}{2}\right] = 1\left[\frac{(3.97)^{1/3}-(3.97)^{-1/3}}{2}\right]$$

$$= 0.476.$$

$$b = \Omega_p\left[\frac{\mu^{1/N}+\mu^{-1/N}}{2}\right] = 1\left[\frac{(3.97)^{1/3}+(3.97)^{-1/3}}{2}\right]$$

$$= 1.107$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \quad ; \quad k = 1,2,3$$

$$\phi_1 = \frac{\pi}{2} + \frac{\pi}{6} = 120^\circ$$

$$\phi_2 = \pi/2 + \pi/2 = 180^\circ$$

$$\phi_3 = \frac{\pi}{2} + \frac{5\pi}{6} = 240^\circ$$

$$s_1 = a\cos\phi_1 + b\sin\phi_1 = 0.476\cos 120 + j\,1.107\sin 120$$
$$= -0.238 + j\,0.9586$$

$$s_2 = a\cos\phi_2 + b\sin\phi_2 = 0.476\cos\pi + j\,1.107\sin\pi$$
$$= -0.476$$

$$s_3 = a\cos\phi_3 + b\sin\phi_3 = 0.476\cos 240 + j\,1.107\sin 240$$
$$= -0.238 - j\,0.9586$$

Denominator of T/F

$$(s+0.476)\left\{(s+0.238)^2+(0.9586)^2\right\}$$

Numerator of T/F: s=0 in Dr

$$(0.476)\ (0.9755) = 0.46463$$

i.e., s=0 in $(s+0.476)\ (s^2+0.476s+0.975)$

$$T/F \quad H(s) = \frac{0.4643}{(s+0.476)\ (s^2+0.476s+0.9755)}$$

The T/F of Bandstop filter can be obtained by following Transformation

$$H(s)\Big|_{\,s \rightarrow \frac{s(1000\pi)}{s^2+24(10^4\pi^2)}}$$

$$H_a(s) = \frac{0.4643\,(s^2+24\times10^4\pi^2)^3}{[1000\pi s + 0.476\,(s^2+24\times10^4\pi^2)]\,[s^2(1000\pi)^2+0.476\,(1000\pi s)\,(s^2+24\times10^4\pi^2)+0.9755\,(s^2+24\times10^4\pi^2)^2]}$$

$$= \frac{0.4643\,(s^2+24\times10^4\pi^2)^3}{(0.476s^2+1000\pi s+1.1275\times10^6)\,(0.9755s^4+4.621\times10^6+5.47\times10^{12}+9.369\times10^6\pi^2 s^2+1495.4\,s^3+3.5\times10^9 s)}$$

$$H_a(s) = \frac{(s^2+24\times10^4\pi^2)^3}{(s^2+6600s+2.3687\times10^6)\,(s^4+1533s^3+1.45\times10^7 s^2+3.589\times10^9 s+5.6\times10^{12})}$$