* Fourier series of DT signals (DTFS)

$$x(n) = \sum_{k=0}^{N-1} C_k \cdot e^{\frac{j2\pi kn}{N}} = \sum_{k=0}^{N-1} C_k \cdot e^{j\omega_0 kn} = \sum_{k=0}^{N-1} C_k \cdot e^{j\omega_k n}$$

C_k - Fourier Co-eff ; $\omega_0 = \frac{2\pi}{N}$ fundamental freq of $x(n)$

$\omega_k = \omega_0 k = \frac{2\pi k}{N}$: k^{th} harmonic freq of $x(n)$

$C_k \cdot e^{j\omega_k n}$ = k^{th} harmonic Component of $x(n)$

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) \cdot e^{\frac{-j2\pi kn}{N}} \quad ; \; k=0 \text{ to } N-1$$

A periodic dT signal with fundamental period N can be decomposed into N harmonically related freq components.

$C_k = |C_k| \angle C_k$ $\quad k = 0$ to $N-1$

$|C_k|$ - mag $\quad \angle C_k$ - phase

plot of harmonic magnitude/phase of DT signal Vs k (or ω_k) is called freq spectrum.

Eg: $\quad x(n) = 3 \cos \frac{\pi n}{4}$

First check for periodicity

$$x(n+N) = 3 \cos \frac{\pi}{4}(n+N) = 3 \cos\left(\frac{\pi n}{4} + \frac{\pi N}{4}\right)$$

For periodicity $\frac{\pi N}{4}$ must be integer multiple of 2π

$$\frac{\pi N}{4} = 2\pi M \Rightarrow N = 8M$$

N is an integer for $M = 1, 2, 3 \ldots$

$\therefore M = 1 \Rightarrow N = 8$ fundamental period.

DTFS: $$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi k n}{N}} \; ; \; k = 0 \text{ to } N-1$$

DTFT : (FT)

(b)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$x(n) = \frac{1}{2\pi} \int_{0}^{2\pi} X(e^{j\omega}) . e^{j\omega n} . d\omega \quad n = -\infty \text{ to } \infty$$

FT exists only for signals that are absolutely summable $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$: Finite energy dT signal

Spectrum : $X(e^{j\omega}) = X_R(e^{j\omega}) + j\, X_I(e^{j\omega})$

$|X(e^{j\omega})|$ vs ω $\quad \angle X(e^{j\omega})$ vs ω.

FT of dT signal is unique in $-\pi$ to π (or 0 to 2π)

\# Analysis of LTI - DTS using DTFT *

i/p $x(n) \rightarrow X(e^{j\omega})$

o/p $y(n) \rightarrow Y(e^{j\omega})$

$$TF = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

Impulse Response : & TF

$y(n) = x(n) * h(n) \Rightarrow Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})$

$$\boxed{H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}} \rightarrow \text{Freq Response}$$

Response of LTI - DTS is $y(n) = F^{-1}\{X(e^{j\omega}) . H(e^{j\omega})$

Freq Response is continuous fn of ω & so it cannot be processed by digital s/m.

prop: 1. Freq Resp is periodic fn of ω with period 2π

2. $h(n)$ - Real

mag is symm ; phase is A. symm in $0 \le \omega \le 2\pi$

3. $h(n)$ - Complex

real ~~part~~ is - symm ; ~~phase~~ imag part - A. symm.

4. $h(n)$ is discrete , $H(e^{j\omega})$ is continuous.

Time Response analysis of DTS

Time domain behaviour of LTI-DTS for different i/p signals.

Two methods for analysing the response of Linear system to a given i/p signal.

i) Using Convolution Sum to find response $y(n)$

$$y(n) = x(n) * h(n)$$

ii) Direct solution of difference equation representing the system.

General form of difference eq'n of N^{th} order LTI DTS is

$$y(n) = -\sum_{k=1}^{N} a_k\, y(n-1) + \sum_{k=0}^{M} b_k\, x(n-k)$$

$\{a_k\}$ & $\{b_k\}$ are Constants/co-eff

Response or o/p of LTI DTS is

$$\boxed{\text{Total Response} = \text{Zero state response} + \text{Zero i/p response}}$$

Zero state response is the response of system to i/p alone when initial state of system is zero (relaxed at time $n=0$). Also called forced response

Zero i/p response is response when i/p is zero and depends on initial state of system. Also called natural response.

Eg: Consider 1st order difference equation

$$y(n) = a\,y(n-1) + x(n)$$

• let $x(n) = 0$; $n < 0$ & $y(-1)$ exists i.e initial Condition.

$y(-1) \neq 0$

$y(n)$ for $n \geq 0$ are

$n=0$ $\quad y(0) = a\,y(-1) + x(0)$

$n=1$ $\quad y(1) = a\,y(0) + x(1)$

$= a\left(a\,y(-1) + x(0)\right) + x(1)$

$= a^2 y(-1) + a\,x(0) + x(1)$

$n=2$ $\quad y(2) = a\,y(1) + x(2)$

$= a\left\{a^2 y(-1) + a\,x(0) + x(1)\right\} + x(2)$

$= a^3 y(-1) + a^2 x(0) + a\,x(1) + x(2)$

So in general for any n

$$y(n) = a^{n+1} y(-1) + a^n x(0) + a^{n-1} x(1) + \dots + x(n)$$

$$\boxed{y(n) = a^{n+1} y(-1) + \sum_{k=0}^{n} a^k x(n-k) \; ; \; n \geq 0}$$

o/p or response consists of two parts
1st term depends on initial Condition
2nd term depends on i/p signal.

* when $y(-1) = 0$, o/p $y(n)$ depends only on i/p.
Hence $y(n)$ is known as zero state response (or) forced response, given by

$$\boxed{y_f(n) = \sum_{k=0}^{n} a^k x(n-k) \; ; \; n \geq 0}$$

* when $x(n) = 0$ & System is non relaxed $y(-1) \neq 0$,
o/p depends only on initial State of System. Hence

$y(n)$ is called <u>zero i/p response</u> or <u>Natural response</u> given by

$$\boxed{y_n(n) = a^{n+1}\, y(-1) \;;\; n \geq 0}$$

I (B)

⇒ Rewriting difference equation of N^{th} order DTS

$$\boxed{\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{k=0}^{M} b_k\, x(n-k)} \quad -(1)$$

N is the order of difference equation

* Solution to LCCDE is $\boxed{y(n) = y_h(n) + y_p(n)}$ — (2)

$y_h(n)$ is homogenous or Complimentary solution

$y_p(n)$ is particular solution

<u>Natural / Zero i/p</u> Response - $y_n(n)$

$y_n(n)$ is solution of eq (1) with $x(n)=0$

i.e., for a DTS system natural response is solution of homogenous equation

$$\boxed{\sum_{k=0}^{N} a_k\, y(n-k) = 0} \quad -(3)$$

Solution is in the form of $\boxed{y_h(n) = \lambda^n}$ — (4)

Substituting (4) in (3) we get $\displaystyle\sum_{k=0}^{N} a_k \cdot \lambda^{n-k} = 0 \;;\; a_0 = 1$

$$\Rightarrow \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_N\lambda^{n-N} = 0$$

$$\lambda^{n-N}\left\{\lambda^N + a_1\lambda^{N-1} + a_2\lambda^{N-2} + \cdots + a_{N-1}\lambda + a_N\right] = 0$$

i.e. $\lambda^N + a_1\lambda^{N-1} + a_2\lambda^{N-2} + \dots + a_{N-1}\lambda + a_N = 0$ — (5)

Eq-(5) is a polynomial equation called Characteristic equation of the system and can be expressed as

$$(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)\dots(\lambda-\lambda_N) = 0 \quad -(6)$$

$\lambda_1, \lambda_2, \lambda_3 \dots \lambda_N$ are roots of characteristic eq (5) called Characteristic roots or Eigen Values of system.

Natural response depends on type of roots.

Real - lead to real exponential

Imaginary - lead to sinusoids

Complex - lead to Exponentially damped sinusoids.

* Distinct Roots : If $\lambda_1, \lambda_2, \dots \lambda_N$ are distinct then it has N solutions $C_1\lambda_1^n, C_2\lambda_2^n, \dots C_N\lambda_N^n$

General solution form is

$$\boxed{y_h(n) = C_1\lambda_1^n + C_2\lambda_2^n + \dots + C_N\lambda_N^n} \quad -(7)$$

$C_1, C_2 \dots C_N$ are arbitrary Constants obtained by applying initial conditions.

Ex: if $\lambda_1 = 2$ $\lambda_2 = 3$ then $y_h(n) = C_1(2)^n + C_2(3)^n$

* Repeated Roots : If root λ_1 is repeated m times and remaining $(N-m)$ roots are distinct. eq (6) can be written as $(\lambda-\lambda_1)^m(\lambda-\lambda_{m+1})(\lambda-\lambda_{m+2})\dots(\lambda-\lambda_N) = 0$

— (8)

the solution form is

$$y_h(n) = \left(C_1 + C_2 n + C_3 n^2 + \dots + C_m n^{m-1}\right)(\lambda_1)^n + C_{m+1}(\lambda_{m+1})^n + C_{m+2}(\lambda_{m+2})^n + \dots + C_N \lambda_N^n$$

Eg: $\lambda_1 = -2$, $\lambda_2 = -2$; $\lambda_3 = 2$ then

$$y_h(n) = (C_1 + C_2 n)(-2)^n + C_3 (2)^n$$

* Complex roots : $\lambda_1 = \lambda = a + jb$
$\lambda_2 = \lambda^* = a - jb$

Solution is $y_h(n) = r^n \left\{A_1 \cos n\theta + A_2 \sin n\theta\right]$

$r = \sqrt{a^2+b^2}$; $\theta = \tan^{-1}(b/a)$

A_1, A_2 are Constants.

prob: Find natural response of system described by difference equation $y(n) + 2y(n-1) + y(n-2) = x(n) + x(n-1)$ with initial condition $y(-1) = y(-2) = 1$

sol^n:- $y(n) + 2y(n-1) + y(n-2) = x(n) + x(n-1)$

Homogeneous eq^n can be obtained by equating i/p to zero.

$$y(n) + 2y(n-1) + y(n-2) = 0 \quad \text{— } A$$

Solution is of the form $y_h(n) = \lambda^n$

$$\Rightarrow \quad \lambda^n + 2\lambda^{n-1} + \lambda^{n-2} = 0$$

$$\lambda^{n-2}\left\{\lambda^2 + 2\lambda + 1\right] = 0$$

$$\lambda^2 + 2\lambda + 1 = 0 \Rightarrow (\lambda+1)^2 = 0$$

$$\lambda_1 = -1 \; ; \; \lambda_2 = -1$$

$\therefore$ Roots are not distinct, (repeated) general form of homogeneous soln

is $y_h(n) = c_1(-1)^n + c_2 n(-1)^n$

$$\left.\begin{array}{ll} n=0; & y(0) = c_1 \\ n=1 & y(1) = -c_1 - c_2 \end{array}\right\} \quad (B)$$

From eq (A) for $n=0 \Rightarrow y(0) + 2y(-1) + y(-2) = 0$

given $y(-1) = y(-2) = 1$

$\Rightarrow y(0) + 2 + 1 = 0 \Rightarrow \boxed{y(0) = -3}$

for $n=1 \Rightarrow y(1) + 2y(0) + y(-1) = 0$

$y(1) + (2\times -3) + 1 = 0$

$\Rightarrow \boxed{y(1) = 5}$

Substituting values in (B)

$c_1 = -3$

$-c_1 - c_2 = 5 \Rightarrow c_2 = -2$

$\therefore$ Natural Response

$y_h(n) = -3(-1)^n - 2n(-1)^n \; ; \; n \geq 0$

$$\boxed{y_h(n) = -3(-1)^n u(n) - 2n(-1)^n u(n)}$$

Note :- For Natural Response initial conditions are not zero. (i.e have to find homogenous solution with given initial conditions)

Forced Response $y_f(n) = y_h(n) + y_p(n)$

$$\Rightarrow y_f(n) = C_1(-1)^n + C_2 \cdot n \cdot (-1)^n + \frac{1}{3}\left(\frac{1}{2}\right)^n \quad -(13)$$

$n=0$ $\quad y(0) = C_1 + \frac{1}{3}$

$n=1$ $\quad y(1) = -C_1 - C_2 + \frac{1}{6}$ $\quad \Big\} - (14)$

From difference equation given

$n=0$; $y(0) + 2y(-1) + y(-2) = x(0) + x(-1)$

$\left(y(-1) = y(-2) = 0\right.$ bcoz homogenous to be found for initial condition set to zero$\left.\right)$

$$\Rightarrow \boxed{y(0) = 1}$$

$n=1$ $\quad y(1) + 2y(0) + y(-1) = x(1) + x(0)$

$$y(1) + 2 = 1 + \frac{1}{2}$$

$$\Rightarrow \boxed{y(1) = -1/2}$$

Substituting in (14) and solving

$$C_1 + \frac{1}{3} = 1 \Rightarrow C_1 = \frac{2}{3}$$

$$-C_1 - C_2 + \frac{1}{6} = -1/2 \Rightarrow C_1 + C_2 = \frac{2}{3}$$

$$\Rightarrow C_2 = 0$$

$$\therefore \boxed{y_f(n) = \frac{2}{3}(-1)^n + \frac{1}{3}\left(\frac{1}{2}\right)^n} ; n \geq 0$$

(or)

$$y_f(n) = \left[\frac{2}{3}(-1)^n + \frac{1}{3}\left(\frac{1}{2}\right)^n\right] u(n)$$

Total Response : $y(n) = y_n(n) + y_f(n)$

Note-1 : $y_n(n)$ is $y_h(n)$ with initial conditions

$y_f(n) = y_h(n) + y_p(n)$

$\hookrightarrow$ with initial conditions set to zero

$$\therefore \quad y(n) = y_n(n) + [y_h(n) + y_p(n)]$$

(or)

$$y(n) = y_h(n) + [y_h(n) + y_p(n)]$$

$\hookrightarrow$ initial conditions set to zero

$\hookrightarrow$ initial conditions $\neq 0$.

$$y(n) = y_n(n) + y_p(n)$$

Note-2

If there is no need to find forced response & natural response separately, the total response can be found in the same way as forced response without zero initial conditions.

i.e total response $y(n) = y_f(n) = y_h(n) + y(n)$

with initial conditions.

Prob: Find forced response of system described by:

$$y(n) + 2y(n-1) + y(n-2) = x(n) + x(n-1)$$

for i/p $x(n) = \left(\frac{1}{2}\right)^n u(n)$

Sol:

$$y_f(n) = y_h(n) + y_p(n)$$

i) $y_h(n)$:- Homogeneous sol is of form $y_h(n) = \lambda^n$ for i/p is zero.

i.e $\lambda^n + 2\lambda^{n-1} + \lambda^{n-2} = 0$

$$\lambda^{n-2}(\lambda^2 + 2\lambda + 1) = 0$$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$\lambda_1 = -1 \quad \lambda_2 = -1$$

$$\boxed{y_h(n) = C_1(-1)^n + C_2 \cdot n \cdot (-1)^n} \quad -(9)$$

ii) $y_p(n)$:- for i/p $x(n) = \left(\frac{1}{2}\right)^n u(n)$ the particular solution form is $\boxed{y_p(n) = k \cdot \left(\frac{1}{2}\right)^n u(n)}$ $-(10)$

Substituting in given difference equation we get

$$k\left(\frac{1}{2}\right)^n + 2k\left(\frac{1}{2}\right)^{n-1} + k\left(\frac{1}{2}\right)^{n-2} = \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n-1} \quad -(11)$$

For $n=2$ none of the terms in (11) will vanish

$$\Rightarrow \quad k\left(\frac{1}{2}\right)^2 + 2k\left(\frac{1}{2}\right) + k = \frac{1}{4} + \frac{1}{2}$$

$$\Rightarrow \quad k = 1/3$$

$$\therefore \quad \boxed{y_p(n) = \frac{1}{3}\left(\frac{1}{2}\right)^n} \quad -(12)$$

Forced Response (Zero state Response)

Forced response is the solution of difference equation for given i/p when initial Conditions are zero.
It Consists of two parts, homogenous solution and particular solution.

$$\boxed{y_f(n) = y_h(n) + y_p(n)}$$ —

$y_h(n)$ Can be obtained from the roots of characteristic equation.

$y_p(n)$ is to satisfy the difference equation for specific i/p signal $x(n)$; $n \geq 0$ i.e $y_p(n)$ is a solution satisfying

$$1 + \sum_{k=1}^{N} a_k\, y_p(n-k) = \sum_{k=0}^{M} b_k\, x(n-k) \quad \text{—(1)}$$

General form of particular solution for different i/p's are

$x(n)$ i/p signal	$y_p(n)$ particular solution
A (step i/p)	K
$A M^n$	$K M^n$
$A n^M$	$k_0 n^M + k_1 n^{M-1} + \ldots + k_M$
$A^n n^M$	$A^n\left[k_0 n^M + k_1 n^{M-1} + \ldots + k_M\right]$
$A\cos\omega n$, $A\sin\omega n$	$C_1 \cos\omega n + C_2 \sin\omega n$

Step Response

Using Convolution o/p is $y(n) = x(n) * h(n)$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

$x(n)$ is i/p ; $h(n)$ is impulse response

Step response is o/p when $x(n) = u(n)$

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \quad ; \quad u(n-k) = \begin{cases} 1 & n \geq k \\ 0 & \text{otherwise} \end{cases}$$

Step response $\quad s(n) = \sum_{k=0}^{\infty} h(k)\, u(n-k)$

$$= \sum_{k=0}^{n} h(k)$$

i.e step response is impulse response

Prob: Find step response of impulse response

$h(n) = \alpha^n u(n)$, $0 < \alpha < 1$

Sol: Step response is given by

$$s(n) = h(n) * u(n)$$

$$s(n) = \sum_{k=-\infty}^{\infty} h(k)\, u(n-k)$$

$$= \sum_{k=-\infty}^{\infty} \left(\alpha^k u(k)\right) u(n-k)$$

$$= \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}$$

Prob: Find step response of $h(n) = \delta(n-2) - \delta(n-1)$..

Sol^n :
$$s(n) = h(n) * u(n)$$

$$s(n) = [\delta(n-2) - \delta(n-1)] * u(n)$$

$$= [\delta(n-2) * u(n)] - [\delta(n-1) * u(n)]$$

$$s(n) = u(n-2) - u(n-1)$$

Prob: Impulse response of LTI system is $h(n) = (\frac{1}{3})^n u(n)$ when $x(n) = u(n)$

Det o/p of s/m $y(n)$ at $n = -2, 2, 4$

Sol^n Convolution $y(n) = x(n) * h(n) = h(n) * x(n)$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} u(k) \left[\frac{1}{3}^{n-k} u(n-k)\right]$$

$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$$\sum_{k=0}^{n} \left(\frac{1}{3}\right)^{n-k}$$

$u(n-k) = \begin{cases} 1 & n \geq k \\ 0 & \text{otherwise} \end{cases}$

$n = -2$; $y(-2) = 0$

$n = 2$; $y(2) = \sum_{k=0}^{2} (1/3)^{2-k} = \frac{1}{9} \sum_{k=0}^{2} 3^k$

$$= \frac{1}{9}\left[\frac{1-3^3}{1-3}\right] = 13/9$$

$n = 4$; $y(4) = \sum_{k=0}^{4} (1/3)^{4-k} = \left(\frac{1}{3}\right)^4 \sum_{k=0}^{4} 3^k$

$$= \frac{1}{81}\left[\frac{1-3^5}{1-3}\right] = \frac{121}{81}$$

Relation b/n DFT & FT

F.T is given by $X(e^{j\omega})$ for a finite duration sequency $x(n)$ having length N.

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n)\, e^{-j\omega n}$$

$X(e^{j\omega})$ is continuous function of ω

DFT of $x(n)$ is $X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$ $k = 0$ to $N-1$

Comparing above two equations

$$X(k) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}} \quad ; k = 0 \text{ to } N-1$$

Relation b/n DFT & ZT

ZT of $x(n)$ is given by

$$X(z) = \sum_{n=0}^{N-1} x(n) \cdot z^{-n}$$

IDFT of $X(k)$ is $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi kn}{N}}$

$$X(z) = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi kn}{N}} \right] z^{-n}$$

$$= \sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} \left(e^{\frac{j2\pi k}{N}} \cdot z^{-1} \right)^{n}$$

$$= \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{\frac{j2\pi k}{N}} \cdot z^{-1}}$$

$$X(k) = X(z) \Big|_{z = e^{\frac{j2\pi k}{N}}} \; ; \; k = 0 \text{ to } N-1$$

Prob: Find Z.T of $x(n) = u(n) - u(n-8)$ and sample it at 6 points on unit circle. Find IDFT & comment on result.

Sol^n:

$$x(n) = u(n) - u(n-8)$$

$$= \{1, 1, 1, 1, 1, 1, 1, 1\} \qquad L = 8.$$

$$X(z) = \sum_{n=0}^{7} x(n) \cdot z^{-n}$$

$$= x(0) + x(1) z^{-1} + x(2) z^{-2} + \dots + x(7) z^{-7}$$

$$X(k) = X(z) \Big|_{z = e^{\frac{j2\pi k}{6}}} \quad \text{for } k = 0, 1, 2, 3, 4, 5$$

$$\text{(or)} \quad z = e^{\frac{j\pi k}{3}}$$

$$= 1 + e^{-\frac{j\pi k}{3}} + e^{-\frac{j2\pi k}{3}} + e^{-j\pi k} + e^{-\frac{j4\pi k}{3}} + e^{-\frac{j5\pi k}{3}} + e^{-\frac{j6\pi k}{3}} + e^{-\frac{j7\pi k}{3}}$$

$$k = 0 \text{ to } 5$$

$$\Rightarrow X(k) = 2 + 2e^{-\frac{j\pi k}{3}} + e^{-\frac{j2\pi k}{3}} + e^{-j\pi k} + e^{-\frac{j4\pi k}{3}} + e^{-\frac{j5\pi k}{3}}$$

IDFT of $X(k)$ is $x'(n) = \{2, 2, 1, 1, 1, 1\}$

$x'(n) \neq x(n)$, time domain aliasing occurred in 1st two points as $X(z)$ is not sampled with sufficient no. of points on unit circle.

$L = 8$; $N = 6$ ∴ as $N < L$ time domain aliasing has occurred.

Properties of DFT

* Circular shift of a sequence

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$

$x_p(n)$ can be visualized as wrapping finite duration sequence $x(n)$ around a circle in anticlockwise direction (positive direction)

Repeatedly transversing samples on circumference of circle, the finite length sequence is periodically repeated on a circular (Modulo N) time axis.

$$x_p(n) = x((n \text{ Modulo } N)) \qquad (N > L)$$

If argument 'n' is b/n 0 & $N-1$ then leave as it is. Otherwise add / substract multiples of 'N' to 'n' until the result is b/n 0 and $N-1$.

Eg: i) $x(-3 \text{ Modulo } 4) = x(1)$

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(-3 - l4) \qquad \text{for } l = +1$$

$$= x(1)$$

ii) $x(10 \text{ Mod } 8) = x(2)$

$$x_p(n) = \sum x(10 - l8) \qquad l = 1$$

$$= x(2)$$

iii) $x(-11 \text{ Mod } 4) = x(1)$

$$x_p(n) = \sum x(-11 - l4) \qquad l = -3$$

$$= x(1)$$

$$\Rightarrow \boxed{x_p(n) = x((n))_N} \qquad x(n \text{ Mod } N)$$

$$\Rightarrow \boxed{X_p(k) = X((k))_N}$$

* Shifted version of $x_p(n)$ can be written as $x_p(n-k)$, k represents the no of times that sequence $x_p(n)$ is shifted to right.

i.e $$x_p'(n) = x_p(n-k) = \sum_{l=-\infty}^{\infty} x(n - lN - k)$$

Finite duration sequence $x'(n) = x_p'(n)$; $0 \le n \le N-1$

$= 0$ otherwise

$x'(n)$ is related to $x(n)$ by a circular shift.

Linear shift of periodic sequence $x_p(n)$ is equal to shifting the sequence $x(n)$ circularly in anticlockwise direction. Finite duration sequence $x'(n)$ can be obtained by circular shift of $x(n)$ in anticlockwise direction.

$$x(n) = \{x(0),\ x(1),\ x(2) \ldots x(N-2),\ x(N-1)\}$$

$$x((n-1))_N = \{x(N-1),\ x(0),\ x(1) \ldots x(N-3),\ x(N-2)\}$$

$$x((n-2))_N = \{x(N-2),\ x(N-1),\ x(0) \ldots x(N-4),\ x(N-3)\}$$

$$\vdots$$

$$x((n-N))_N = \{x(0),\ x(1),\ x(2) \ldots x(N-2),\ x(N-1)\}$$

$$\Rightarrow \boxed{x((n-N))_N = x(n)}$$

$$\Rightarrow \boxed{x((n-m))_N = x(N-m+n)}$$

Statement: If $DFT\{x(n)\} = X(k)$

then $DFT\{x((n-m))_N\} = e^{\frac{-j2\pi km}{N}} \cdot X(k)$

Proof: $$DFT\{x((n-m))_N\} = \sum_{n=0}^{N-1} x((n-m))_N\, e^{\frac{-j2\pi kn}{N}}$$

$$= \sum_{n=0}^{m-1} x((n-m))_N \, e^{\frac{-j2\pi kn}{N}} + \sum_{n=m}^{N-1} x((n-m))_N \cdot e^{\frac{-j2\pi kn}{N}}$$

$$x((n-m))_N = x(N-m+n)$$

Consider 1st term

$$\sum_{n=0}^{m-1} x((n-m))_N \, e^{\frac{-j2\pi kn}{N}} = \sum_{n=0}^{m-1} x(N-m+n) \, e^{\frac{-j2\pi kn}{N}}$$

let $N-m+n = l$, then

$$= \sum_{l=N-m}^{N-1} x(l) \, e^{\frac{-j2\pi k(-N+m+l)}{N}}$$

$$= \sum_{l=N-m}^{N-1} x(l) \cdot e^{\frac{-j2\pi k(l+m)}{N}} \qquad \left(e^{j2\pi k} = 1 \text{ for } k = 0, 1, 2 \ldots \right)$$

$^{\prime}$$\!/\!/$ly 2nd term

$$\sum_{n=m}^{N-1} x((n-m))_N \, e^{\frac{-j2\pi kn}{N}} = \sum_{l=0}^{N-1-m} x(l) \cdot e^{\frac{-j2\pi k(m+l)}{N}}$$

$$\therefore \; DFT\{x((n-m))_N\} = \sum_{l=N-m}^{N-1} x(l) \, e^{\frac{-j2\pi k(m+l)}{N}} + \sum_{l=0}^{N-m-1} x(l) \cdot e^{\frac{-j2\pi k(m+l)}{N}}$$

$$= e^{\frac{-j2\pi km}{N}} \sum_{l=0}^{N-1} x(l) \cdot e^{\frac{-j2\pi kl}{N}}$$

$$= e^{\frac{-j2\pi km}{N}} \sum_{n=0}^{N-1} x(n) \cdot e^{\frac{-j2\pi kn}{N}} \qquad \left(\text{replacing dummy var } l = n\right)$$

$$= e^{\frac{-j2\pi k m}{N}} X(k)$$

$$\therefore \quad DFT\left\{x((n-m))_N\right\} = e^{\frac{-j2\pi k m}{N}} \cdot X(k)$$

Filter Method or Section Convolution

The response of an LTI system for any arbitrary signal is given by linear convolution of i/p and impulse response of the system. If any one of the sequences is very much larger than the other, then it is very difficult to compute linear convolution for the following reasons.

* Entire sequence should be available before convolution can be carried out. This makes long delay in getting the o/p.
* Large amounts of memory is required to store the sequences.

There are two methods to overcome above problems. The largest sequence is sectioned or splitted into the size of smaller sequences. Convolution of each section of longer sequence and smaller sequence is performed. The o/p sequences obtained from convolutions of all sections are combined to get the overall o/p sequence. There are two methods of sectioned convolution.

i) Over lap - Save
ii) Over lap - Add } Methods.

i) Overlap Save Method

The results of linear convolution are obtained by circular convolution. Each section of longer sequence and smaller sequence are converted to size of o/p sequence. Normally longer sequence is divided into sections of size same as smaller sequence.

Let $h(n)$ size is 'M' and i/p $x(n)$ is long data sequence. Assume 'L' is size that ~~new~~ ~~data~~ sequence obtained by subdividing longer size sequence. The new data sequence is $N = L+M-1$. N point DFT is used for both and o/p is

$$Y_{new}(k) = X_{new}(k) \cdot H(k) \quad k=0 \text{ to } N-1$$

To match lengths of each data block, $h(n)$ is added with L-1 zeroes. O/p data sequence is obtained by IDFT. As data length is N, first M-1 points of $y_{new}(n)$ are corrupted by aliasing and must be discarded. Last L-points of $y_{new}(n)$ are exactly same as the result from linear convolution. To avoid aliasing, last (M-1) data points of each data block is saved and these (M-1) data points become first (M-1) data points of subsequent data block.

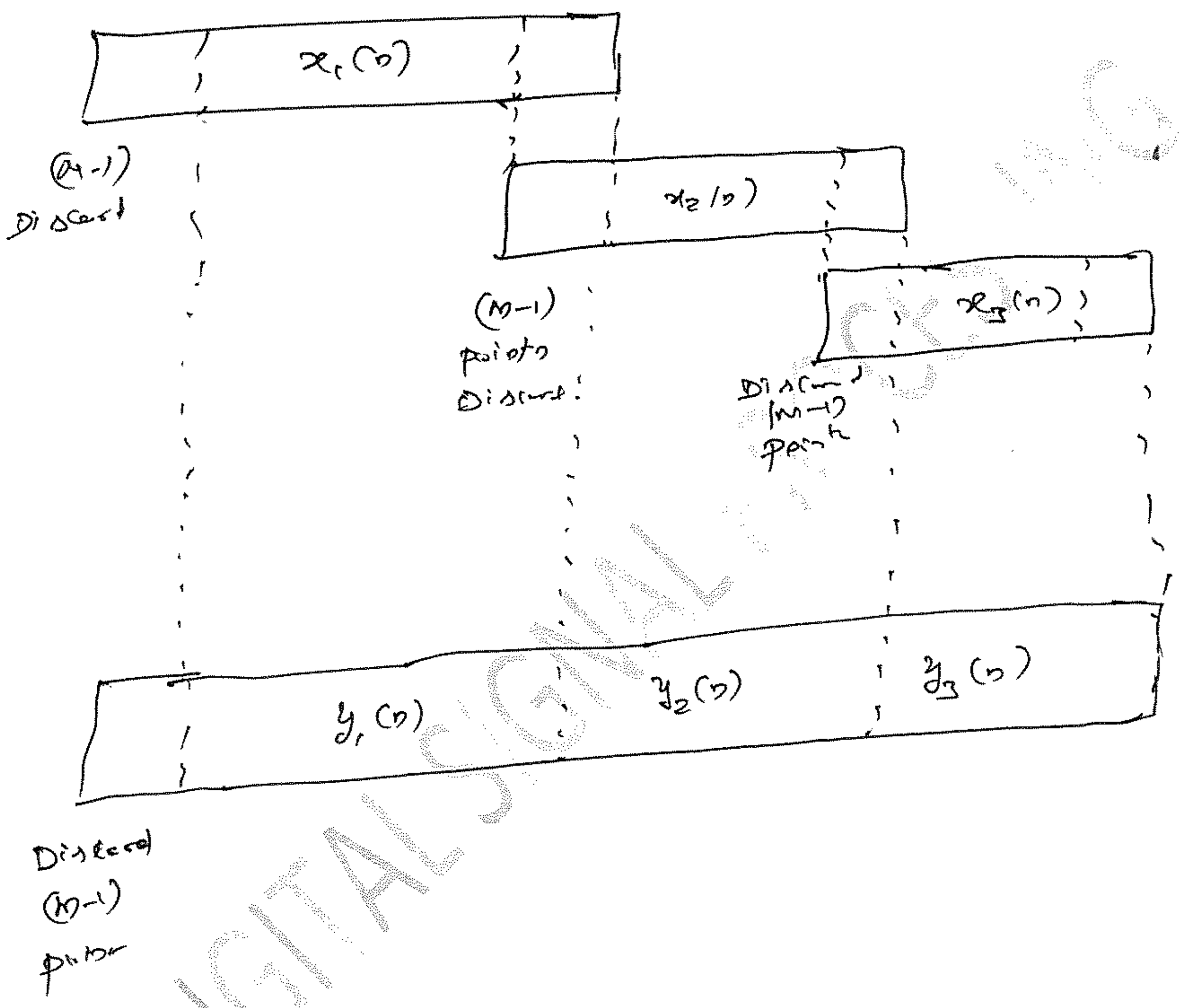
$$x_1(n) = \underbrace{0, 0, 0 \ldots 0}_{(M-1) \text{ zeros}}, x(0)\, x(1) \ldots x(L-1)$$

$$x_2(n) = \underbrace{x(L-M+1), x(L-M+2) \ldots, x(L-1)}_{(M-1) \text{ new data points from } x_1(n)}, \underbrace{x(L) \ldots x(2L-1)}_{L \text{ New data points}}$$

and so on

O/p of each subblock is $y_i(n) = x_i(n) \circledN h(n)$

First $(M-1)$ points in each subblock o/p has to be discarded and construct the actual o/p of the system which is equal to Linear Convolution of system.



Prob: Determine the o/p of Linear FIR filter whose impulse response $h(n) = \{1, 2, 3\}$ and i/p signal $x(n) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ using overlap save Method.

Sol^n:
$$x(n) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$h(n) = \{1, 2, 3\} \Rightarrow M = 3$$

Divide i/p data sequence $x(n)$ in to $L=3$ data subblocks. length of $h(n)$ is $M=3$

Add $(M-1)$ zeroes to 1^{st} sub block of data

$$x_1(n) = \{ \underbrace{0, 0}_{\substack{M-1 \\ \text{zeroes}}}, \underbrace{1, 2, 3}_{\substack{L=3 \\ \text{data points}}} \}$$

$\therefore \quad N = L+M-1 = 3+3-1 = 5$

$$x_2(n) = \{ \underbrace{2, 3}_{\substack{(M-1)\text{ data} \\ \text{from } x_1(n) \\ \text{data block}}}, \underbrace{4, 5, 6}_{\text{New data}} \}$$

$$x_3(n) = \{ \underbrace{5, 6}_{\substack{(M-1)\text{ data} \\ \text{from } x_2(n)}}, \underbrace{7, 8, 9}_{\text{New data}} \}$$

$$x_4(n) = \{ \underbrace{8, 9}_{\substack{(M-1)\text{ data} \\ \text{from } x_3(n)}}, \underbrace{0, 0, 0}_{\substack{\text{zeros to} \\ \text{make length} = 5}} \}$$

To perform circular convolution length of $h(n)$ must be 5

$\therefore \quad h(n) = \{1, 2, 3, 0, 0\}$

then $\quad x_1(n) \; N \; h(n) = y_1(n)$

$x_2(n) \; N \; h(n) = y_2(n)$

$x_3(n) \; N \; h(n) = y_3(n)$

$x_4(n) \; N \; h(n) = y_4(n)$

* $y_1(n) = x_1(n) \circledast h(n)$

$$\begin{bmatrix} 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ 1 & 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 12 \\ 9 \\ 1 \\ 4 \\ 10 \end{bmatrix}$$

$y_1(n) = \{12, 9, 1, 4, 10\}$

* $y_2(n) = x_2(n) \circledast h(n)$

$$\begin{bmatrix} 2 & 6 & 5 & 4 & 3 \\ 3 & 2 & 6 & 5 & 4 \\ 4 & 3 & 2 & 6 & 5 \\ 5 & 4 & 3 & 2 & 6 \\ 6 & 5 & 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 29 \\ 25 \\ 16 \\ 22 \\ 28 \end{bmatrix}$$

$y_2(n) = \{29, 25, 16, 22, 28\}$

* $y_3(n) = x_3(n) \circledast h(n)$

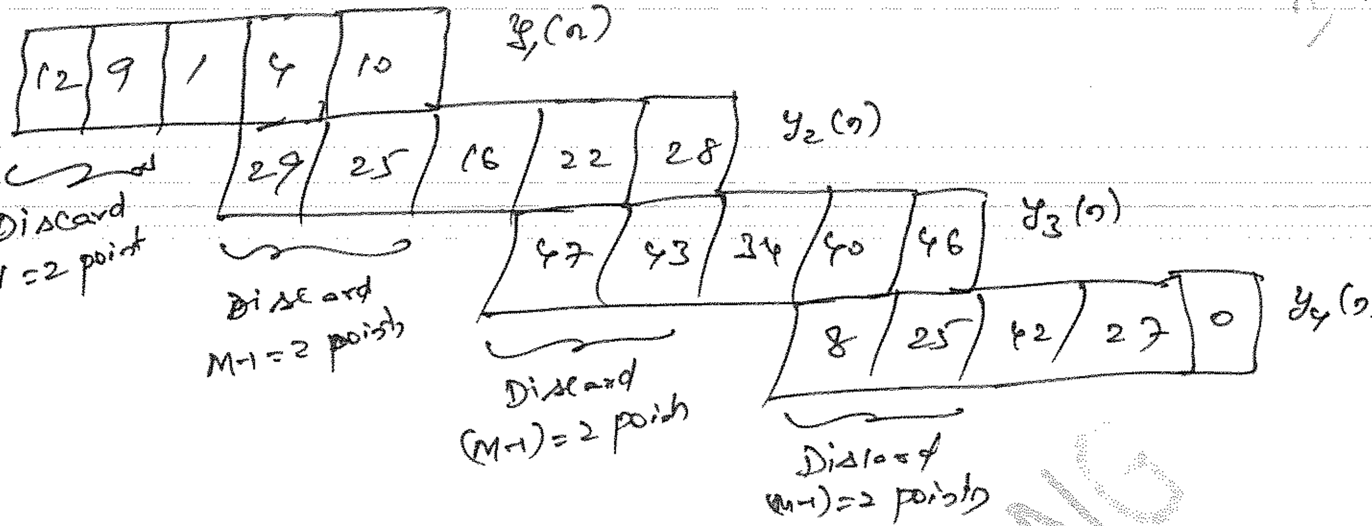
$$\begin{bmatrix} 5 & 9 & 8 & 7 & 6 \\ 6 & 5 & 9 & 8 & 7 \\ 7 & 6 & 5 & 9 & 8 \\ 8 & 7 & 6 & 5 & 9 \\ 9 & 8 & 7 & 6 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 47 \\ 43 \\ 34 \\ 40 \\ 46 \end{bmatrix}$$

$y_3(n) = \{47, 43, 34, 40, 46\}$

* $y_4(n) = x_4(n) \circledast h(n)$

$$\begin{bmatrix} 8 & 0 & 0 & 0 & 9 \\ 9 & 8 & 0 & 0 & 0 \\ 0 & 9 & 8 & 0 & 0 \\ 0 & 0 & 9 & 8 & 0 \\ 0 & 0 & 0 & 9 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 25 \\ 42 \\ 27 \\ 0 \end{bmatrix}$$

$y_4(n) = \{8, 25, 42, 27, 0\}$



$$y(n) = \{1, 4, 10, 16, 22, 28, 34, 40, 46, 42, 27\}$$

Note! Aliasing of data points are completely removed

(ii) Overlap - Add Method

Consider a long i/p data segment. Assume the length of impulse response of system to be 'M'. Subdivide the i/p data segment to size L so that new data sequence of length $N = L + M - 1$. The impulse response of the system is increased in length by appending L-1 zeros to match length of each data subblock.

Here each data block is appended (M-1) zeros at the end. Data blocks are represented as

$$x_1(n) = \{x(0), x(1), \ldots x(L-1), \underbrace{0, 0, 0}_{\substack{(M-1)\text{ zeros} \\ \text{added}}}\}$$

$$x_2(n) = \{\underbrace{x(L), x(L+1) \ldots x(2L-1)}_{L\text{- New data points}}, \underbrace{0, 0, 0}_{\substack{(M-1)\text{ zeros} \\ \text{added}}}\}$$

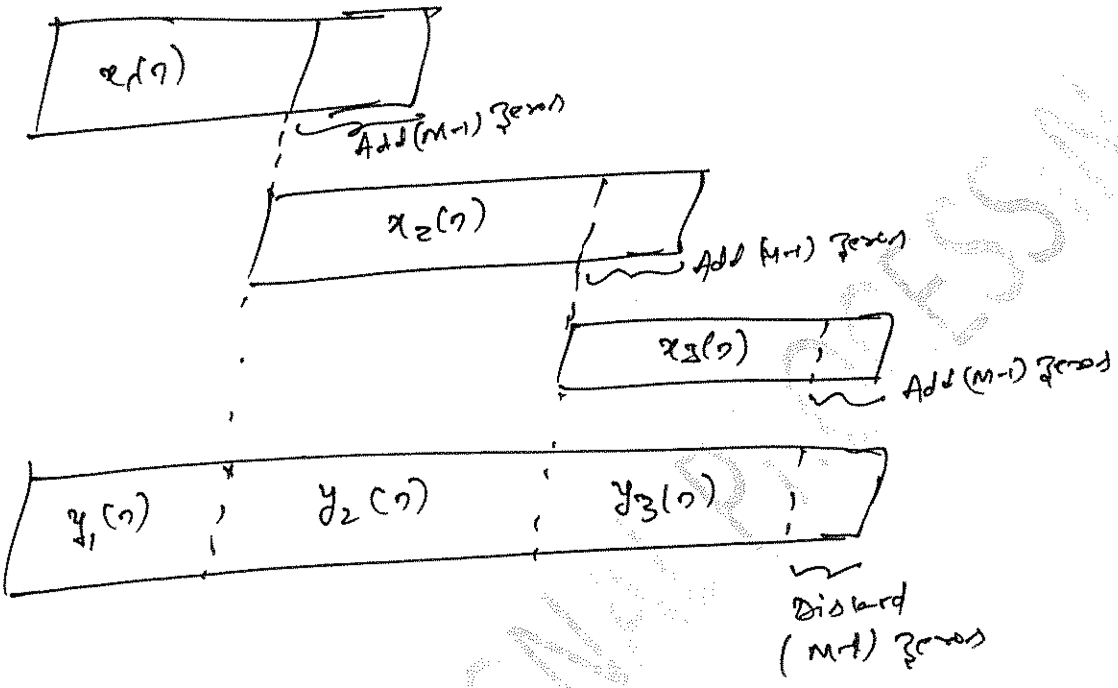
and so on

O/p of each subblock is given by

(8)

$$y_i(n) = x_i(n) \; (N) \; h(n)$$

Last $(M-1)$ points in each subblock o/p is to be ~~discarded~~ and construct the actual o/p of system, which is equal to Linear Convolution of system.



Prob: Det the o/p of Linear FIR filter whose impulse response is $h(n) = \{1, 2, 3\}$ and i/p signal is

$x(n) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Sol^n:

$$x(n) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$h(n) = \{1, 2, 3\} \Rightarrow M = 3$$

$x(n)$ is divided into 3 data subblocks of size '3'.

Add $(M-1)$ zeros at end of each subblocks.

$$x_1(n) = \{1, 2, 3, \underbrace{0, 0}_{(M-1)\text{ zero's}}\}$$

$$x_2(n) = \{4, 5, 6, \underbrace{0, 0}_{(M-1)\text{ zero's}}\} \qquad x_3(n) = \{7, 8, 9, 0, 0\}$$

To perform circular convolution h(n) length must be equal to length of data subblock.

$$h(n) = \{1, 2, 3, \underbrace{0, 0}_{\text{zeros added}}\}$$

Circular convolution of each data subblock with impulse response.

$$y_1(n) = x_1(n) \circledast h(n)$$
$$y_2(n) = x_2(n) \circledast h(n)$$
$$y_3(n) = x_3(n) \circledast h(n)$$

$$y_1(n) = x_1(n) \circledast h(n)$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4+0+0+0 \\ 5+8+0+0 \\ 6+10+12+0 \\ 0+12+15+0+0 \\ 0+0+18+0+0 \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \\ 28 \\ 27 \\ 18 \end{bmatrix}$$

$$y_1(n) = \{4, 13, 28, 27, 18\}$$

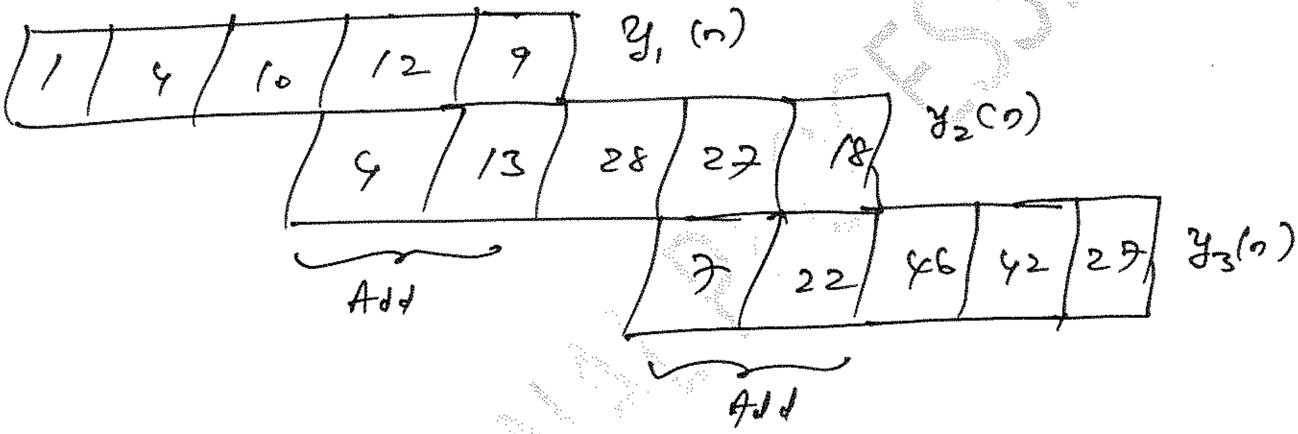
$$y_3(n) = x_3(n) \circledast h(n)$$

$$\begin{bmatrix} 7 & 0 & 0 & 9 & 8 \\ 8 & 7 & 0 & 0 & 9 \\ 9 & 8 & 7 & 0 & 0 \\ 0 & 9 & 8 & 7 & 0 \\ 0 & 0 & 9 & 8 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7+0+0+0+0 \\ 8+14+0+0+0 \\ 9+16+21+0+0 \\ 0+18+24+0+0 \\ 0+0+27+0 \end{bmatrix} = \begin{bmatrix} 7 \\ 22 \\ 46 \\ 42 \\ 27 \end{bmatrix}$$

$$y_3(n) = \{7, 22, 46, 42, 27\}$$

$$y_2(n) = x_2(n) \circledast h(n)$$

$$\begin{bmatrix} 4 & 0 & 0 & 6 & 5 \\ 5 & 4 & 0 & 0 & 6 \\ 6 & 5 & 4 & 0 & 0 \\ 0 & 6 & 5 & 4 & 0 \\ 0 & 0 & 6 & 5 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4+0+0+0+0 \\ 5+8+0+0+0 \\ 6+10+12+0+0 \\ 0+12+15+0+0 \\ 0+0+18+0+0 \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \\ 28 \\ 27 \\ 18 \end{bmatrix}$$



$$y(n) = \{1, 4, 10, 16, 22, 28, 34, 40, 46, 42, 27\}$$