UNIT-IV

FREQUENCY RESPONSE ALALYSIS

Consider a linear system with a sinusoidal input $r(t) = A\sin\omega t \longrightarrow (1)$

Under steady-state, the system output as well as the signals at other points in the system are sinusoidal. The steady-state may be written as

$$c(t) = B\sin(\omega t + \phi) \longrightarrow (2)$$

The magnitude and phase relationship between the sinusoidal input and the steady-state output of a system is termed as frequency response.

In linear time-invariant systems, the frequency response is independent of the amplitude and phase of the input signal.

The frequency response test on a system is normally performed by keeping the amplitude 'A' fixed and determining 'B' and 'ϕ' for a suitable range of frequencies.

The frequency response is easily evaluated from the sinusoidal transfer function which can be obtained by replacing 's' by 'jω' in the system transfer function T(s). The transfer function thus obtained T(jω) is a complex function of frequency and has both magnitude and phase angle. These characteristics are conveniently represented by graphical plots.

Advantages of Frequency Response Analysis

The ease and accuracy of measurements are some of the advantages of the frequency response method.

(1) wherever it is not possible to obtain the form of the transfer function of a system through analytical techniques, the necessary information to compute its transfer function can be extracted by performing the frequency response test on the system.

The step response test can also be performed easily but the extraction of transfer function from step response data is quite a laborious procedure.

(2) The design and parameter adjustment of the open-loop transfer function of a system for a specified closed loop performance is carried out somewhat more easily in frequency domain than in time domain.

(3) The effects of noise disturbance and parameter variations are relatively easy to visualize and assess through frequency response.

(4) The Absolute and relative stability of the closed loop systems can be estimated from the knowledge of their open loop frequency response.

(5) The practical testing of systems can be easily carried with available sinusoidal signal generators and precise measurement equipments.

(6) The frequency response analysis and designs can be extended to certain non linear control systems.

Disadvantages of Frequency Response Analysis:

(1) For systems with very large time-constants, the frequency response test is cumbersome to perform as the time required for the output to reach steady-state for each frequency of the test signal is excessively long. Therefore, the frequency response test is not recommended for systems with very large time constants.

(2) Frequency response obviously can not be performed on non-interruptable systems. under such circumstances a single shot test (step or impulse) is more convenient.

Frequency Domain Specifications: The performance and characteristics of a system in frequency domain are measured in terms of frequency domain specifications. The requirements of a system to be designed are usually specified in terms of these specifications.

The frequency domain specifications are

(1) Resonant peak, M_r (2) Resonant Frequency ω_r

(3) Bandwidth ω_b (4) cut-off rate

(5) Gain Margin (6) phase margin

Let us consider a second order system shown in figure.

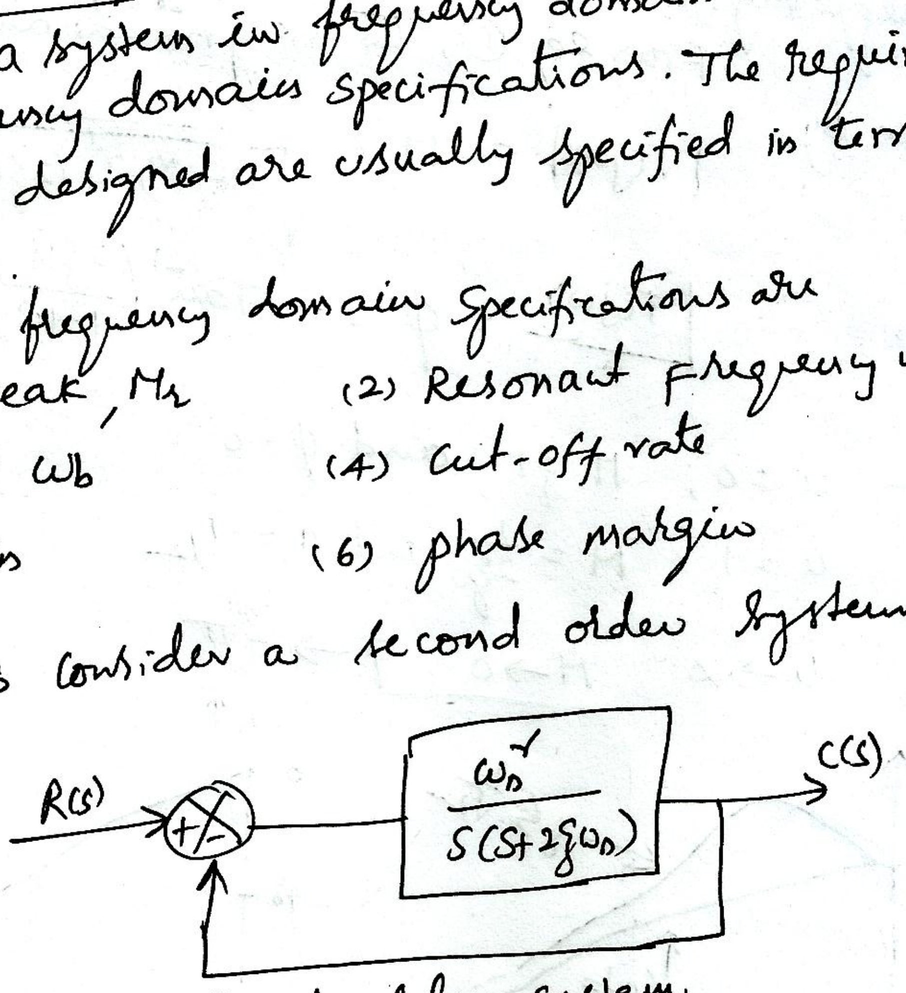


Figure: Second order system

$\therefore$ The closed loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s(s+2\xi\omega_n)+\omega_n^2} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$\therefore$ The sinusoidal transfer function of the system is obtained by replace 's' by 'jω'.

$$\therefore T(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{\omega_n^2}{(j\omega)^2+2\xi\omega_n(j\omega)+\omega_n^2}$$

$$= \frac{\omega_n^2}{\omega_n^2+j2\xi\omega\omega_n-\omega^2}$$

$$= \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2+j2\xi\left(\frac{\omega}{\omega_n}\right)} = \frac{1}{(1-u^2)+j2\xi u} \longrightarrow (I)$$

where $u = \frac{\omega}{\omega_n}$ is the normalized driving signal frequency.

From eq (I), the magnitude and phase angle are given by

$$|T(j\omega)| = M = \frac{1}{\sqrt{(1-u^2)^2+(2\xi u)^2}} \longrightarrow (2)$$

$$\angle T(j\omega) = \phi = -\tan^{-1}\left(\frac{2\xi u}{1-u^2}\right) \longrightarrow (3)$$

if $u=0$; $M=1$ and $\phi=0$

$u=1$ $M=\frac{1}{2\xi}$ and $\phi=-\pi/2$.

$u\to\infty$ $M\to 0$ $\phi\to -\pi$

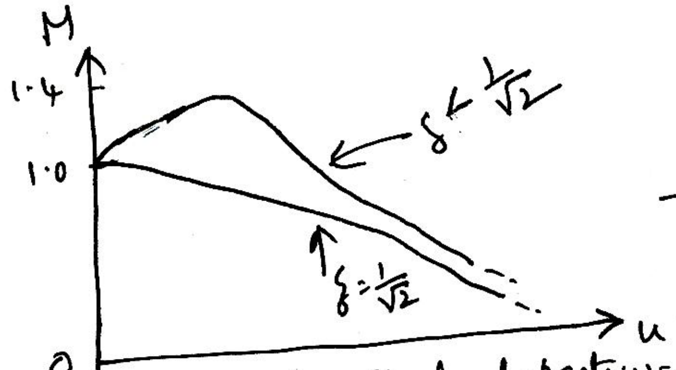


Figure : Magnitude characteristics

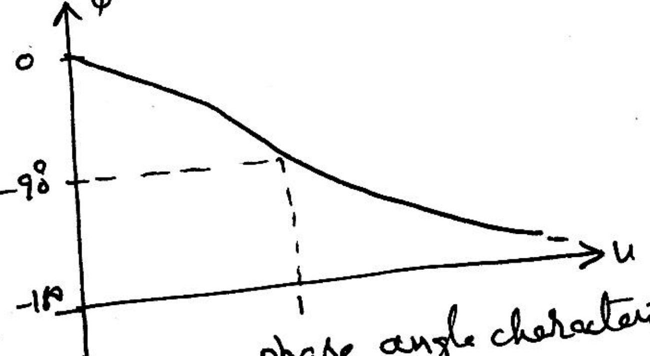


Figure : phase angle characteristics

(1) **Resonant Frequency**: The frequency where magnitude M has a peak value is known as the resonant frequency. At this frequency, the slope of the magnitude curve is zero. Let ω_r be the resonant frequency and $u_r = \frac{\omega_r}{\omega_n}$ be the normalized resonant frequency. Then

$$\left.\frac{dM}{du}\right|_{u=u_r} = 0 \Rightarrow \frac{d}{du}\left\{\frac{1}{\sqrt{(1-u^2)^2+(2\xi u)^2}}\right\}\Bigg|_{u=u_r} = 0$$

$$\Rightarrow -\frac{1}{2}\left[(1-u^2)^2+(2\xi u)^2\right]^{-3/2}\left\{2(1-u^2)(-2u)+4\xi u(2\xi)\right\}\Bigg|_{u=u_r} = 0$$

$$\Rightarrow -\frac{1}{2}\left\{\frac{-4(1-u_r^2)u_r+8\xi^2 u_r}{\left[(1-u^2)^2+(2\xi u)^2\right]^{3/2}}\right\} = 0$$

$$\therefore\ 2(1-u_r^2)u_r - 4\xi^2 u_r = 0$$

or $\quad -u_r^3 - u_r - 2\xi^2 u_r = 0$

or $\quad u_r^2 - 1 - 2\xi^2 = 0$

or $\quad u_r^2 = 1-2\xi^2 \quad \therefore\ u_r = \sqrt{1-2\xi^2} \longrightarrow$ (i)

$\therefore$ Denormalized resonant frequency $\omega_r = u_r \cdot \omega_n = \omega_n\sqrt{1-2\xi^2}$

(2) **Resonant peak**: The maximum value of the magnitude of the closed loop transfer function is known as resonant peak. The magnitude is maximum at resonant frequency ω_r

$$\therefore\ M_r = M\Big|_{\omega=\omega_r} = \frac{1}{\sqrt{(1-u^2)^2+(2\xi u)^2}}\Bigg|_{u=u_r}$$

$$= \frac{1}{\sqrt{\left[1-(1-2\xi^2)\right]^2+(2\xi)^2(1-2\xi^2)}}$$

$$M_r = \frac{1}{\sqrt{(2\xi^2)^2 + 4\xi^2(1-2\xi^2)}}$$

$$= \frac{1}{\sqrt{4\xi^2(\xi^2) + 4\xi^2(1-2\xi^2)}} = \frac{1}{\sqrt{4\xi^2(1-2\xi^2+\xi^2)}}$$

$$= \frac{1}{\sqrt{4\xi^2(1-\xi^2)}} = \frac{1}{2\xi\sqrt{1-\xi^2}} \longrightarrow (i)$$

The phase angle at resonant frequency ω_r is given by

$$\phi_r = \mathrm{Tan}^{-1}\left(\frac{2\xi u}{1-u^2}\right)\Bigg|_{u=u_r} = \mathrm{Tan}^{-1}\left(\frac{2\xi u_r}{1-u_r^2}\right)$$

$$= \mathrm{Tan}^{-1}\left(\frac{2\xi\sqrt{1-2\xi^2}}{1-1+2\xi^2}\right) = \mathrm{Tan}^{-1}\left(\frac{2\xi\sqrt{1-2\xi^2}}{2\xi^2}\right)$$

$$= \mathrm{Tan}^{-1}\left(\frac{\sqrt{1-2\xi^2}}{\xi}\right)$$

(3) Bandwidth: The range of frequencies over which magnitude is equal to or greater than $\frac{1}{\sqrt{2}}$ is defined as bandwidth ω_b. The frequency at which magnitude M has a value of $\frac{1}{\sqrt{2}}$ is called cut-off frequency ω_c.

In general, the bandwidth of a control system indicates the noise-filtering characteristics of the system. Also, bandwidth gives a measure of transient response properties.

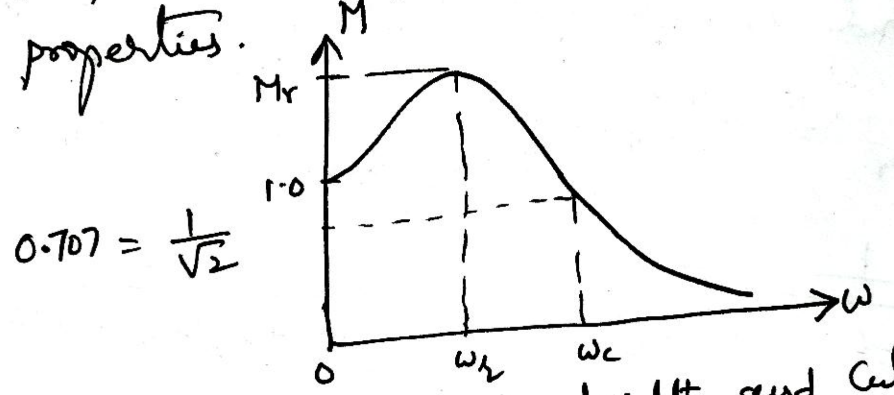


Figure: Bandwidth and Cut-off frequency.

The normalized bandwidth $u_b = \frac{\omega_b}{\omega_n}$ of the second-order systems can be determined as follows. 4

$$M\Big|_{u=u_b} = \frac{1}{\sqrt{(1-u_b)^2 + (2\xi u_b)^2}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow (1-u_b)^2 + (2\xi u_b)^2 = 2$$

$$(1 - 2u_b^2 + u_b^4) + 4\xi^2 u_b^2 = 2$$

$$u_b^4 - 2(1-2\xi^2)u_b^2 - 1 = 0$$

$$\therefore u_b^2 = \frac{2(1-2\xi^2) \pm \sqrt{[2(1-2\xi^2)]^2 - 4(1)(-1)}}{2}$$

$$= \frac{2(1-2\xi^2) \pm \sqrt{4 - 16\xi^2 + 16\xi^4 + 4}}{2}$$

$$= 1 - 2\xi^2 \pm \sqrt{2 - 4\xi^2 + 4\xi^4}$$

$\therefore$ Normalized Bandwidth $u_b = \left[1 - 2\xi^2 + \sqrt{2 - 4\xi^2 + 4\xi^4}\right]^{1/2}$

Denormalized Bandwidth $\omega_b = \omega_n u_b$

The bandwidth is a measure of the ability of a feedback system to reproduce the input signal, noise rejection characteristics, and rise time. A large bandwidth corresponds to a small rise time or fast response.

Figure: Bandwidth Vs Damping factor

(4) Cut-off rate: The slope of log-magnitude curve near the cut-off frequency is called cut-off rate. The cut-off rate indicates the ability of a feedback system to distinguish the signal from noise.

(5) Gain Margin (GM): The gain margin is defined as the value of gain, to be added to the system, in order to bring the system to the verge of instability.

The gain margin is given by the reciprocal of the magnitude of open loop transfer function at phase cross over frequency.

The frequency at which the phase of the open loop transfer function is -180° is called the phase cross-over frequency.

$$\therefore \text{Gain Margin } GM = \left.\frac{1}{|G(j\omega)|}\right|_{\omega=\omega_p}$$

The gain margin in dB can be expressed as

$$GM = 20\log\left|\frac{1}{|G(j\omega)|}\right|_{\omega=\omega_p}$$

where $\left.\angle G(j\omega)\right|_{\omega=\omega_p} = -180^\circ$

The gain margin indicates the additional gain that can be provided to system without affecting the stability of the system.

(6) Phase Margin: The phase margin is defined as the additional phase lag to be added at the gain cross over frequency in order to bring the system to the verge of instability.

The gain cross over frequency ω_g is the frequency at which the magnitude of the open loop transfer function is unity or 0dB.

phase Margin $PM = -(180 + \phi_{gc})$

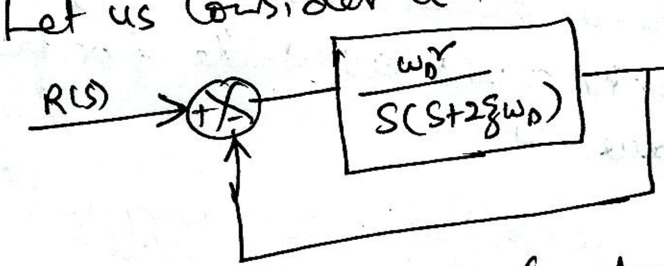
where $\phi_{gc} = \angle G(j\omega)\big|_{\omega=\omega_{gc}}$ & $|G(j\omega)|\big|_{\omega=\omega_{gc}} = 1$

The margin indicates the additional phase lag that can be provided to the system without affecting stability.

For stable systems both gain margin and phase margin are positive.

Correlation between Time and Frequency Response:

Let us Consider a second order system given by



$\therefore$ The closed loop transfer function $\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2 + 2\xi\omega_n S + \omega_n^2}$

For an under damped system ($\xi < 1$),

The damped natural frequency $\omega_d = \omega_n\sqrt{1-\xi^2} \rightarrow (1)$

peak overshoot $M_p = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}} \rightarrow (2)$

Resonant peak $M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}$

Resonant frequency $\omega_r = \omega_n\sqrt{1-2\xi^2}$

For $\xi > \frac{1}{\sqrt{2}}$, the resonant peak does not exist and the correlation breaks down.

$$\frac{\omega_r}{\omega_d} = \sqrt{\frac{1-2\xi^2}{1-\xi^2}} \rightarrow (I)$$

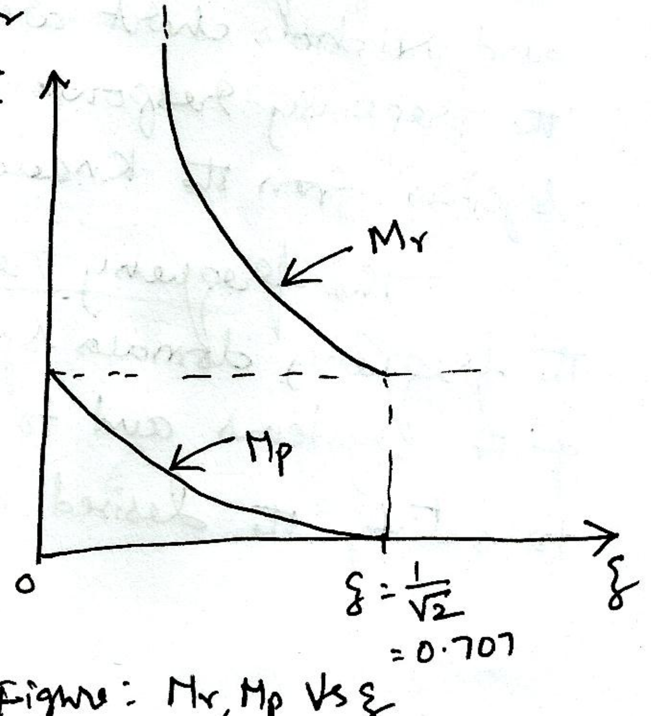


Figure: M_r, M_p Vs ξ

From the figure, it is clear that the correlation breaks down for $\xi > \frac{1}{\sqrt{2}}$.

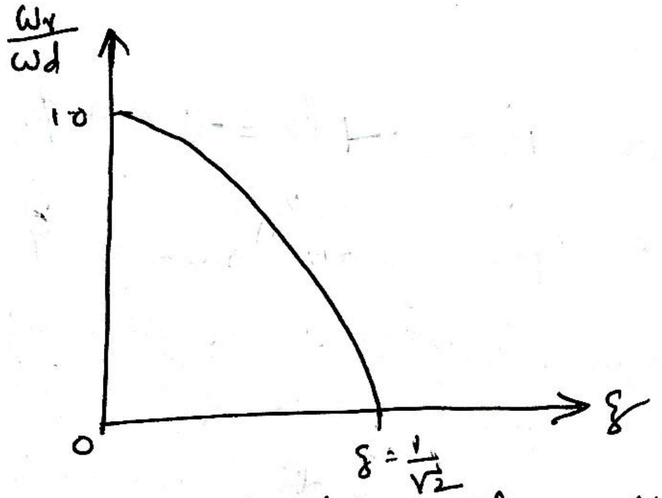


Figure: Correlation between ω_r & ω_d

Frequency Response plots

The frequency response analysis of control systems can be carried either analytically or graphically. The various graphical techniques available for frequency response analysis are

(1) Bode plot (2) polar plot (3) Nyquist plot

(4) Nichols chart (5) M and 'N' circles.

The Bode plot, polar plot and Nyquist plot are usually drawn for open loop systems. From open loop response plot, the performance and stability of closed loop system are estimated. The M and N circles and Nichols chart are used to graphically determine the frequency response of unity feedback closed loop system from the knowledge of open loop response.

The frequency response plots are used to determine the frequency domain specifications, to study the stability of the systems and to adjust the gain of the system to satisfy the desired specifications.

BODE PLOTS: The Bode plot is a frequency response plot of the sinusoidal transfer function of a system. Bode plot consists of two graphs. One is the plot of the magnitude in dB versus $\log\omega$. The other is a plot of the phase angle of sinusoidal transfer function versus $\log\omega$. These plots are called Bode plots in honour of H.W. Bode, who did the basic work in this area.

Let us consider a system with open loop transfer function

$$G(s) = \frac{K\,s(1+sT_1)}{(1+sT_2)(s^2+2\xi\omega_n s+\omega_n^2)}$$

The sinusoidal transfer function $G(j\omega)$ can be obtained by replacing 's' by $j\omega$. The main advantage of the Bode plot is that multiplication of magnitudes can be converted into addition.

$$\therefore G(j\omega) = \frac{K\,j\omega(1+j\omega T_1)}{(1+j\omega T_2)\left((j\omega)^2+j2\xi\omega_n\omega+\omega_n^2\right)}$$

$$= \frac{K\,j\omega(1+j\omega T_1)}{(1+j\omega T_2)\,\omega_n^2\left[1-\left(\frac{\omega}{\omega_n}\right)^2+j2\xi\frac{\omega}{\omega_n}\right]}$$

$$= \frac{K\,j\omega(1+j\omega T_1)}{(1+j\omega T_2)\,\omega_n^2\left[1-u^2+j2\xi u\right]}\; ; \quad \text{where } u=\frac{\omega}{\omega_n}$$

(i) If $G_1(j\omega) = K$; $\quad |G_1(j\omega)| = K \quad \angle G_1(j\omega) = 0^\circ$

$\therefore 20\log|G_1(j\omega)| = 20\log K$

(ii) If $G_2(j\omega) = j\omega$; $\quad |G_1(j\omega)| = \omega$; $\quad \angle G_1(j\omega) = 90^\circ$

$\therefore$ Magnitude in dB is $20\log\omega$.

(iii) If $G_3(j\omega) = (1+j\omega T_1)$

$|G_3(j\omega)| = \sqrt{1+\omega^2T_1^2}$; $\quad \phi = \tan^{-1}\left(\frac{\omega T_1}{1}\right)$

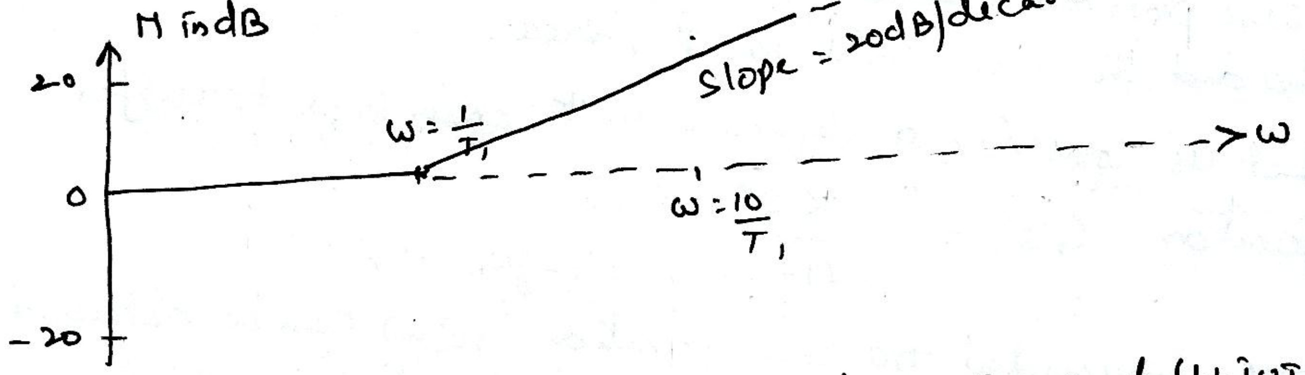
Magnitude in dB is

$$20\log|G_3(j\omega)| = 20\log\sqrt{1+\omega^2T_1^2}$$

If $\omega T >> 1$, M in dB $= 20\log\sqrt{\omega^2 T_1^2}$

$$= 20\log(\omega T_1)$$

If $\omega T_1 << 1$, M in dB $= 20\log 1 = 0$ dB



Therefore, the log magnitude versus $\log\omega$ curve of $(1+j\omega T_1)$ can be approximated by two straight line asymptotes, one a straight line at 0 dB for the frequency range $0 < \omega \leq 1/T$ and the other a straight line with a slope 20 dB/decade for the frequency $1/T \leq \omega < \infty$. The frequency $\omega = 1/T$ at which the two asymptotes meet is called the corner frequency or the break frequency.

(iv) If $G_4(j\omega) = \frac{1}{(1+j\omega T_2)}$

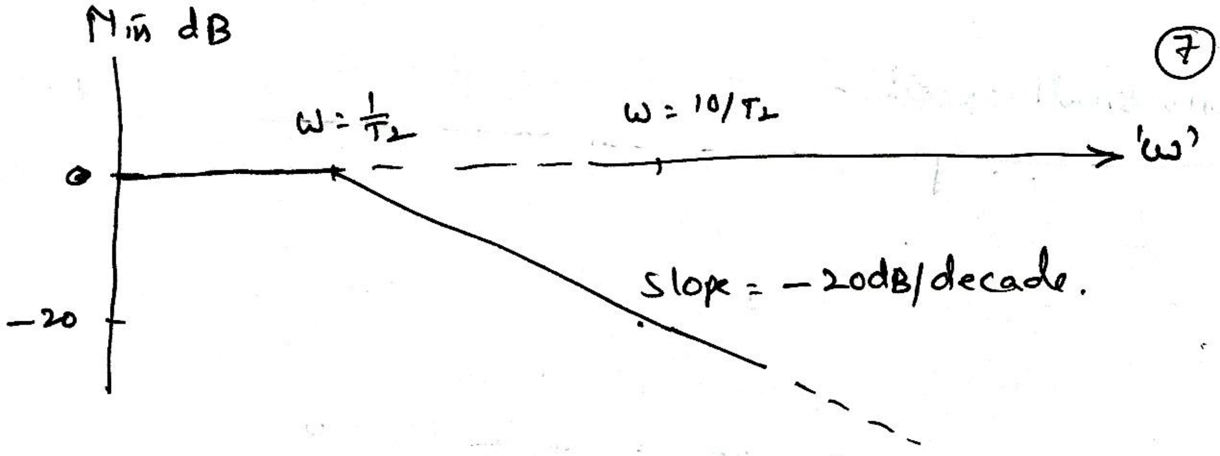
$$|G_4(j\omega)| = \frac{1}{\sqrt{1+\omega^2T_2^2}} \; ; \quad \angle G_4(j\omega) = -\tan^{-1}(\omega T_2)$$

Magnitude in dB $= 20\log|G_4(j\omega)| = 20\log\frac{1}{\sqrt{\omega^2T_2^2+1}}$

$$= -20\log\sqrt{1+\omega^2T_2^2}$$

for $\omega T_2 << 1$, Magnitude $= -20\log(1) = 0$ dB

for $\omega T_2 \quad 1$, Magnitude $= -20\log\sqrt{\omega^2T_2^2} = -20\log(\omega T_2)$



where $\omega = \frac{1}{T_2}$ is known as corner frequency or Break over frequency.

(iv) If $G_5(j\omega) = \frac{1}{(1-u^2+j2\xi u)}$ where $\boxed{\omega/\omega_n = u}$

$$|G_5(j\omega)| = \frac{1}{\sqrt{(1-u^2)^2+(2\xi u)^2}} \; ; \; \phi = -\tan^{-1}\left(\frac{2\xi u}{1-u^2}\right)$$

∴ Magnitude in dB = $20\log|G_5(j\omega)| = 20\log\frac{1}{\sqrt{(1-u^2)^2+(2\xi u)^2}}$

By making two assumptions, we can plot the bode plot

(i) if $u^2 << 1$; $20\log\frac{1}{\sqrt{(1-u^2)^2+(2\xi u)^2}} = 20\log 1 = 0$

if $u^2 >> 1$; $20\log\frac{1}{\sqrt{(1-u^2)^2+(2\xi u)^2}} = 20\log\frac{1}{\sqrt{u^4}}$

$$= -40\log u$$

∴ Magnitude is zero up to $u = 1$, after that magnitude curve is a line with slope −40dB/decade

Note: Decade	:	$\omega_2 = 10\omega_1$
Octave	:	$\omega_2 = 2\omega_1$

In the above case $u = \frac{\omega}{\omega_n} = 1$ is the break over or corner frequency.

Error in Magnitude at corner frequency

(1) If $G(s) = (1+sT_1)$

$$G(j\omega) = (1+j\omega T_1)$$

$$|G(j\omega)| = \sqrt{1+\omega^2 T_1^2}$$

$\therefore$ for $\omega T_1 << 1$ Magnitude in dB $= 0$

$\omega T_1 >> 1$ Magnitude in dB $= 20\log(\omega T_1)$

Actually at $\omega T_1 = 1$

$$M \text{ in dB} = 20\log\sqrt{1+1^2} = 20\log\sqrt{2}$$

$$= 10\log 2 = 10(0.3010) = \underline{\underline{3\,dB}}$$

(2) If $G(j\omega) = \frac{1}{1+j\omega T_2}$

Magnitude in dB $: 20\log\frac{1}{\sqrt{1+\omega^2 T_2^2}}$

where $\omega = \frac{1}{T_2}$ is the corner frequency. and

if $\omega T_2 << 1$; M in dB $= 20\log 1 = 0$

for $\omega T_2 >> 1$, M in dB $= 20\log\frac{1}{\sqrt{\omega^2 T_2^2}} = -20\log(\omega T_2)$

At $\omega T_2 = 1$ or at $\omega = 1/T_2$

$$M \text{ in dB} = 20\log\frac{1}{\sqrt{1+1}} = -20\log\sqrt{2}$$

$$= -10\log 2 = \underline{\underline{-3\,dB}}$$

(3) $G(s) = \frac{1}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{1}{\omega_n^2}\cdot\frac{1}{[(1-u^2)+j2\xi u]}$

if $G(ju) = \frac{1}{(1-u^2)+j2\xi u}$

where $u=1$ is the corner or break over frequency.

at $u=1$ or at $\frac{\omega}{\omega_n}=1$ or at $\omega = \omega_n$

Magnitude in dB (at $u=1$) $= 20\log\left[\frac{1}{\sqrt{(1-u^2)^2+(2\xi u)^2}}\right]_{u=1}$ ⑧

$$= 20\log \frac{1}{\sqrt{(1-1)^2+[2\xi(1)]^2}}$$

$$= -20\log\sqrt{(2\xi)^2}$$

$$= \underline{-20\log 2\xi}$$ for Complex Conjugate poles.

where 'ξ' is the damping factor.

(4) If $G(s) = (s^2+2\xi\omega_n s+\omega_n^2)$, the error at corner frequency $u = \frac{\omega}{\omega_n} = 1$ is $\underline{(20\log 2\xi)}$

<u>Note</u>: In the Construction of the Bode plot, the following factors may appear

(1) Constant gain 'K'

(2) poles at the origin $\frac{1}{(j\omega)^r}$

(3) zeros at the origin $(j\omega)^r$

(4) poles on the real axis $\frac{1}{(1+j\omega T_1)}$

(5) Zeros on the real axis is $(1+j\omega T_2)$

(6) Complex Conjugate poles $\frac{1}{s^2+2\xi\omega_n s+\omega_n^2}$

(7) Complex Conjugate zeros $(s^2+2\xi\omega_n s+\omega_n^2)$

<u>Imp</u>: To plot the Bode diagram, we need the transfer function in time-constant form.

ie $$G(j\omega) = \frac{K(1+j\omega T_1)(1+j\omega T_2)}{(j\omega)^r((1-u^2)+j2\xi u)}$$

procedure to Construct the Bode plot:

(1) obtain the sinusoidal transfer function from the given transfer function.

(2) Identify the corner frequencies of poles and zeros from the time-constant form of $G(j\omega)$.

(3) sketch the asymptotic bode plot, Then make corrections at corner frequencies.

① Draw the Bode plot for the transfer function

$$G(s) = \frac{64(s+2)}{s(s+0.5)(s^2+3.2s+64)}$$

(Sol) To draw the Bode plot, we need the transfer function in time-constant form.

$$G(s) = \frac{64 \times 2(1+s/2)}{s \times 0.5(1+s/0.5)64(1+\frac{3.2}{64}s+s^2/64)}$$

Therefore, the sinusoidal transfer function is given by

$$G(j\omega) = \frac{4(1+j\omega/2)}{j\omega(1+j2\omega)(1-u^2+j2\xi u)} \quad ; \text{ where } u = \frac{\omega}{8}, \ \xi = 0.2$$

Factor	Corner frequency	Asymptotic log magnitude characteristic
$4/j\omega$	None	Magnitude = $20\log 4/\omega$. straight line of slope -20dB/decade, with magnitude 0 dB at $\omega=4$
$\frac{1}{1+j2\omega}$	$\omega_{c_1} = \frac{1}{2} = 0.5$ rad/sec	straight line of 0dB up to $\omega<\omega_{c_1}$ and straight line of slope -20dB/decade for $\omega>\omega_{c_1}$

$1+j\frac{\omega}{2}$	$\omega_{c_2} = 2$ rad/sec	straight line of 0dB for $\omega < \omega_{c_2}$ and a straight line with slope +20 dB/decade for $\omega > \omega_{c_2}$
$1+j2(0.2)\left(\frac{\omega}{8}\right)-\left(\frac{\omega}{8}\right)^2$	$\omega_{c_3} = 8$ rad/sec $\xi = 0.2$;	straight line of 0dB for $\omega < \omega_{c_3}$ and straight line of slope −40dB/decade for $\omega > \omega_{c_3}$

$$\phi = \tan^{-1}(\omega/2) - 90^\circ - \tan^{-1}(2\omega) - \tan^{-1}\left(\frac{2\xi u}{1-u^2}\right)$$

where $u = \frac{\omega}{8}$; $\xi = 0.2$

Note: The least corner frequency is 0.5. So that, we can choose the frequency scale from 0.1 onwards.

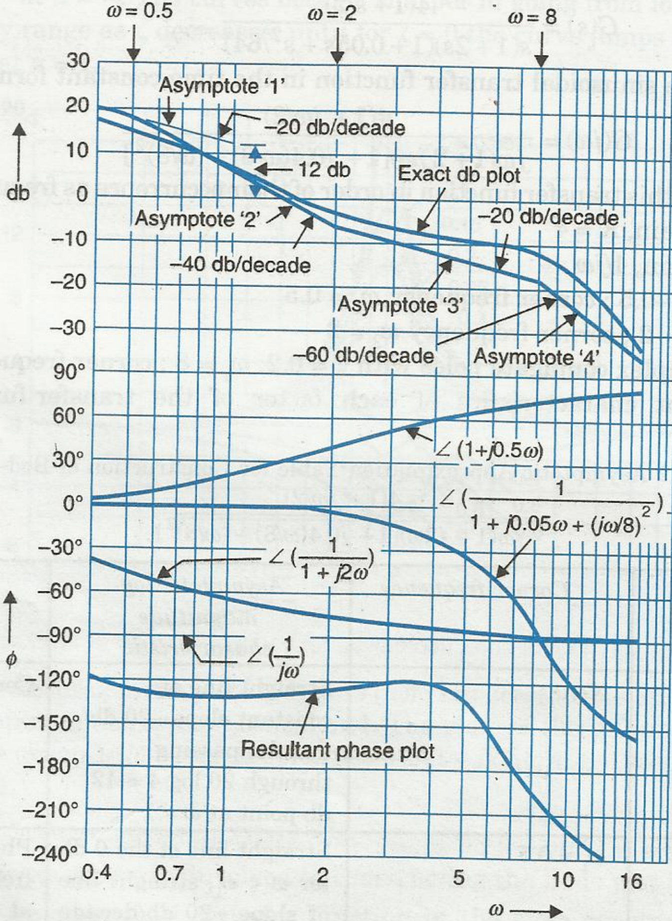


Fig. 8.16. Bode plot of $\dfrac{4(1+j\omega/2)}{j\omega(1+j2\omega)[1+j0.4(\omega/8)-(\omega/8)^2]}$.

Problem 8.1 Sketch the Bode Plots and determine the gain cross-over and phase cross-over frequencies.

$$G(s) = \frac{10}{s(1+0.5s)(1+0.1s)}$$ **(Pune University)**

Solution

Corner frequencies The corner frequencies are 2 and 10.

Magnitude Plot

Ser. No.	*Factor*	*Corner frequency*	*Asymptotic log-magnitude Characteristic*
1	$\frac{1}{s}$	None	Straight line of constant slope (–20 db/dec) passing through at $\omega = 1$
2	$\frac{1}{(1+0.5s)}$	$\omega_1 = 2$	Straight line of constant slope (–20 db/dec) originating from $\omega_1 = 2$
3	$\frac{1}{(1+0.1s)}$	$\omega_2 = 10$	Straight line of constant slope (–20 db/dec) originating from $\omega_2 = 10$
4	10	None	Straight line of constant slope of 0 db/dec starting from 20 log 10 = 20 db point

Magnitude plots for individual factors are shown by dotted lines. Resultant line is shown by a firm line (Fig. 8.1).

Phase Plot $\phi = -90° - \tan^{-1} 0.5\omega - \tan^{-1} 0.1\omega$

Ser. No.	ω	ϕ
1	0	–90°
2	0.1	–93.43°
3	1	–122.3°
4	2	–146.31°
5	5	–184.76°
6	10	–213.7°
7	15	–228.7°

Magnitude and phase plots are shown in Fig. 8.1. From the plots

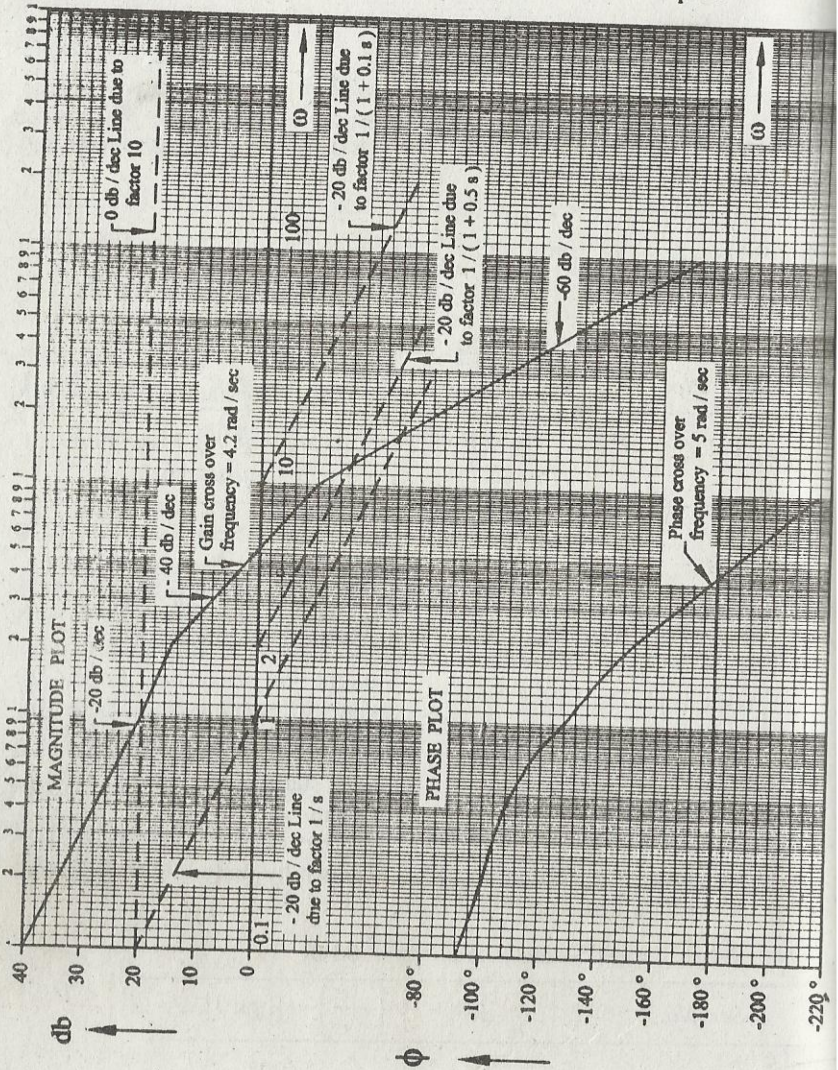


Fig. 8.1

1. Gain crossover frequency = 4.2 rad/sec **Ans.**
2. Phase crossover frequency = 4.5 rad/sec. **Ans.**

Problem 8.2 Sketch the Bode plot for the transfer function

$$G(s) = \frac{K s^2}{(1+0.2 s)(1+0.02 s)}$$

Determine the system gain K for the gain cross-over frequency to be 5 rad/sec.

Solution

Let, $K = 1$, then

$$G(s) = \frac{s^2}{(1+0.2 s)(1+0.02 s)}$$

Corner frequencies The corner frequencies are 5 and 50 rad/sec.

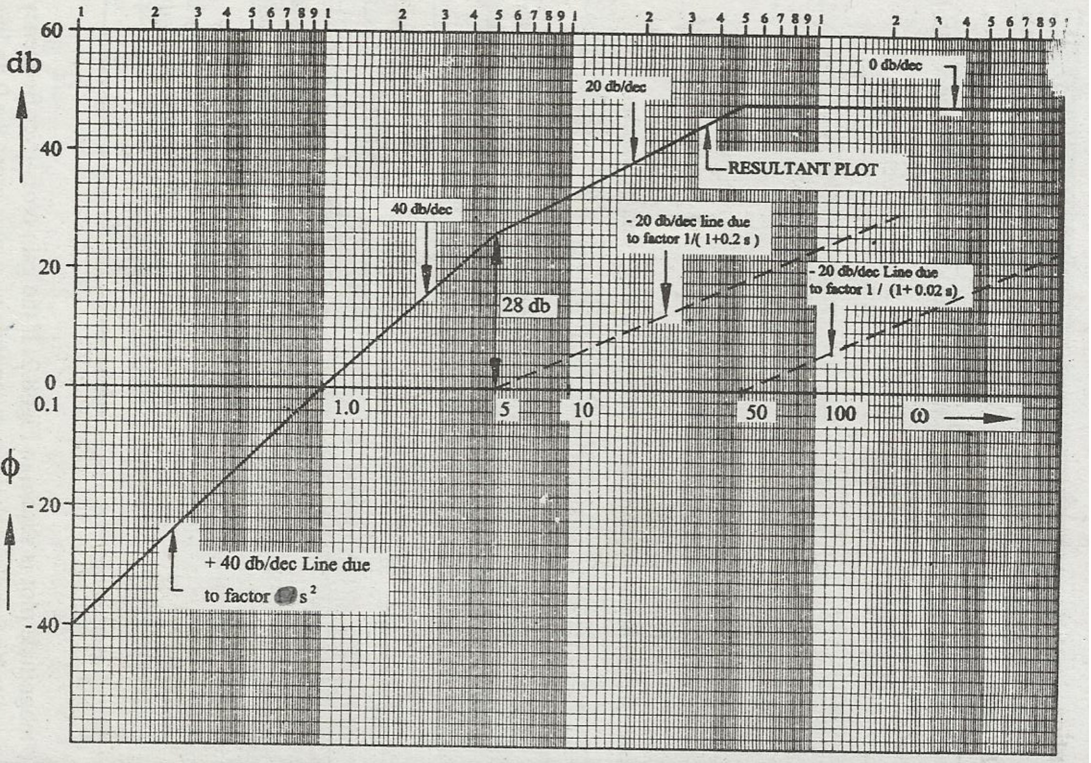
Magnitude Plot

Ser. No.	*Factor*	*Corner frequency*	*Asymptotic log-magnitude Characteristic*
1	s^2	None	Straight line of constant slope 40 db/dec passing through $\omega = 1$
2	$\frac{1}{(1+0.2 s)}$	$\omega_1 = 5$	Straight line of constant slope –20 db/dec originating from $\omega = 5$
3	$\frac{1}{(1+0.02 s)}$	$\omega_2 = 50$	Straight line of constant slope –20 db/dec originating from $\omega = 50$

The magnitude plot is shown in Fig. 8.2. If the gain cross-over frequency is required to be 5 rad/sec, then the magnitude plot must cross the 0 db line at 5 rad/sec. For this, the plot has to be brought down by 28 db. Hence

$$20 \log K = -28$$

$$\therefore \quad K = 0.04$$ **Ans.**



db
60
40
20
0
0.1
φ
- 20
- 40
0 db/dec
20 db/dec
RESULTANT PLOT
40 db/dec
28 db
- 20 db/dec line due
to factor 1/(1+0.2 s)
- 20 db/dec Line due
to factor 1 / (1+ 0.02 s)
+ 40 db/dec Line due
to factor s 2
1.0
5
10
50
100
ω

Problem 8.4 Draw the Bode plot for a system having

$$G(s)\,H(s) = \frac{100}{s(s+1)(s+2)}.\ \text{Find}$$

(a) Gain Margin

(b) Phase margin

(c) Gain cross over frequency

(d) Phase cross over frequency **(Pune University)**

Solution

$$G(s)\,H(s) = \frac{50}{s(s+1)(1+0.5s)}$$

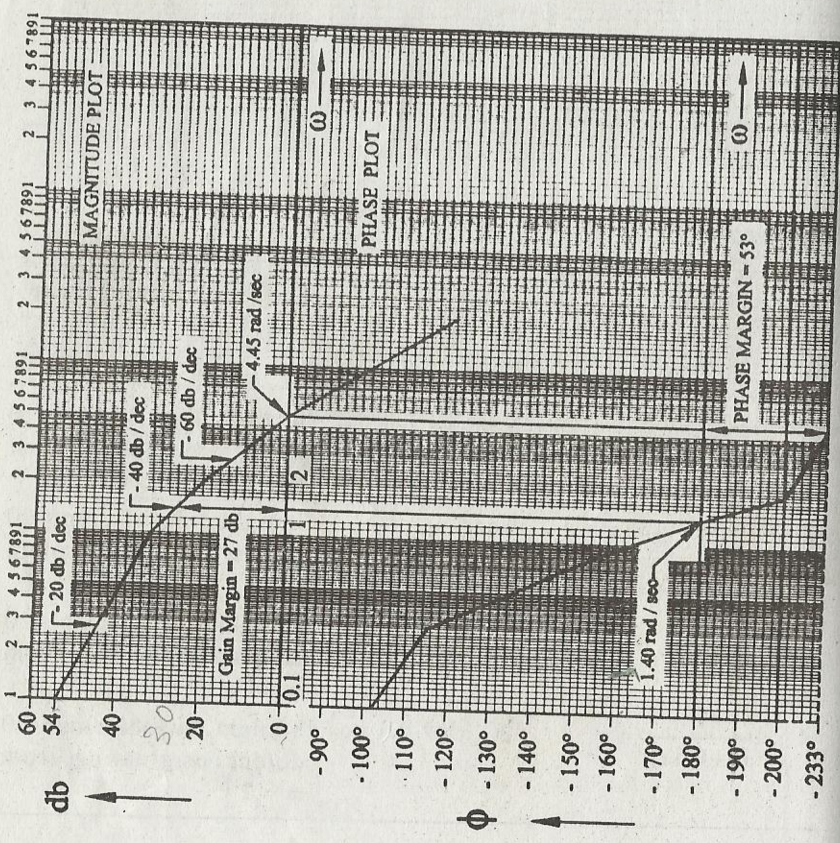
Magnitude Plot

Ser. No.	*Factor*	*Corner frequency rad/sec*	*Asymptotic log-magnitude Characteristic*
1	50	None	Straight line of slope 0 db/dec starting from point 20 log 50 = 34 db
2	$\frac{1}{s}$	None	Straight line of slope 20 db/dec passing through $\omega = 1$
3	$\frac{1}{(1+s)}$	1	Straight line of slope–20 db/dec originating from $\omega = 1$
4	$\frac{1}{(1+0.5\,s)}$	2	Straight line of slope–20 db/dec originating from $\omega = 2$

Phase Plot $\phi = -90° - \tan^{-1}\omega - \tan^{-1} 0.5\omega$

Ser. No.	ω *rad/sec*	ϕ
1	0	−90°
2	0.1	−98.6°
3	0.2	−107°
4	0.5	−130.6°
5	1	−161.6°
6	1.3	−175.5°
7	1.4	−179.5°
8	1.5	−183.2°
9	2	−198.4°
10	4.45	−233°

Magnitude and Phase plots as shown in Fig. 8.4



Result

1. Gain Crossover frequency : 4.45 rad/sec
2. Phase Crossover frequency : 1.40 rad/sec
3. Gain Margin : 27 db
4. Phase Margin : 53°.

Problem 8.4 The Open-loop transfer function of a certain unity feedback system is

$$G(s) = \frac{K}{s(s+2)(s+10)}$$

Construct Bode plots and determine.

(a) Limiting value of K for system to be stable

(b) Value of K for gain margin to be 10 db

(c) Value of K for phase margin to be 50° **(Pune University)**

Solution $$G(s) = \frac{0.025\,K}{s(1+0.5s)(1+0.05s)}$$

Let = 0.025 K = 1, then $$G(s) = \frac{1}{s(1+0.5s)(1+0.05s)}$$

Magnitude Plot

Ser. No.	*Factor*	*Corner frequency rad/sec*	*Asymptotic log-magnitude Characteristic*
1	$\frac{1}{s}$	None	Straight line of 0 db/dec passing through ω = 1
2	$\frac{1}{(1+0.5\,s)}$	2	Straight line of –20 db/dec originating from ω = 2
3	$\frac{1}{(1+0.05\,s)}$	20	Straight line of –20 db/dec originating from ω = 4

Phase Plot $\phi = -90° - \tan^{-1} 0.5\,\omega - \tan^{-1} 0.05\,\omega$

Ser. No.	*ω rad/sec*	ϕ
1	0	–90°
2	1	–119°
3	2	–141°
4	2.5	–148.5°
5	3	–155°
6	4	–165°
7	4.5	–168.7°
8	5	–172°
9	6	–178.3°
10	6.5	–181°
11	10	–195°

Bode plots are shown in Fig. 8.5

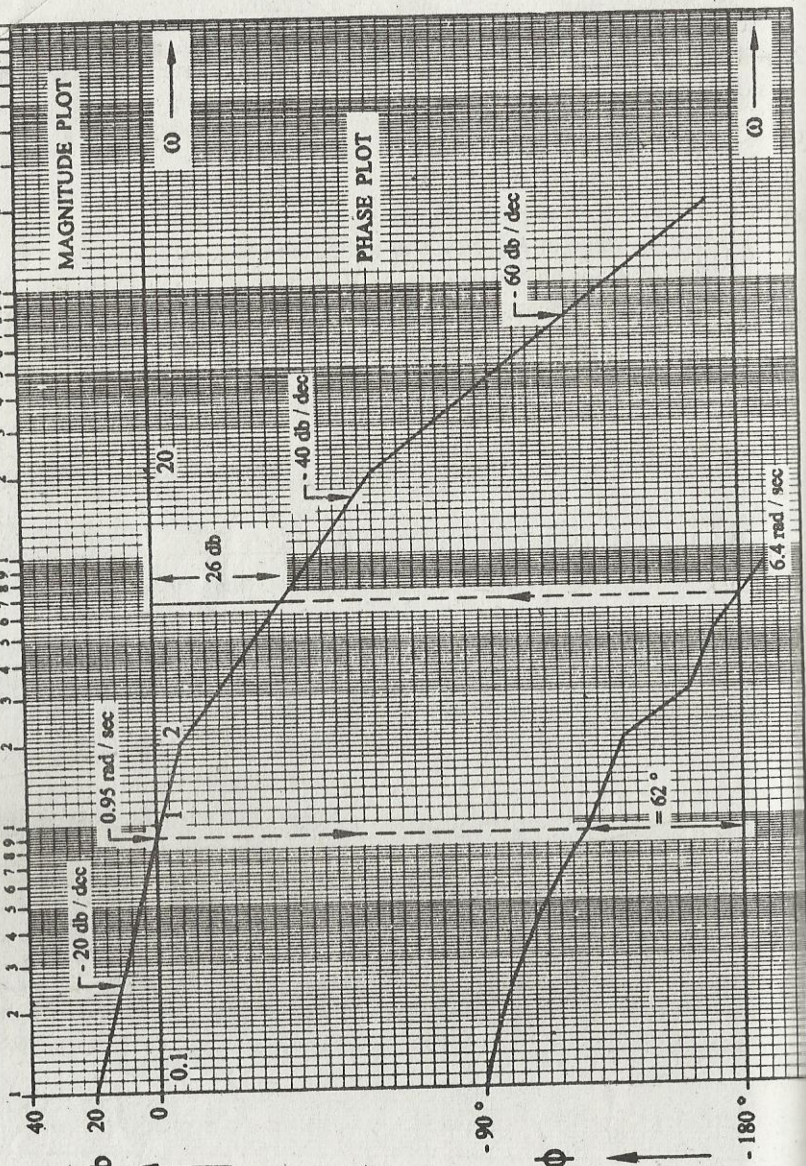


Fig. 8.5

(a) From the curves, the gain margin is 26 db

$$\therefore \qquad 20 \log K_1 = 26$$

or $$K_1 = 19.95$$

or $\quad 0.025\,K = 19.95$

or $\quad K = 798$ **Ans.**

(b) For the gain margin to be 10 db, the graph has to be lifted up by 26 – 10 = 16 db

$\therefore \quad 20 \log K_1 = 16$

or $\quad K_1 = 6.3$

or $\quad 0.025\,K = 6.3$

or $\quad K = 252$ **Ans.**

(c) For the phase margin to be 50°, the value of ω at –180° + 50° = –130° is 1.9 rad/sec. Gain Margin at 1.9 rad/sec is 5.5 db. Therefore, to have phase margin of 50°, magnitude plot has to be lifted up by 5.5 db, so that gain cross over frequency is 1.9 rad/sec

$\therefore \quad 20 \log K_1 = 5.5$

or $\quad K_1 = 1.88$

or $\quad 0.025\,K = 1.88$ or $K = \dfrac{1.88}{0.025} = \mathbf{75.35}$ **Ans.**

Extraction of Transfer function from Bode Diagrams:

(1) Find the open loop transfer function of a system whose approximate plot is shown in figure

(Sol) The corner frequencies are

$\omega_{c_1} = 2.5;\ \omega_{c_2} = 10;\ \omega_{c_3} = 25$ rad/sec

- change in magnitude in dB

$= \text{slope (Number of decades between two frequencies)}$

$= -20\ (\log 2.5 - \log 1)$

$= -7.95$

$\therefore$ Magnitude $= -12 - (-7.95)$

$= -12 + 7.95$

$= -4.05\,\text{dB}$

$\therefore 20 \log K = -4.05$

$\log K = \dfrac{-4.05}{20}$

$\therefore K = 10^{-\frac{4.05}{20}}$

$= 0.63$

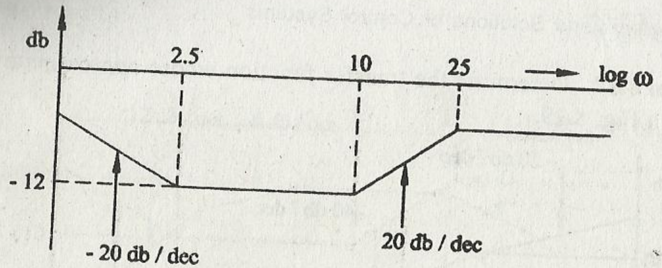


Fig. 8.18

$$= -20 (\log 2.5 - \log 1)$$
$$= -20 \log 2.5 = -7.95$$

Magnitude $= -12 + 7.95 \text{ db} = -4.05 \text{ db}$

$\therefore$ $20 \log K = -4.05$

or $K = 0.63$

Since first line has a slope of – 20 db/dec and starts from a point – 4.05 db at ω = 1 rad/sec the factor contributing this is

$$= \frac{K}{s} = \frac{0.63}{s}$$

Plot between ω = 2.5 and ω = 10 is having a slope of 0 db/dec. At ω = 2.5 the slope has changed from – = 20 db/dec and this can only happen due to a factor in the numerator and is

$$= \left(\frac{s}{2.5} + 1\right) = (1 + 0.4s)$$

At ω = 10, the slope has changed from 0 db/dec to + 20 db/dec and is due to a factor in the numerator and is

$$= \left(\frac{s}{10} + 1\right) = (1 + 0.1s)$$

At ω = 25, the slope has changed from + 20 db/dec to 0 db/dec and is due to a factor in the denominator and is

$$= \left(\frac{s}{25} + 1\right) = (1 + 0.4s)$$

The open-loop transfer function is this

$$G(s) = \frac{0.63(1 + 0.4s)(1 + 0.1s)}{(1 + 0.04s)}$$

Problem 8.17 Determine the transfer function whose approximate plot is shown in Fig. 8.19.

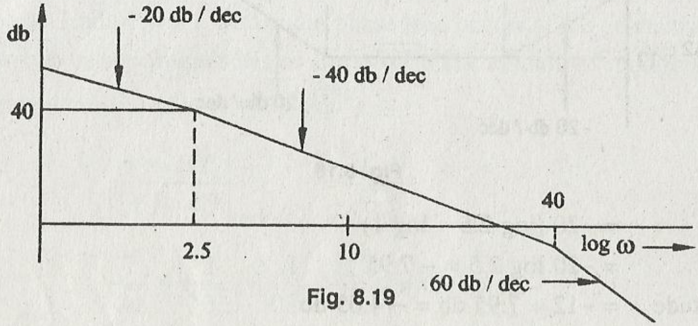


Fig. 8.19

Solution

Corner frequencies are 2.5 and 40 rad/sec

$$20 \log K = 40 + 20 \log 2.5 = 47.95$$

or
$$K = 250$$

At $\omega = 2.5$ rad/sec slope changes from -20 db/dec to -40 db/dec due to a factor $\dfrac{1}{\left(1+\dfrac{s}{2.5}\right)}$. At $\omega = 40$ rad/sec slope changes from -40 db/dec to -60 db/dec due to a factor $\dfrac{1}{\left(1+\dfrac{s}{40}\right)}$. Also, since initial slope is -20 db/dec, it is due to factor. $1/s$. Therefore open-loop transfer function is

$$G(s) = \frac{250}{s\left(1+\dfrac{s}{2.5}\right)\left(1+\dfrac{s}{40}\right)} = \frac{250}{s(1+0.4s)(1+0.025s)}$$

Problem 8.18 Determine the open-loop transfer function of a system whose approximate plot is shown in Fig. 8.20.

Solution

First line is having a slope of 12 db/oct (40 db/dec). Therefore, there is a s^2 term in the numerator. At $\omega = 0.5$ rad/sec slope changes to 6 db/oct (20 db/dec) due to a term in the denominator equal to $\left(1+\dfrac{s}{0.5}\right)$

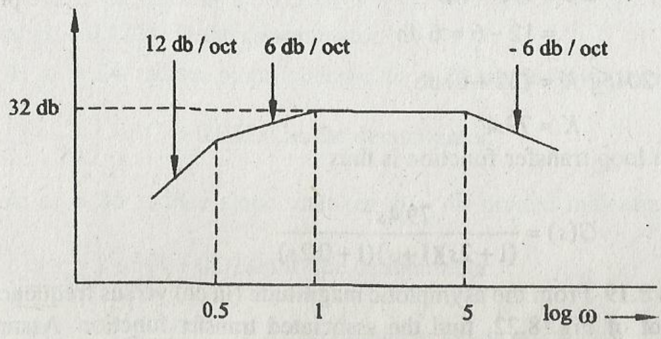


Fig. 8.20

At $\omega = 1$ rad/sec slope becomes 0 db/dec due to a term in the denominator equal to $(1 + s)$.

At $\omega = 5$ rad/sec slope becomes $-$ 6 db/oct ($-$ 20 db/dec) due to a term in the denominator equal to $\left(1+\frac{s}{5}\right)$

$$G(s) = \frac{Ks^2}{\left(1+\frac{s}{0.5}\right)(1+s)\left(1+\frac{s}{5}\right)}$$

Calculation of 'K'

Refer Fig. 8.21

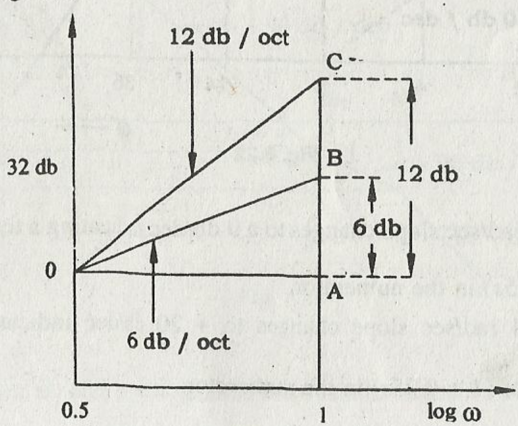


Fig. 8.21

Difference between $\omega = 0.5$ and $\omega = 1$ rad/sec is one octave i.e. $AB = 6$ db since the slope of line OB is 6 db/oct. OC is the extended line having a slope of 12 db/oct.

$$\therefore \qquad AC = 12 \text{ db}$$

$$BC = AC - AB$$

$$= 12 - 6 = 6 \text{ db}$$

$$\therefore \qquad 20 \log K = (32 + 6) \text{ db}$$

or

$$K = 79.4$$

The open loop transfer function is thus

$$G(s) = \frac{79.4s^2}{(1+2s)(1+s)(1+0.2s)}$$ **Ans.**

Problem 8.19 From the asymptotic magnitude (in db) versus frequency (log scale) plot of Fig. 8.22, find the associated transfer function. Assume no right half plane poles or zeros present. **(Pune University)**

Solution

1. Slope of the first line is – 20 db/dec indicating a term $\frac{1}{s}$

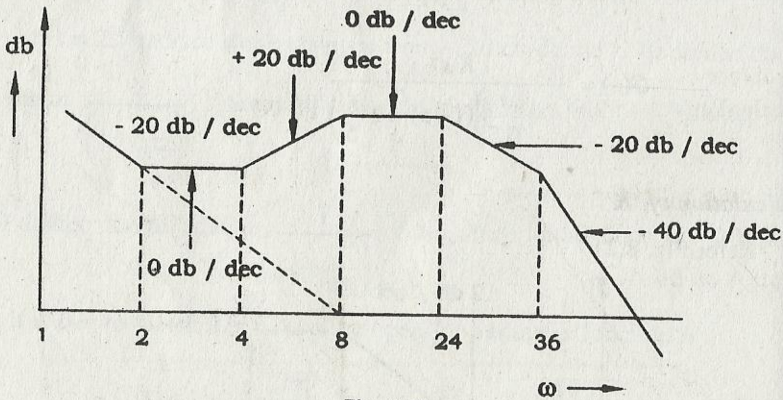


Fig. 8.22

2. At $\omega = 2$ rad/sec slope changes to a 0 db/dec indicating a term $\left(1+\frac{s}{2}\right)$ or $(1 + 0.5s)$ in the numerator.
3. At $\omega = 4$ rad/sec slope changes to + 20 db/dec indicating a term $\left(1+\frac{s}{4}\right)$ or $(1 + 0.25s)$ in the numerator.

4. At $\omega = 8$ rad/sec slope changes to 0 db/dec indicating a term $\left(1+\frac{s}{8}\right)$ or $(1 + 0.125s)$ in the denominator.

5. At $\omega = 24$ rad/sec slope changes to $-$ 20 db/dec indicating a term $\left(1+\frac{s}{24}\right)$ or $(1 + 0.042s)$ in the denominator.

6. At $\omega = 36$ rad/sec slope changes to $-$ 40 db/dec indicating a term $\left(1+\frac{s}{40}\right)$ or $(1 + 0.028s)$ in the denominator.

Transfer function is thus $\dfrac{K(1+0.5s)(1+0.25s)}{s(1+0.125s)(1+0.042s)(1+0.028s)}$

Calculation of 'K'

$$20 \log K = 20 \log 8$$

or
$$K = 8$$

$\therefore$
$$G(s) = \frac{8(1+0.5s)(1+0.025s)}{s(1+0.125s)(1+0.042s)(1+0.028s)}$$

Problem 8.20 Derive the transfer function of the system from the data given on the Bode diagram shown in Fig. 8.23 below. (AMIE)

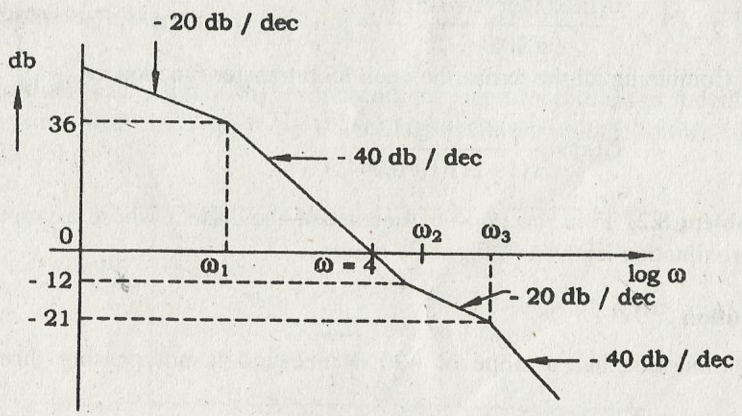


Fig. 8.23

Solution

Between ω_1 and $\omega = 4$ rad/sec there is a decrease of 36 db

$\therefore$
$$-36 = -40(\log 4 - \log \omega_1)$$

or
$$\omega_1 = 0.5036 \cong 0.5 \text{ rad/ sec}$$

Calculation of 'K' $\quad 20\log K = 36 + 20\log 0.5$

or $\quad K = 31.62$

Calculation of 'ω_2' $\quad -12 = -40(\log\omega_2 - \log 4)$

or $\quad \omega_2 = 8$ rad/sec

Calculation of 'ω_3' $\quad -21 + 12 = -20(\log\omega_3 - \log 8)$

or $\quad \omega_3 = 22.5$ rad/sec

First line has a slope of – 20 db/dec indicating a term $\frac{1}{s}$ and since it is not passing through $\omega = 1$ rad/sec, the term is $\frac{K}{s}$ or $\frac{31.62}{s}$.

At $\omega_1 = 0.5$ rad/sec slope changes to –40 db/dec indicating a term $\frac{1}{\left(1+\frac{s}{0.5}\right)}$, or $\frac{1}{(1+2s)}$.

At $\omega_2 = 8$ rad/sec, slope changes to –20 db/dec indicating a term $\left(1+\frac{s}{8}\right)$ or $(1+0.125s)$

At $\omega_3 = 22.5$ rad/sec, slope changes to –40 db/dec indicating a term $\frac{1}{\left(1+\frac{s}{22.5}\right)}$ or $\frac{1}{(1+0.044s)}$

Combining all the terms, the open-loop transfer function is

$$G(s) = \frac{31.62(1+0.125s)}{s(1+2s)(1+0.044s)}$$

Problem 8.21 Find the transfer function of the system whose asymptotic approximation is given in Fig. 8.24 below.

Solution

First line has a slope of –20 db/dec and is not passing through $\omega = 1$ rad/sec. Therefore, it indicates a term $\frac{K}{s}$

$$20\log K = -9 \quad \text{or} \quad K = 0.35$$

$\therefore$ the term is $\frac{0.35}{s}$

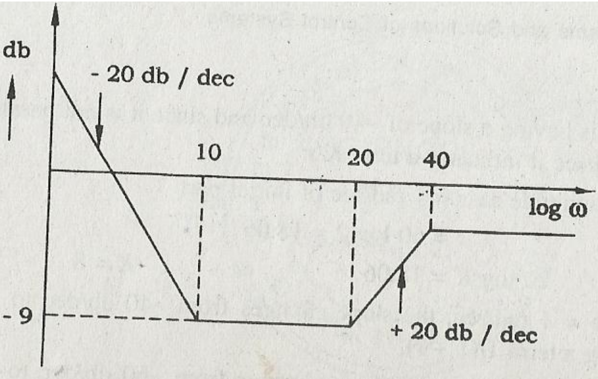


Fig. 8.24

At $\omega = 1$ rad/sec, slope changes to 0 db/dec indicating a term $(1 + s)$.

At $\omega = 20$ rad/sec, slope changes to +20 db/dec indicating a term $\dfrac{1}{\left(1+\dfrac{s}{20}\right)}$ or $(1+0.05s)$.

At $\omega = 40$ rad/sec, slope changes to 0 db/dec indicating a term $\dfrac{1}{\left(1+\dfrac{s}{40}\right)}$ or $\dfrac{1}{(1+0.025s)}$.

Combining all terms, we get $G(s) = \dfrac{0.35(1+s)(1+0.05s)}{s(1+0.025s)}$

Problem 8.22. Obtain the expression for open-loop transfer function for a system with unity feedback whose log-magnitude plot is shown in Fig. 8.25 below:

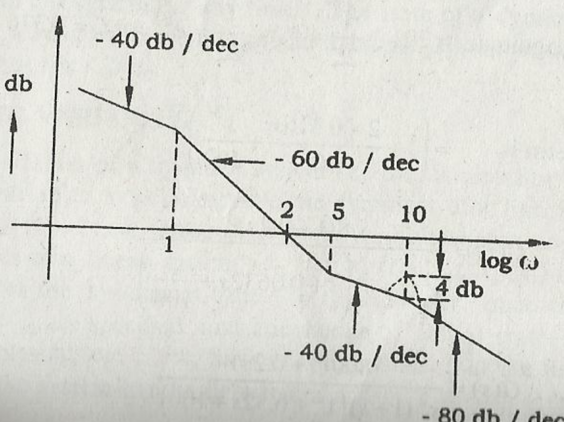


Fig. 8.25

Solution

First line is having a slope of –40 db/dec and since it is not passing through $\omega = 1$ rad/sec it indicates a term K/s^2

Magnitude at $\omega = 1$ rad/sec of initial part

$$= 60 \log 2 = 18.06$$

$$\therefore \quad 20 \log K = 18.06 \qquad \text{or} \qquad K = 8$$

At $\omega = 1$ rad/sec, the slope changes from –40 db/dec to –60 db/dec indicating a term $1/(1 + s)$.

At $\omega = 5$ rad/sec, the slope changes from –60 db/dec to –40 db/dec indicating a term $\left(1+\frac{s}{5}\right)$ or $(1+0.2s)$.

At $\omega = 10$ there is a term of the form $\left\{\left(1+\frac{2\zeta s}{\omega_n}+\frac{s^2}{\omega_n^2}\right)\right\}^{-1}$

because the slope changes from –40 db/dec to –80 db/dec and also a peak of 4 db is shown

$$\omega_n = 10 \text{ rad/sec}$$

Value of $\left\{1+\frac{2\zeta s}{\omega_n}+\left(\frac{s}{\omega_n}\right)^2\right\}^{-1}$ at $\omega = \omega_n$

$$= \left\{\sqrt{\left(1-\frac{10}{10}\right)^2+\left(\frac{2\times\zeta\times10}{10}\right)^2}\right\}^{-1} = \frac{1}{2\zeta}$$

$$\therefore \quad \text{log magnitude} = 20\log\frac{1}{2\zeta} = 4, \text{ or } \frac{1}{2\zeta} = e^{1/5}, \text{ or } \zeta = 0.316$$

$$\therefore \quad \text{the term is} \quad = \left(1+\frac{2\times0.316s}{10}+\frac{s^2}{100}\right)^{-1}$$

$$\therefore \quad G(s) = \frac{8(1+0.2s)}{s^2(1+s)\left(1+0.0632s+\frac{s^2}{100}\right)}$$

or

$$G(s) = \frac{800(1+0.2s)}{s^2(1+s)(s^2+6.32s+100)}$$

Minimum phase, All pass and Non-Minimum phase systems:

(1) All pass System: A system having a pole-zero pattern which is antisymmetric about the imaginary axis, ie, for every pole in left half s-plane, there is a zero in the mirror image position. The transfer function of all pass system is given by

$$G(j\omega) = \left(\frac{1-j\omega T}{1+j\omega T}\right) \longrightarrow ①$$

$$\text{Magnitude} = |G(j\omega)| = \frac{\sqrt{1+\omega^2T^2}}{\sqrt{1+\omega^2T^2}} = 1$$

$$\text{phase angle } \phi = -\tan^{-1}(\omega T) - \tan^{-1}(\omega T) = -2\tan^{-1}(\omega T)$$

Thus, the all pass system has a magnitude of unity and phase angle varies from 0 to $-180°$ as ω is increased from 0 to ∞.

(2) Non-Minimum phase System: If a system has poles in the left half s-plane and zeros in both the left and right half s-plane, such a system is said to be non-minimum phase system. The transfer function of such a system is given by

$$G_1(j\omega) = \frac{(1-j\omega T)}{(1+j\omega T_1)(1+j\omega T_2)} \longrightarrow ②$$

$$\text{Magnitude} = |G_1(j\omega)| = \frac{\sqrt{1+\omega^2T^2}}{\sqrt{1+\omega^2T_1^2}\cdot\sqrt{1+\omega^2T_2^2}}$$

$$\text{phase angle } \phi = -\tan^{-1}(\omega T) - \tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$$

Non-minimum phase system is a combination of both allpass and minimum phase systems. The transfer function of non-minimum phase system is also given by

$$G_1(j\omega) = G_2(j\omega)\,G(j\omega)$$

where $G_2(j\omega)$ is minimum phase system.

<u>Minimum-phase</u> <u>System</u>: If all the poles and zeros of a system lie in left half s-plane, the system is said to be minimum-phase system. The transfer function of minimum phase system is given by

$$G_2(j\omega) = \frac{(1+j\omega T)}{(1+j\omega T_1)(1+j\omega T_2)} \longrightarrow (3)$$

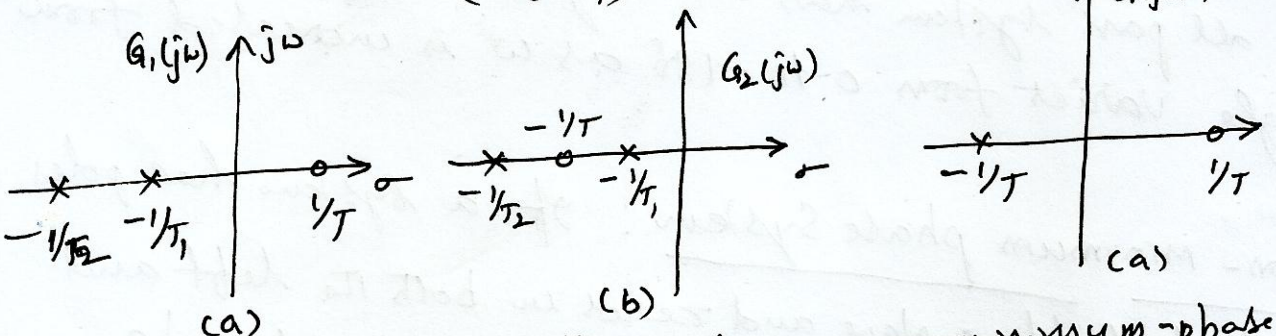


Figure: <u>pole-zero patterns for</u> (a) <u>nonminimum-phase function</u> (b) <u>minimum-phase function</u> (c) <u>all-pass function</u>

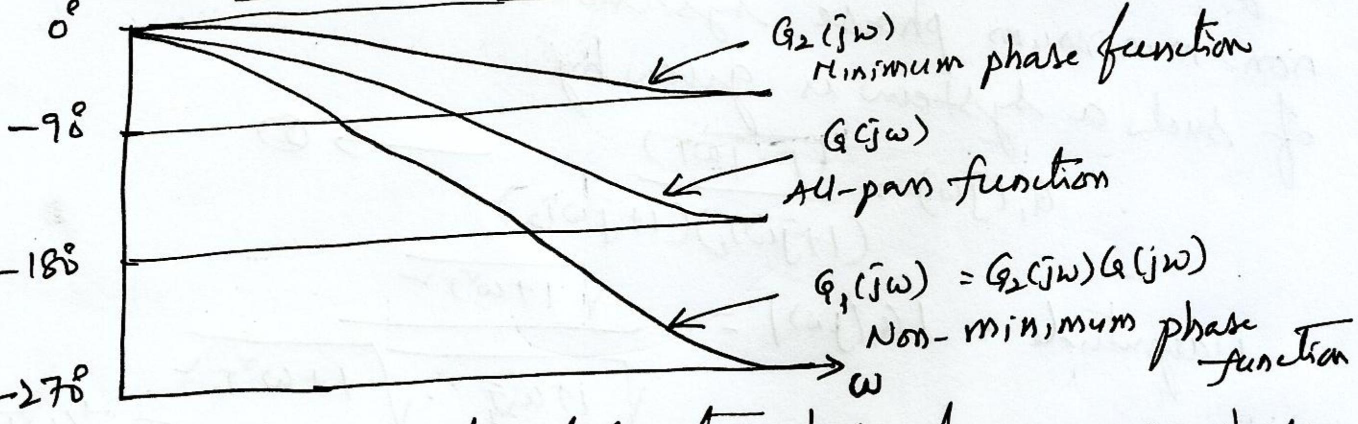


Figure: <u>phase angle characteristics</u> of <u>minimum-phase</u> <u>All-pass</u> and <u>nonminimum-phase</u> <u>functions</u>.

Polar plots : Let us consider a simple RC network (20)

shown in figure.

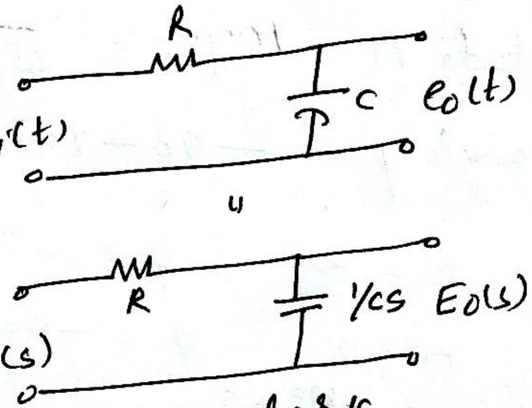


Figure : RC network.

The transfer function $G(s) = \dfrac{E_o(s)}{E_i(s)} = \dfrac{1/cs}{R+1/cs} = \dfrac{1}{1+RCs}$

where $T = RC$ is the time constant

$\therefore$ Transfer function $G(s) = \dfrac{1}{1+sT}$

The sinusoidal TF $G(j\omega) = \dfrac{1}{1+j\omega T} \longrightarrow (1)$

Magnitude $|G(j\omega)| = M = \dfrac{1}{\sqrt{1+\omega^2T^2}}$

phase angle $\angle G(j\omega) = \phi = -\tan^{-1}(\omega T)$

If $\omega = 0 \quad M = 1; \quad \phi = 0$

$\omega = \frac{1}{T} \quad M = \frac{1}{\sqrt{2}} \quad \phi = -45^\circ$

$\omega \to \infty \quad M \to 0 \quad \phi \to -90^\circ$

As the input frequency ω is varied from 0 to ∞, the magnitude M and phase angle ϕ change and hence the tip of the phase $G(j\omega)$ traces a locus in the complex plane. The locus thus obtained is known as polar plot.

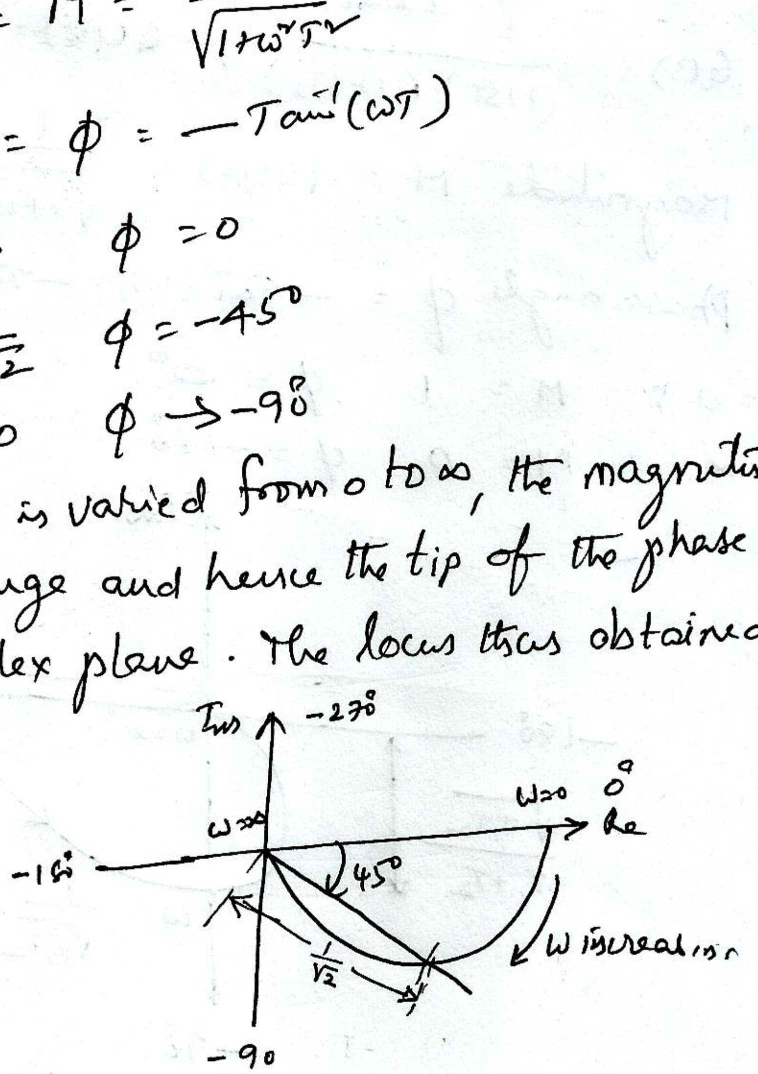


Figure : polar plot of $\dfrac{1}{1+j\omega T}$

② Sketch the polar plot of $G(j\omega) = \dfrac{1}{j\omega(1+j\omega T)}$

(Sol) Magnitude $M = |G(j\omega)| = \dfrac{1}{\omega\sqrt{1+\omega^2T^2}}$

phase angle $\phi = -90^\circ - \tan^{-1}(\omega T)$

$\omega = 0 \qquad M = \infty \quad ; \quad \phi = -90^\circ$

$\omega = \infty \qquad M = 0 \qquad \phi = -90-90 = -180^\circ$

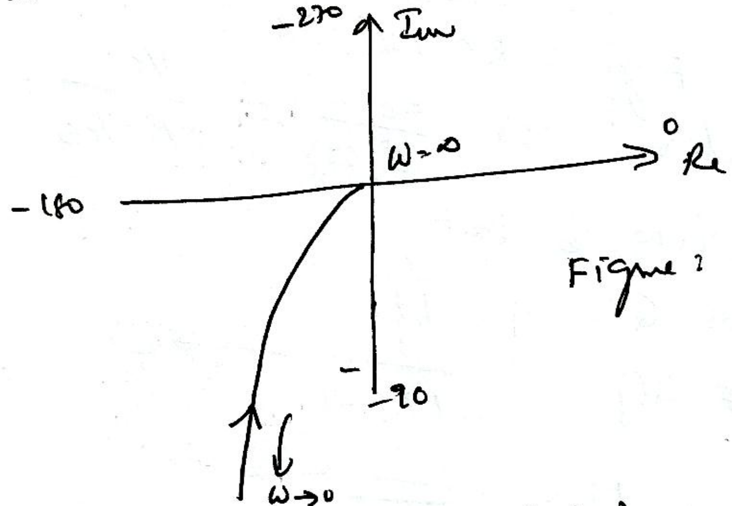


Figure: polar plot of $\dfrac{1}{j\omega(1+j\omega T)}$

(3) $G(s) = \dfrac{1}{(1+sT_1)(1+sT_2)}$; $G(j\omega) = \dfrac{1}{(1+j\omega T_1)(1+j\omega T_2)}$

∴ Magnitude $M = |G(j\omega)| = \dfrac{1}{\sqrt{1+\omega^2T_1^2}\sqrt{1+\omega^2T_2^2}}$

Phase angle $\phi = -\tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$

$\omega = 0 ; \quad M = 1 \quad \phi = 0^\circ$

$\omega = \infty \quad M = 0 \quad \phi = -180^\circ$

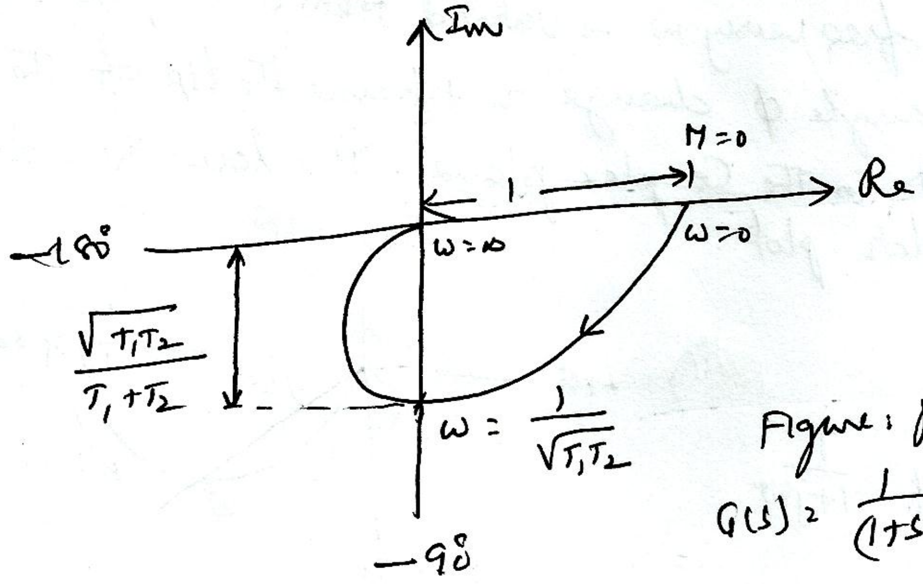


Figure: polar plot of $G(s) = \dfrac{1}{(1+sT_1)(1+sT_2)}$

(3) $G(s) = \dfrac{1}{(1+sT_1)(1+sT_2)(1+sT_3)}$;

$$G(j\omega) = \frac{1}{(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$$

Magnitude $M = \dfrac{1}{\sqrt{1+\omega^2 T_1^2}\sqrt{1+\omega^2 T_2^2}\sqrt{1+\omega^2 T_3^2}}$

$$\phi = -\tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2) - \tan^{-1}(\omega T_3)$$

$\omega = 0$; $M = 1$; $\phi = 0$

$\omega \to \infty$ $M \to 0$ $\phi \to -270$

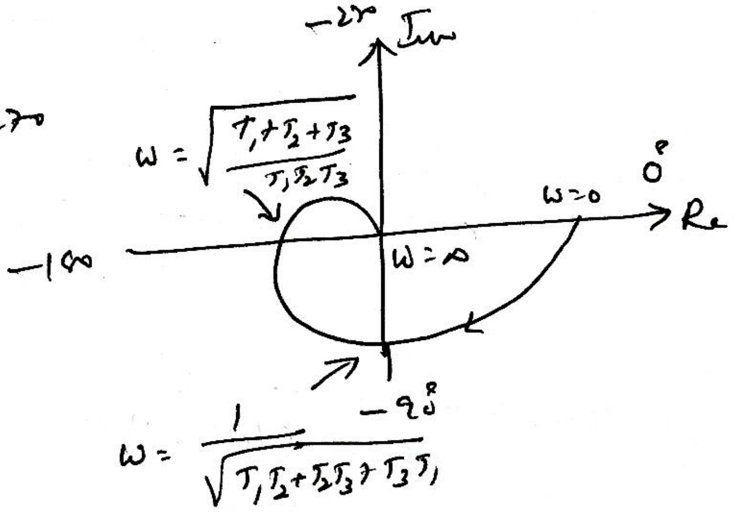


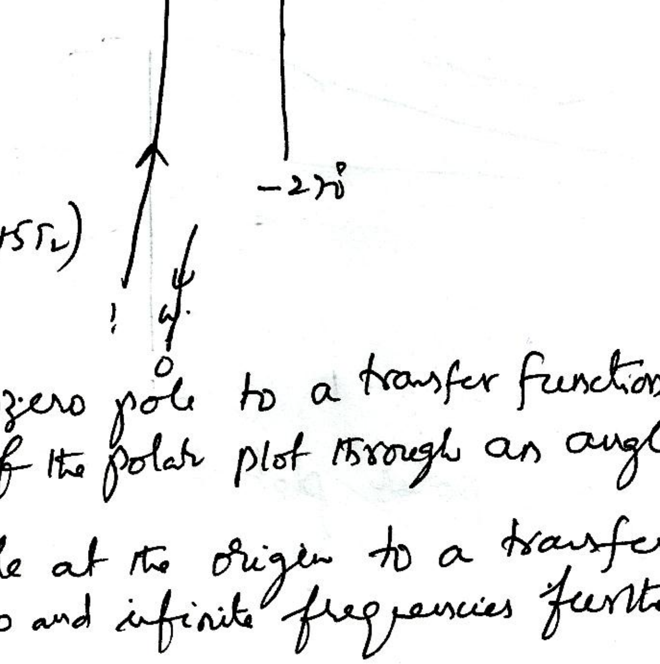
Figure: polar plot of $G(s) = \dfrac{1}{(1+sT_1)(1+sT_2)(1+sT_3)}$

(4) $G(j\omega) = \dfrac{1}{j\omega(1+j\omega T_1)(1+j\omega T_2)}$

$M = \dfrac{1}{\omega\sqrt{1+\omega^2 T_1^2}\sqrt{1+\omega^2 T_2^2}}$; $\phi = -90 - \tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$

$\omega = 0$ $M = \infty$ $\phi = -90°$

$\omega \to \infty$ $M \to 0$ $\phi = -270$



polar plot of $G(s) = \dfrac{1}{s(1+sT_1)(1+sT_2)}$

Note: (1) Addition of nonzero pole to a transfer function results in further rotation of the polar plot through an angle of $-90°$ as $\omega \to \infty$.

(2) Addition of a pole at the origin to a transfer function rotates the polar plot at zero and infinite frequencies further by an angle of $-90°$.

(5) $G(s) = \dfrac{1}{s^2(1+sT_1)(1+sT_2)}$ $\therefore G(j\omega) = \dfrac{1}{(j\omega)^2(1+j\omega T_1)(1+j\omega T_2)}$

$\therefore M = \dfrac{1}{\omega^2\sqrt{1+\omega^2T_1^2}\sqrt{1+\omega^2T_2^2}}$; $\phi = -180 - \tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$

$\omega = 0 \quad M = \infty; \quad \phi = -180^\circ$

$\omega = \infty \quad M = 0 \quad \phi = -360^\circ$

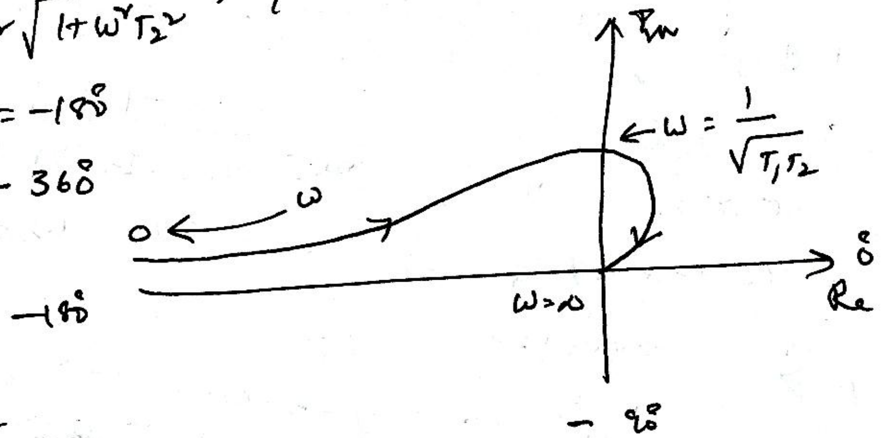


Figure: polar plot of
$G(s) = \dfrac{1}{s^2(1+sT_1)(1+sT_2)}$

(6) $G(s) = \dfrac{1}{s^2(1+sT_1)(1+sT_2)(1+sT_3)}$ $\therefore G(j\omega) = \dfrac{1}{(j\omega)^2(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$

$M = \dfrac{1}{\omega^2\sqrt{1+\omega^2T_1^2}\sqrt{1+\omega^2T_2^2}\sqrt{1+\omega^2T_3^2}}$; $\phi = -180^\circ - \tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2) - \tan^{-1}(\omega T_3)$

$\omega = 0; \quad M = \infty; \quad \phi = -180^\circ$

$\omega \to \infty \quad M \to 0 \quad \phi \to -450^\circ$

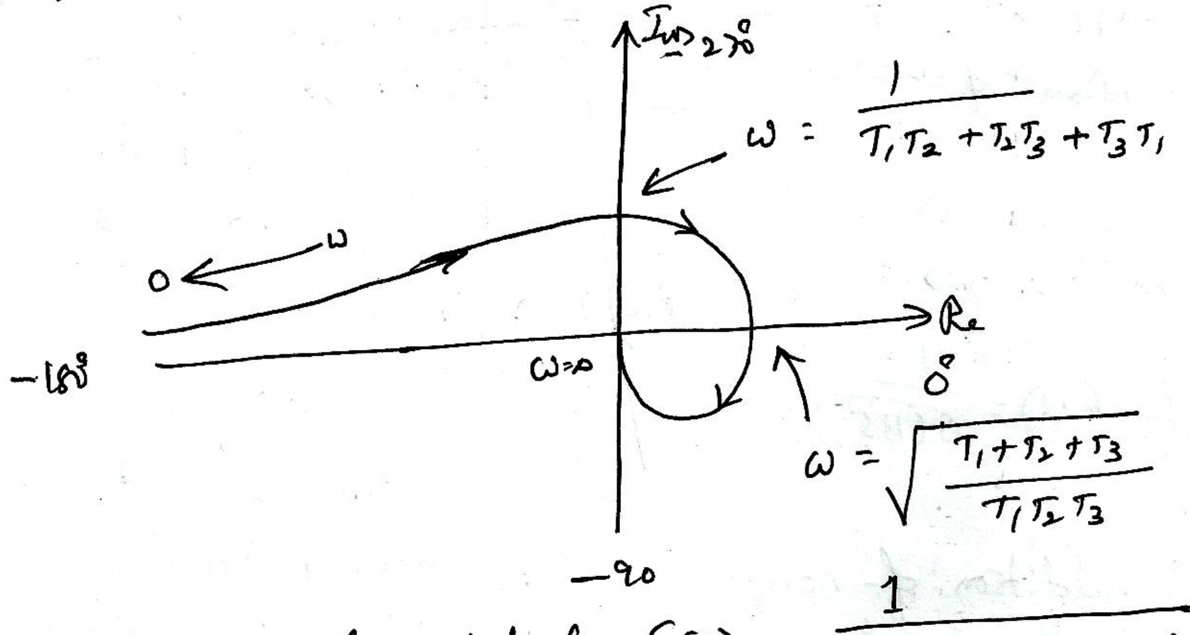


Figure: polar plot of $G(s) = \dfrac{1}{s^2(1+sT_1)(1+sT_2)(1+sT_3)}$

Principle of Argument of Cauchy: Let us consider a function $q(s)$ given by

$$q(s) = \frac{(s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_m)}{(s-\beta_1)(s-\beta_2)\cdots(s-\beta_n)} \rightarrow ①$$

Let 's' is a complex variable represented by $s = \sigma + j\omega$ on the complex plane. Then $q(s)$ is also complex and may be defined as $q(s) = u + jv$

A function $q(s)$ is analytic in the s-plane provided the function and all the derivatives of it exists. The points in the s-plane where the function or its derivatives does not exist, are called singular points. The poles of a function are singular points.

The equation ① indicates that for every point 's' in the s-plane at which $q(s)$ is analytic, we can find a corresponding point $q(s)$ in the $q(s)$-plane. Alternatively, it can be stated that the function $q(s)$ maps the points in the s-plane into $q(s)$-plane. It follows that for a contour in the s-plane which does not go through any singular point, there corresponds a contour in the $q(s)$-plane as shown in figure. The region to the right of a closed contour is considered enclosed by the contour when the contour is traversed in the clockwise direction. Thus the shaded area is enclosed by the closed contour

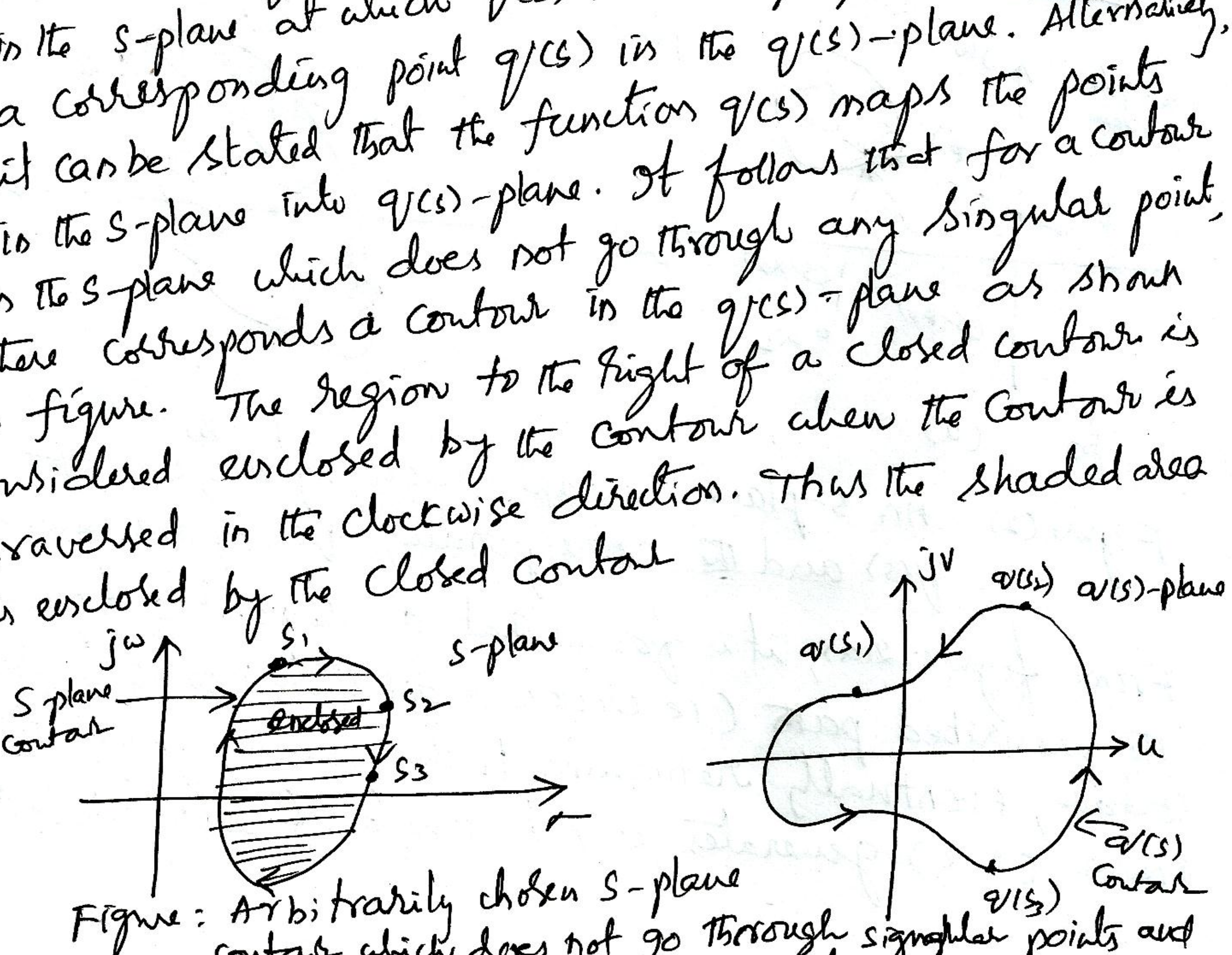


Figure: Arbitrarily chosen s-plane contour which does not go through singular points and the corresponding $q(s)$ plane contour.

we are not interested in the exact shape of the $q(s)$-plane contour. An important fact that concerns is the encirclement of the origin by the $q(s)$-plane contour. To investigate this, consider an s-plane contour which encloses only one of the zeros of $q(s)$, say $s=\alpha_1$, while all the poles and remaining zeros are distributed in the s-plane outside the contour. For any non-singular point 's' on the s-plane contour, there corresponds a point $q(s)$ on the $q(s)$-plane contour, From eq ①, the point $q(s)$ is given by

$$|q(s)| = \frac{|s-\alpha_1|\,|s-\alpha_2|\,\cdots}{|s-\beta_1|\,|s-\beta_2|\,\cdots} \longrightarrow ②$$

$$\angle q(s) = \angle s-\alpha_1 + \angle s-\alpha_2 + \cdots - \angle s-\beta_1 - \angle s-\beta_2 \cdots$$

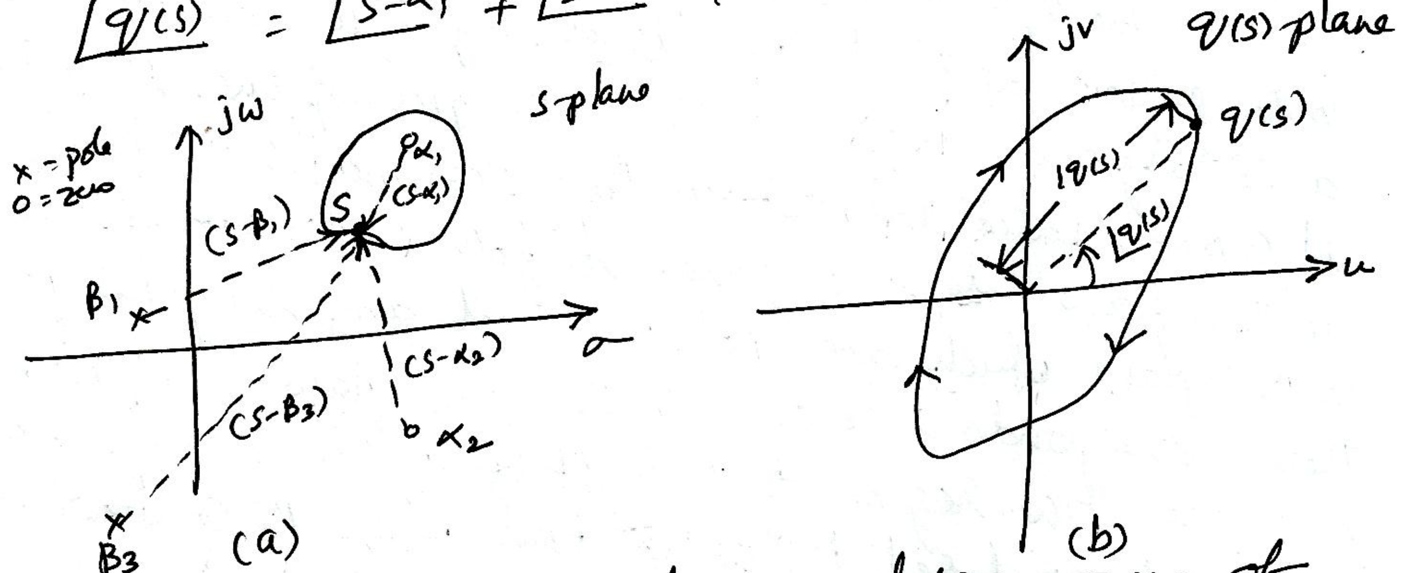


Figure (2): An s-plane contour enclosing a zero of $q(s)$ and the corresponding $q(s)$-plane contour.

From figure 2(a), it is found that as the point s follows the prescribed path (ie clockwise direction) on the s-plane contour, eventually returning to the starting point, the phasor $(s-\alpha_1)$ generates a net angle of -2π, while the other phasors generate zero net angles. Therefore, the $q(s)$-phasor undergoes a net phase change of -2π. This implies that the tip of the $q(s)$-phasor must describe a closed contour about the origin of the $q(s)$-plane in the clockwise direction as shown in figure 2(a).

The exact shape of the closed contour in the $q(s)$-plane is not interest to us, but it is sufficient for us to observe that this contour encircles the origin once. If the contour in the s-plane is so chosen that it does not enclose any zero or pole, the corresponding contour in $q(s)$-plane then will not encircle the origin.

If the s-plane contour encloses two zeros, say at $s=\alpha_1$ and $s=\alpha_2$, the $q(s)$-plane contour encircles the origin twice in the clockwise direction as shown in figure (3). Generalizing, we can say that for each zero of $q(s)$ enclosed by the s-plane contour, the corresponding $q(s)$-plane contour encircles the origin once in the clockwise direction.

If the s-plane contour encloses a pole at $s=\beta_1$, then the phasor $(s-\beta_1)$ generates an angle of -2π as 's' traverses the prescribed path. Since $(s-\beta_1)$ is in the denominator, the $q(s)$-plane contour experiences an angle change of $+2\pi$, which means one counter-clockwise encirclement of the origin.

Thus, if there are 'p' poles and 'z' zeros of $q(s)$ enclosed by the s-plane contour, then the corresponding $q(s)$-plane contour must encircle the origin Z-times in the clockwise direction and 'p' times in the counter clockwise direction, resulting in a net encirclement of the origin, (P-Z) times in the counter clockwise direction.

For example, in case of 1 zero and 3 poles enclosed by the s-plane contour, the net encirclement of the origin by by the q(s)-plane contour is 2π(3-1) = 4π rad, i.e, two counterclockwise revolutions as shown below.

This relation between the enclosure of poles and zeros of q(s) by the s-plane contour and the encirclements of the origin by the q(s)-plane contour is commonly known as the principle of argument.

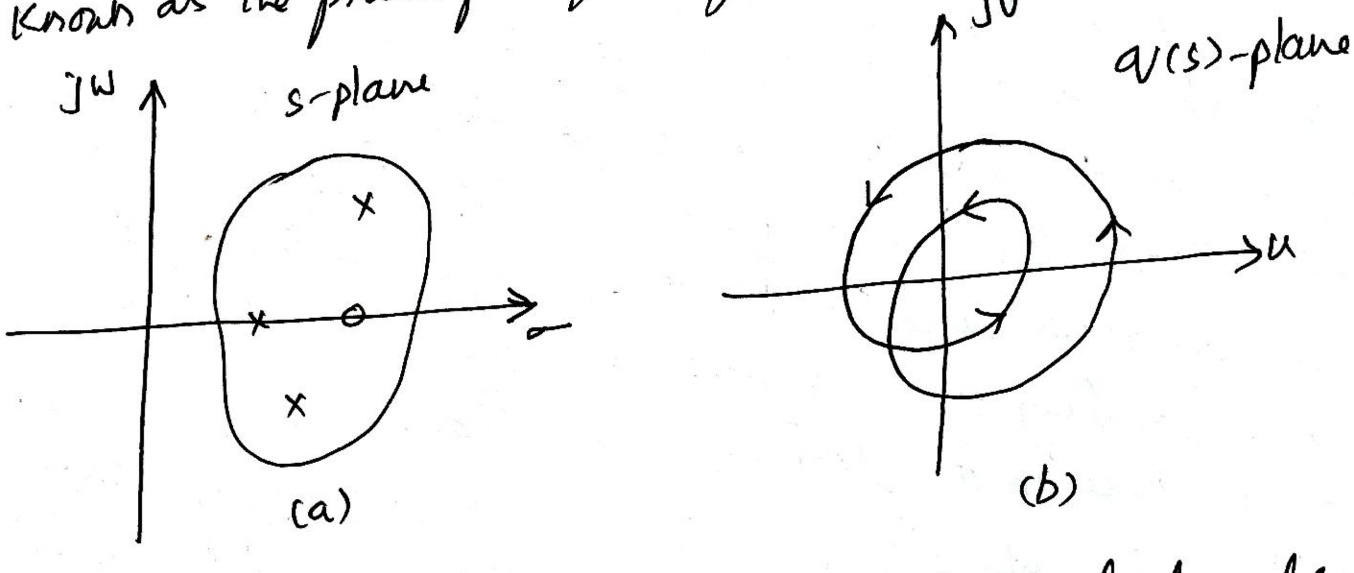


Figure: Mapping of the s-plane contour which encloses 1 zero and 3 poles

Nyquist stability criterion: Consider a feedback system shown in figure:

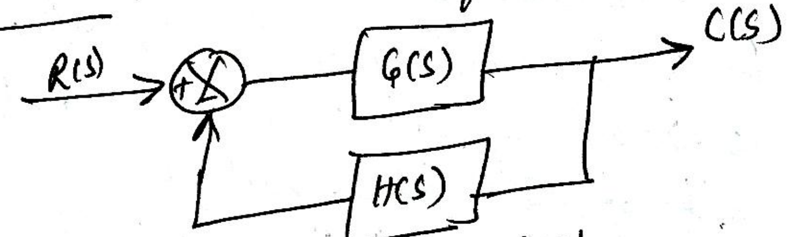


Figure: Feedback system

The characteristic equation of the system is

$$q(s) = 1 + G(s)H(s) = 0.$$

The pole-zero form of the open-loop transfer function is

$$G(s)H(s) = K\frac{(s+z_1)(s+z_2) \cdots (s+z_m)}{(s+p_1)(s+p_2) \cdots (s+p_n)}; \quad m \leq n \quad \rightarrow (1)$$

$$\therefore q(s) = 1 + \frac{K(s+z_1)(s+z_2) \cdots (s+z_m)}{(s+p_1)(s+p_2) \cdots (s+p_n)} \longrightarrow (2)$$

$$= \frac{(s+p_1)(s+p_2) \cdots (s+p_n) + K(s+z_1)(s+z_2) \cdots (s+z_m)}{(s+p_1)(s+p_2) \cdots (s+p_n)}$$

$$= \frac{(s+z_1')(s+z_2') \cdots (s+z_n')}{(s+p_1)(s+p_2) \cdots (s+p_n)} \longrightarrow (3)$$

From the above equation it is seen that the zeros of $q(s)$ at $s = -z_1', -z_2' \cdots -z_n'$ are the roots of the characteristic equation and the poles of $q(s)$ at $-p_1, -p_2 \cdots -p_n$ are same as the open-loop poles of the system.

For the system to be stable, the roots of the characteristic equation and hence the zeros of $q(s)$ must lie in the left half of the s-plane.

It is important to note that even if some of the open-loop poles lie in the right-half s-plane, all the zeros of $q(s)$, ie, the closed loop poles may lie in the left half s-plane. That is even an open-loop unstable system may lead to a closed-loop stable operation.

In order to investigate the presence of any zeros of $q(s) = 1 + G(s)H(s)$ in the right half s-plane, let us choose a contour which completely encloses right half of the s-plane. Such a contour 'C' is called Nyquist contour is shown in figure. The Nyquist contour is directed clockwise and comprises of an infinite line segment C_1 along the $j\omega$-axis and an arc C_2 of infinite radius.

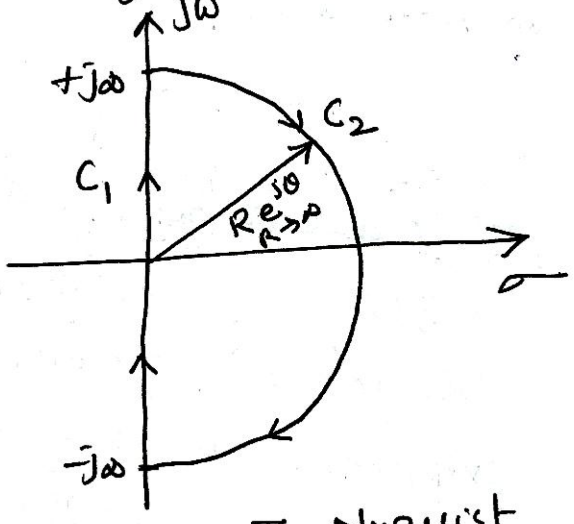


Figure: The Nyquist Contour.

Along C_1,

$s = j\omega$ with s varying from $-j\infty$ to $+j\infty$

Along C_2, $s = \underset{R \to \infty}{Re^{j\theta}}$ with 'θ' varying from $+\frac{\pi}{2}$ to 0 to $-\frac{\pi}{2}$

The Nyquist contour so defined encloses all the right half s-plane zeros and poles of $q(s) = 1 + G(s)H(s)$. Let there are Z zeros and P poles of $q(s)$ in the right half s-plane. As 's' moves along the Nyquist contour in the s-plane, a closed contour Γ_q traversed in the $q(s)$-plane which encloses the origin $N = P - Z$ times in the counter clockwise direction

For the system to be stable, there should be no zeros of $q(s) = 1 + G(s)H(s)$ in the right half s-plane, ie $Z = 0$. This condition is met if $N = P$. That is, for a system (closed-loop) to be stable, the number of counter clockwise encirclements of the origin of the $q(s)$-plane by the contour Γ_q should equal the number of the right half s-plane poles of $q(s)$ which are the poles of open-loop transfer function $G(s)H(s)$.

In the special case of $P=0$, the closed loop system is stable if $N = P = 0$

It is easily observed that $G(s)H(s) = [1+G(s)H(s)] - 1$

Therefore, it follows that the contour Γ_{GH} of $G(s)H(s)$ corresponding to the Nyquist contour in the s-plane is the same as contour Γ_q of $1+G(s)H(s)$ drawn from the point $(-1+j0)$. Thus the encirclement of the origin by the contour Γ_q is equivalent to the encirclement of the point $(-1+j0)$ by the contour Γ_{GH} as shown in figure.

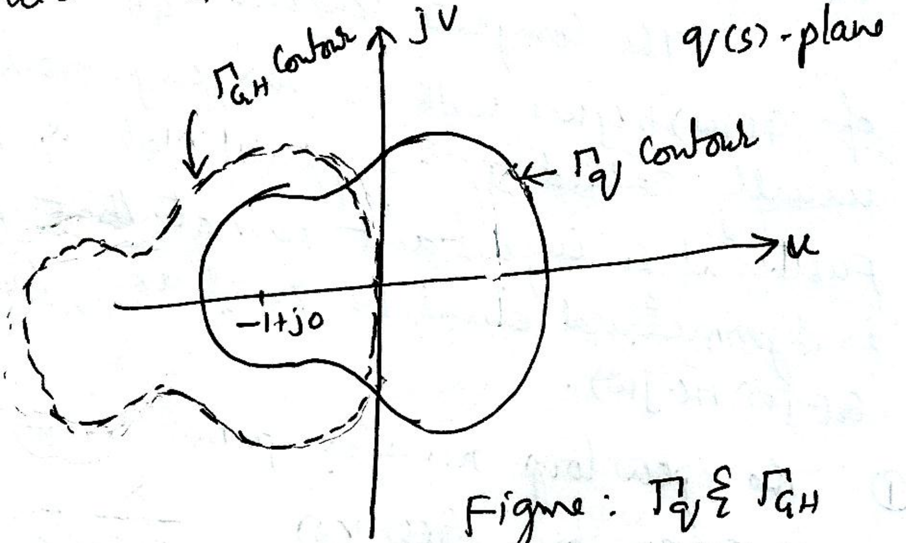


Figure: Γ_q & Γ_{GH} Contours.

The Nyquist stability criterion, now can be stated as;

If the Contour Γ_{GH} of the open-loop transfer function $G(s)H(s)$ corresponding to the Nyquist contour in the s-plane encircles the point $(-1+j0)$ in the counter clock wise direction as many times as the number of right half s-plane poles of $G(s)H(s)$, the closed loop systems is stable.

If the open loop system is stable, then the corresponding closed loop systems is stable, if the contour Γ_{GH} of $G(s)H(s)$ does not encircle $(-1+j0)$ point, ie, the net encirclement is zero.

The mapping of the Nyquist contour into the contour Γ_{GH} is carried out as follows:

(1) The mapping of the imaginary axis is carried out by substitution of $s = j\omega$ in $G(s)H(s)$. This converts the mapping function into a frequency function of $G(j\omega)H(j\omega)$

(2) In physical systems $(m \le n)$, $\lim_{\substack{s = Re^{j\theta} \\ R \to \infty}} G(s)H(s) =$ real constant (it is zero if $m < n$). Thus the infinite arc of the Nyquist contour maps into a point on the real axis.

The complete contour Γ_{GH} is thus the polar plot of $G(j\omega)H(j\omega)$ with 'ω' varying from $-\infty$ to ∞. This is usually called the Nyquist plot or locus of $G(s)H(s)$. Further it is important to note that the Nyquist plot is symmetrical about the real axis, since $G^*(j\omega)H^*(j\omega) = G(-j\omega)H(-j\omega)$.

① The open loop transfer function of a system (feedback) is given by $G(s)H(s) = \dfrac{K}{(1+T_1 s)(1+T_2 s)}$. Sketch the Nyquist plot and comment on stability of closed loop system.

(Sol)

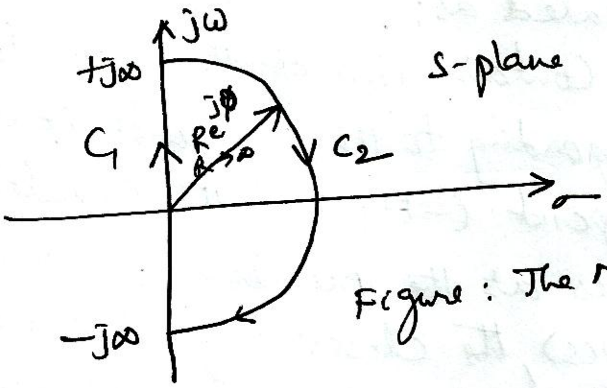


Figure: The Nyquist contour

(1) The mapping of the imaginary axis C_1 into $q(s)$ plane is carried by substituting $s = j\omega$ in $G(s)H(s)$

$$G(j\omega)H(j\omega) = \frac{K}{(1+j\omega T_1)(1+j\omega T_2)}$$

$$M = |G(j\omega)H(j\omega)| = \frac{K}{\sqrt{1+\omega^2 T_1^2}\sqrt{1+\omega^2 T_2^2}}$$

$$\phi = \angle G(j\omega)H(j\omega) = -\tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$$

$\omega = 0$ $\quad M = K;$ $\quad \phi = 0$

$\omega \to \infty$ $\quad M \to 0$ $\quad \phi \to -180^\circ$

(2) The maping of semicircular arc C_2 is carried out by replacing $s = \underset{R\to\infty}{Re^{j\phi}}$ $(\phi \to +90^\circ \to 0\ 90^\circ)$

$$\therefore \underset{R\to\infty}{Lt} \frac{K}{(1+T_1 Re^{j\phi})(1+T_2 Re^{j\phi})} = 0\, e^{-j2\phi}$$

where semicircular arc is mapped into a point (origin) in $G(s)$-plane

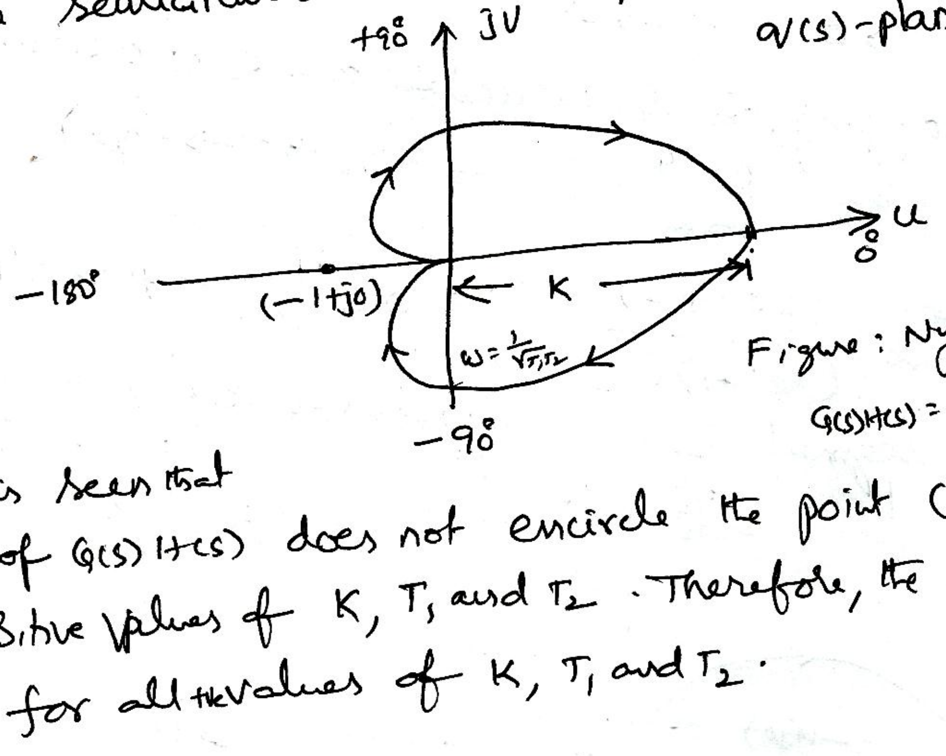
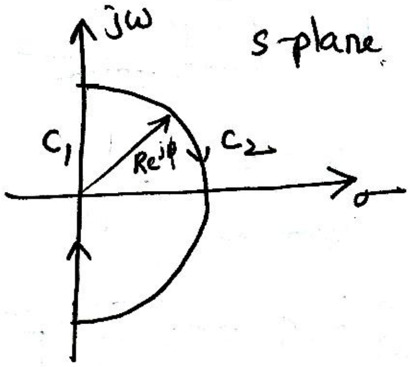


Figure : Nyquist plot of $G(s)H(s) = \frac{K}{(1+sT_1)(1+sT_2)}$

It is seen that the plot of $G(s)H(s)$ does not encircle the point $(-1+j0)$ for any positive values of K, T_1 and T_2. Therefore, the system is stable for all the values of K, T_1 and T_2.

(2) The open loop transfer function of a unity feedback system is given by $G(s)H(s) = \dfrac{(s+2)}{(s+1)(s-1)}$. Draw the Nyquist plot and determine stability of closed loop system.

(sol)
$$G(j\omega)H(j\omega) = \frac{(j\omega+2)}{(j\omega+1)(j\omega-1)}$$



$$\text{Magnitude } M = \frac{\sqrt{4+\omega^2}}{\sqrt{1+\omega^2}\sqrt{1+\omega^2}}$$

$$\phi = \tan^{-1}(\omega/2) - \tan^{-1}(\omega) - \tan^{-1}(-\omega)$$
$$= \tan^{-1}(\omega/2) - \cancel{\tan^{-1}(\omega)} - \{\pi - \cancel{\tan^{-1}(\omega)}]$$
$$= -\pi + \tan^{-1}(\omega/2)$$

$\omega = 0 \qquad M = \sqrt{4} = 2; \qquad \phi = -\pi$ or -180°

$\omega \to \infty \qquad M \to 0 \qquad \phi \to -90^\circ$

(2) The mapping of semicircular arc C_2 is carried out by replacing s by $\underset{R\to\infty}{Re^{j\phi}}$ $(\phi \to +\pi/2 \to 0 \to -\pi/2)$

$$\therefore \underset{R\to\infty}{Lt} \frac{(Re^{j\phi}+2)}{(Re^{j\phi}+1)(Re^{j\phi}-1)} = 0e^{-j\phi}; \quad -\phi \to -\pi/2 \to 0 \to \pi/2$$

Thus, the segment C_2 is mapped into origin in $q(s)$-plane

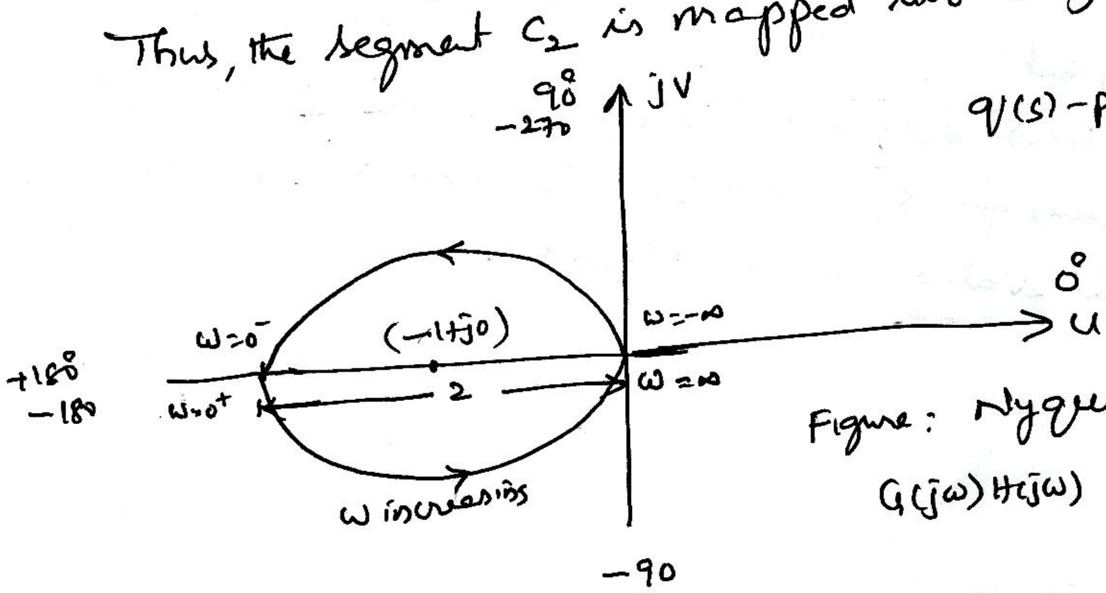


Figure: Nyquist plot of $G(j\omega)H(j\omega) = \dfrac{(j\omega+2)}{(j\omega+1)(j\omega-1)}$

The contour encircles $(-1+j0)$ point one time in counter clock wise direction $\therefore N = 1$; No of right side open loop poles $P = -1$
$\therefore N = P - Z \Rightarrow Z = P - N = 1 - 1 = 0$. No zeros of $G(s)H(s)$ lies on RHS. Hence the system is stable.

The Nyquist plot encircles the $(-1+j0)$ point one time in counterclockwise direction. Therefore $N = 1$ 27

The number of RHS open loop poles $P = 1$.

$\therefore$ The number of zeros of $G(s)H(s)$ on RHS $= Z$

where $N = P - Z$

$\therefore Z = P - N = 1 - 1 = 0$

None of the zeros of $G(s)H(s)$ lie on RHS, Therefore the closed loop system is stable.

Open loop poles on the $j\omega$ axis: If $G(s)H(s)$ and therefore $1+G(s)H(s)$ has any poles on the $j\omega$-axis, the Nyquist contour should not pass through those poles. To study stability in this case, the Nyquist contour must be modified so as to bypass any $j\omega$-axis poles. This is accomplished by indenting the Nyquist contour around the $j\omega$-axis poles along a semicircular of radius ϵ where $\epsilon \to 0$ as shown in figure.

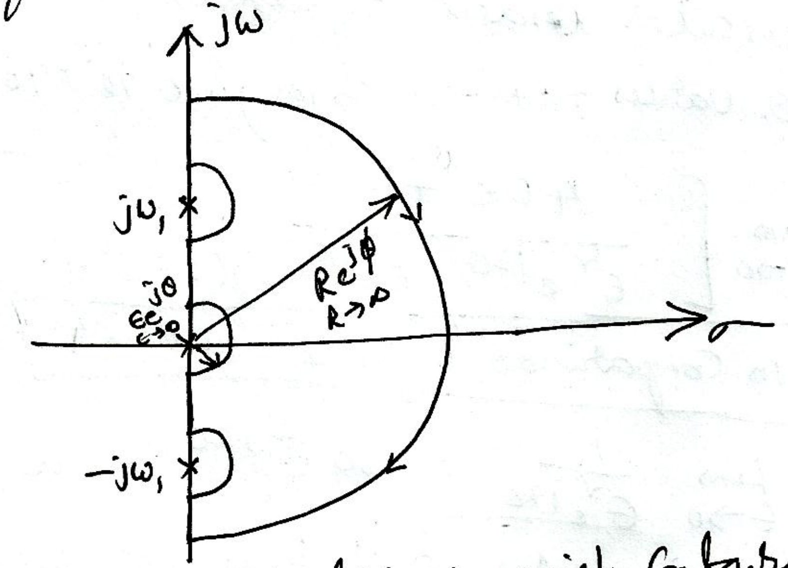


Figure: Indented Nyquist Contour for $j\omega$-axis open loop poles.

① Consider a system with open loop transfer function

$$G(s)H(s) = \frac{(4s+1)}{s^2(s+1)(2s+1)}$$. Determine stability of the system from Nyquist stability criterion

(Sol)

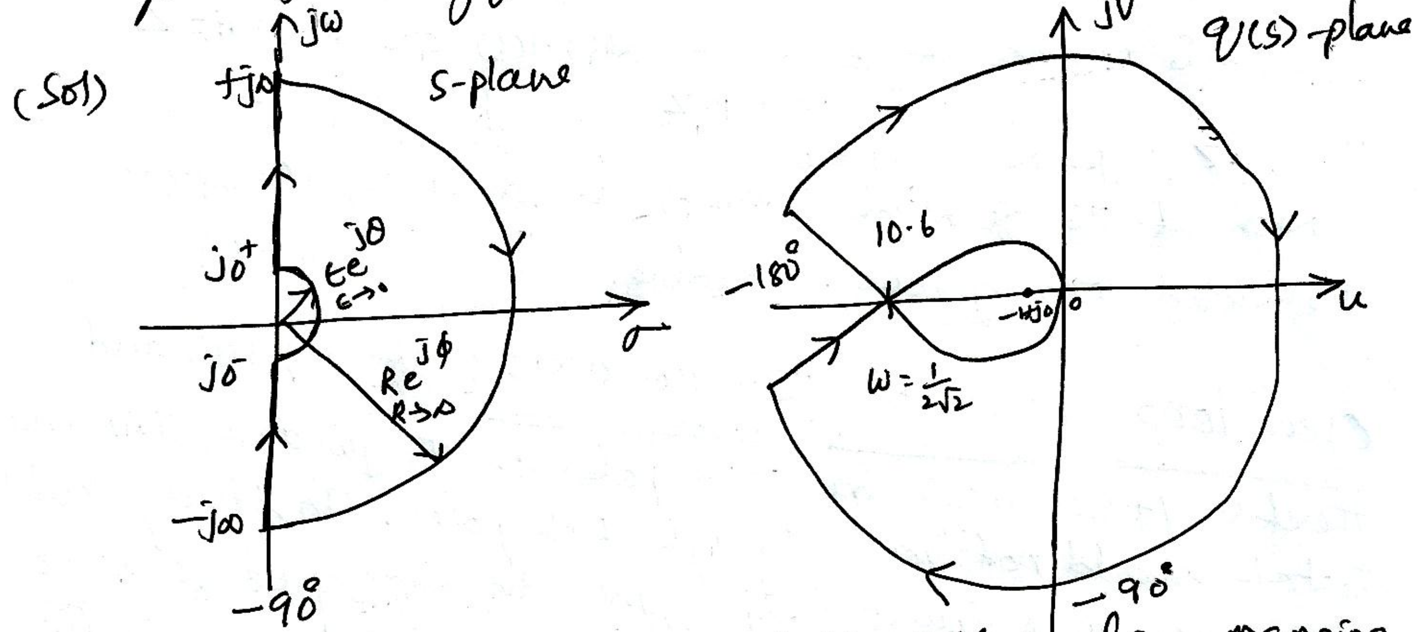


Figure: Nyquist Contour and the corresponding mapping for $G(s)H(s) = (4s+1)/s^2(s+1)(2s+1)$

(1) Semicircular indent represented by $s = \lim_{\epsilon \to 0} \epsilon e^{j\theta}$ (where θ varies from -90° through 0 to +90°) is mapped into

$$\lim_{\epsilon \to 0}\left[\frac{4\epsilon e^{j\theta}+1}{\epsilon^2 e^{j2\theta}(\epsilon e^{j\theta}+1)(2\epsilon e^{j\theta}+1)}\right] = \lim_{\epsilon \to 0}\left(\frac{1}{\epsilon^2 e^{j2\theta}}\right)$$

$\boxed{\text{Note : in Comparison with 1} \quad 1+\epsilon \simeq 1}$

$$\therefore \lim_{\epsilon \to 0} \frac{1}{\epsilon^2 e^{j2\theta}} = \infty e^{-j2\theta} = \infty\,(180° \to 0 \to -180°)$$

Thus the semicircular indent is mapped into an infinite circle in q(s)-plane

(2) Along the jω axis $G(j\omega)H(j\omega) = \dfrac{1+j4\omega}{(j\omega)^2(1+j\omega)(1+j2\omega)}$

$$M = \frac{\sqrt{1+(4\omega)^2}}{\omega^2\sqrt{1+\omega^2}\sqrt{1+(2\omega)^2}}$$

phase angle $\phi = \tan^{-1}(4\omega) - 180° - \tan^{-1}(\omega) - \tan^{-1}(2\omega)$ (28)

of $\omega = 0$, Magnitude $M = \infty$ $\phi = -180°$

$\omega = \infty$ $M = 0$; $\phi = 90 - 180 - 90 - 90 = -270°$

(3) The infinite semi circular arc represented by $S = \lim_{R\to\infty} Re^{j\phi}$ (ϕ varies from $+90°$ through 0 to $-90°$) is mapped into $= \lim_{R\to\infty} \frac{(1+4Re^{j\phi})}{R^2 e^{j2\phi}(1+Re^{j\phi})(1+Re^{j\phi})} = 0e^{-j3\phi}$

$= 0(-270° \to 0 \to 270°)$

Thus the infinite semicircular arc is mapped into a point in $G(s)$ plane.

The $G(j\omega)H(j\omega)$ locus intersects the Real axis at a point where $\angle G(j\omega)H(j\omega) = -180°$

$$\Rightarrow +\tan^{-1}(4\omega) - 180 - \tan^{-1}(\omega) - \tan^{-1}(2\omega) = -180°$$

$$\Rightarrow \tan^{-1}(4\omega) = \tan^{-1}(\omega) + \tan^{-1}(2\omega)$$

Taking tan on both sides

$$\tan(\tan^{-1}4\omega) = \tan(\tan^{-1}\omega + \tan^{-1}2\omega)$$

$$4\omega = \frac{\omega + 2\omega}{1-2\omega^2} = \frac{3\omega}{1-2\omega^2}$$

$$\Rightarrow 4\cancel{\omega}(1-2\omega^2) = 3\cancel{\omega} \Rightarrow 2\omega^2 = 1 - \frac{3}{4} = \frac{1}{4}$$

$$1-2\omega^2 = 3/4$$

$$\therefore \omega^2 = \frac{1}{8} \text{ and } \omega = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}}$$

$\therefore$ The magnitude at $\omega = \frac{1}{2\sqrt{2}}$ is

$$|G(j\omega)H(j\omega)|_{\omega=\frac{1}{2\sqrt{2}}} = \left.\frac{\sqrt{1+(4\omega)^2}}{\omega^2\sqrt{1+\omega^2}\sqrt{1+(2\omega)^2}}\right|_{\omega=\frac{1}{2\sqrt{2}}} = 10.6$$

The mapped Contour encircles $(-1+j0)$ point '2' times in clock wise direction $\therefore N = -2$ and $P = 0$ $\therefore Z = 0-(-2) = 2$

Therefore, the system is unstable.

① Consider a unity feedback system with open loop transfer function $G(s) = \frac{1}{s(1+0.2s)(1+0.05s)}$. Sketch the polar plot and determine GM & PM.

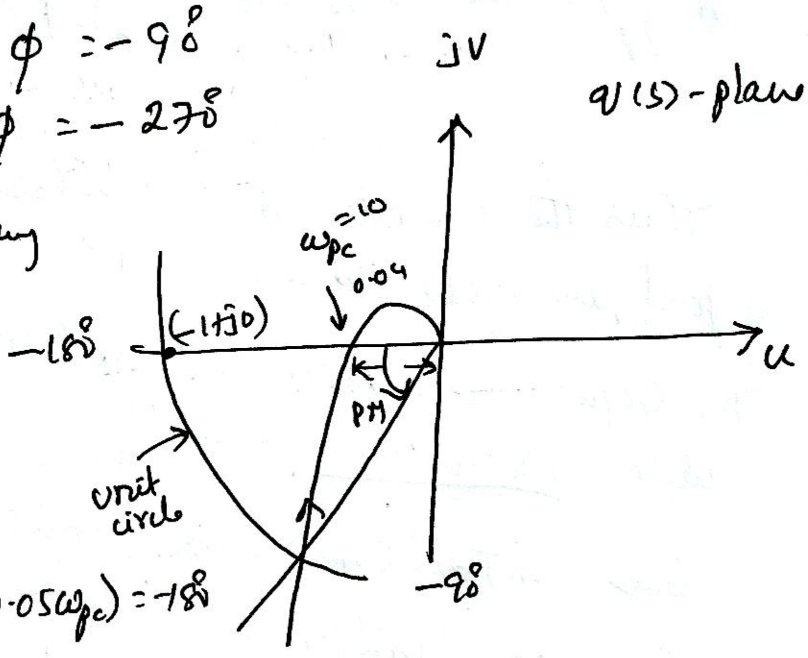
(Sol) $G(j\omega)H(j\omega) = \frac{1}{j\omega(1+0.2j\omega)(1+0.05j\omega)}$

$\therefore$ Magnitude $|G(j\omega)H(j\omega)| = M = \frac{1}{\omega\sqrt{1+(0.2\omega)^2}\sqrt{1+(0.05\omega)^2}}$

phase angle $\phi = -90 - \tan^{-1}(0.2\omega) - \tan^{-1}(0.05\omega)$

$\omega = 0 \quad M = \infty \quad \phi = -90^\circ$

$\omega = \infty \quad M = 0 \quad \phi = -270^\circ$



The phase cross over frequency can be determined as follows

$\left. \angle G(j\omega)H(j\omega) \right|_{\omega=\omega_{pc}} = -180^\circ$

$\Rightarrow -90 - \tan^{-1}(0.2\omega_{pc}) - \tan^{-1}(0.05\omega_{pc}) = -180^\circ$

$\Rightarrow \tan^{-1}(0.2\omega_{pc}) = 90 - \tan^{-1}(0.05\omega_{pc})$

Taking tan on both sides

$\tan(\tan^{-1} 0.2\omega_{pc}) = \tan(90 - \tan^{-1} 0.05\omega_{pc})$

$0.2\omega_{pc} = \cot \tan^{-1}(0.05\omega_{pc}) = \frac{1}{0.05\omega_{pc}}$

$\omega_{pc}^2 = \frac{1}{0.2 \times 0.05}$

$\Rightarrow \omega_{pc} = 10$ rad/sec

$\therefore$ Gain Margin $= 20\log\left[\frac{1}{|G(j\omega)H(j\omega)|}\right]_{\omega=\omega_{pc}} = 20\log\left(\frac{1}{0.04}\right)$

$= 28$ dB

To find phase margin draw a circle with radius '1' and origin as a centre, then identify the intersection of polar plot and circular arc. $\therefore$ Phase Margin $= 76^\circ$

Compensation Techniques: If the performance of a control system is not upto expectations as per desired specifications, then it is required that some change in the system is needed to obtain the desired performance. The change can be in the form of adjustment of forward path gain or inserting a compensating device in control systems.

For example, the steady state error in a control system can be reduced by increasing forward path gain, but on the other hand this increase in forward path gain results in making the system more oscillatory or sometimes unstable.

Thus the gain adjustment improves the steady state accuracy of the system at the cost of driving the system towards instability. In such cases a compensation network is introduced in the system. The compensation network can be introduced in forward path as shown in figure.

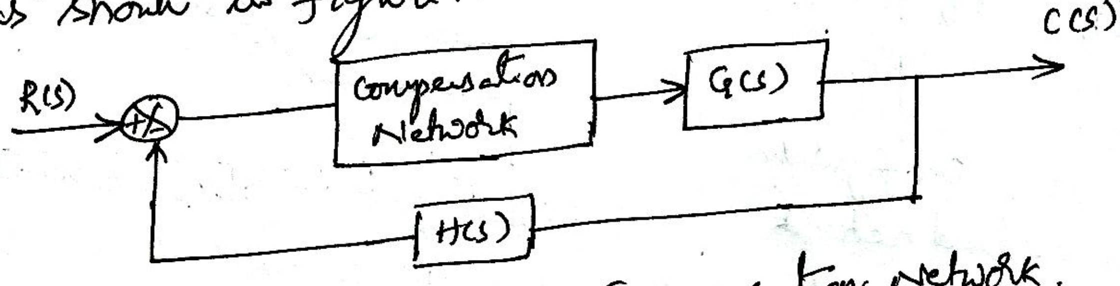


Figure: System with Compensation Network.

There are three types of compensators

(1) phase lead compensator
(2) phase lag compensator
(3) Lead-lag compensator

① **phase-lead compensator**: For phase-lead network the output leads the input. Let us consider a phase lead network shows in figure (1)

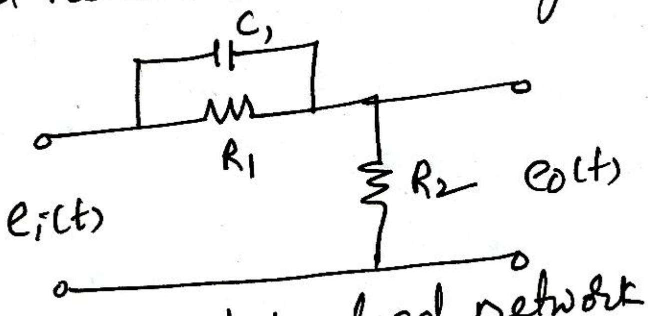


Figure: phase-lead network

The transfer function of phase lead network is given by

$$\frac{E_o(s)}{E_i(s)} = G(s) = \frac{\alpha(1+sT_1)}{(1+\alpha s T_1)}$$

where $\alpha = \frac{R_2}{R_1+R_2} < 1$ and $T_1 = R_1 C_1$

The sinusoidal transfer function $G(j\omega) = \dfrac{\alpha(1+j\omega T_1)}{(1+j\omega\alpha T)}$

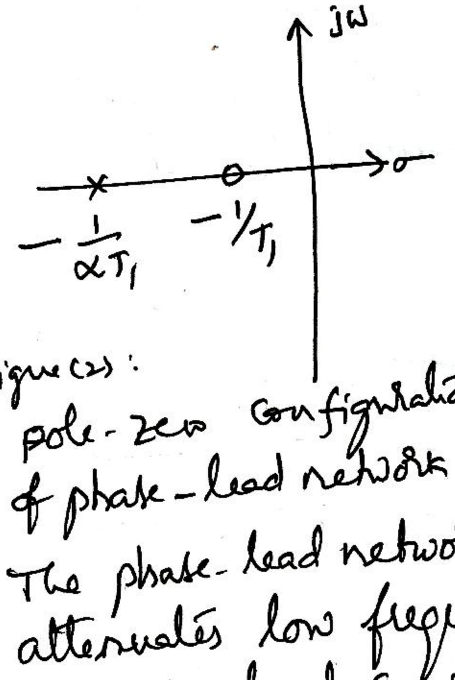


Figure (2): pole-zero configuration of phase-lead network

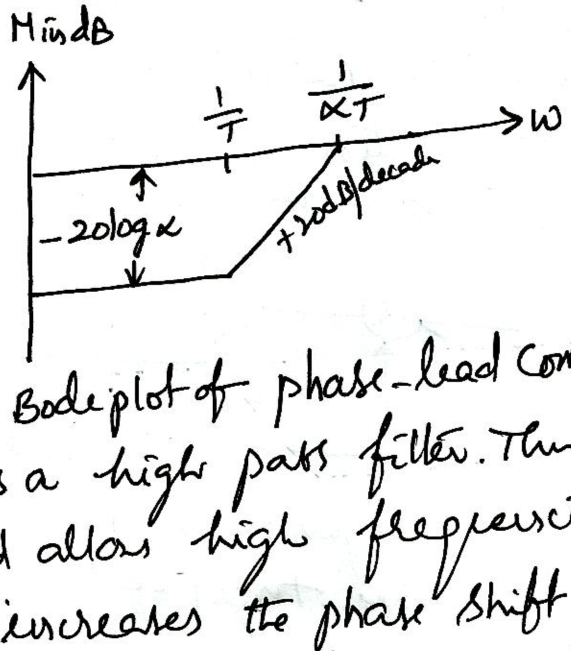


Fig: Bode plot of phase-lead compensator

The phase-lead network acts as a high pass filter. Thus it attenuates low frequencies and allows high frequencies. The phase-lead compensator increases the phase shift of the system. The phase-lead compensator shifts the gain cross over frequency to a higher value and therefore increases bandwidth, speed of the response and reduces overshoot but the steady state error does not show much improvement.

- **Phase-lag Compensator**: For phase-lag network, the output lags the input. The phase-lag network is shown in figure (1)

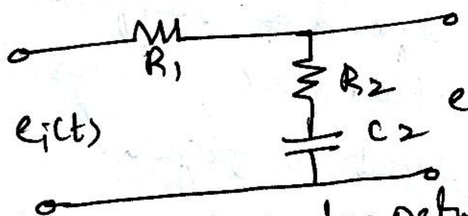


Figure : phase-lag network

The transfer function of phase-lag network is given by

$$\frac{E_o(s)}{E_i(s)} = \frac{1+sT_2}{1+s\beta T_2}$$

where $\beta = \frac{R_1+R_2}{R_2} > 1$; Time constant $T_2 = R_2C_2$

The sinusoidal transfer function is given by

$$\frac{E_o(j\omega)}{E_i(j\omega)} = G(j\omega) = \frac{1+j\omega T_2}{1+j\omega\beta T_2}$$

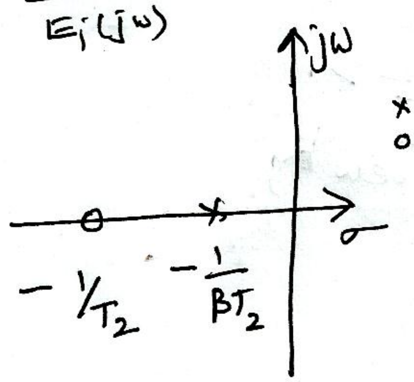


Figure : pole-zero configuration of phase-lag compensator

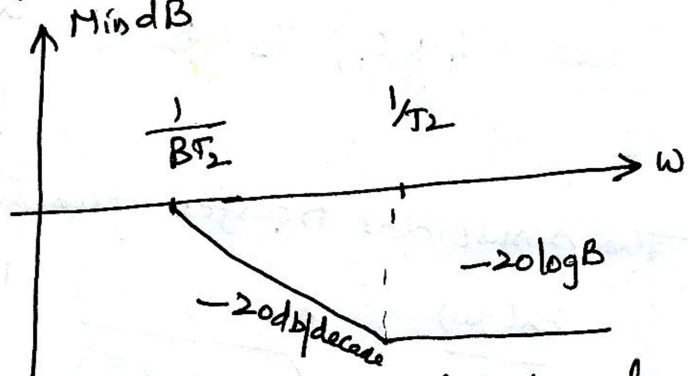


Figure : Bode plot of phase-lag compensator

When the phase-lag network is introduced in cascade with forward transfer function, the phase-shift will be reduced. The phase-lag compensator shifts the gain cross over frequency to lower value and thus decreases bandwidth and speed but improves the steady state error. The phase-lag compensator acts as a lowpass filter and thus allows low frequency signals and attenuates high frequency signals.

(3) **Lead-lag Compensator**: If phase-lead and phase-lag compensators are simultaneously used, then the speed of response and steady state error are simultaneously improved. The phase lead-lag network is shown in figure.

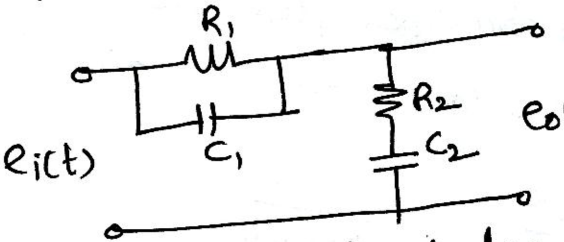


Figure: phase lead-lag network

The transfer function of lead-lag network is given by

$$\frac{E_o(s)}{E_i(s)} = \frac{\alpha(1+sT_1)}{(1+s\alpha T_1)} \frac{(1+sT_2)}{(1+s\beta T_2)}$$

where $T_1 = R_1C_1$; $T_2 = R_2C_2$; $\alpha = \frac{R_2}{R_1+R_2} < 1$

$$\beta = \frac{R_1+R_2}{R_2} > 1$$

The sinusoidal transfer function is given by

$$\frac{E_o(j\omega)}{E_i(j\omega)} = \frac{\alpha(1+j\omega T_1)}{(1+j\omega\alpha T_1)} \frac{(1+j\omega T_2)}{(1+j\omega\beta T_2)}$$

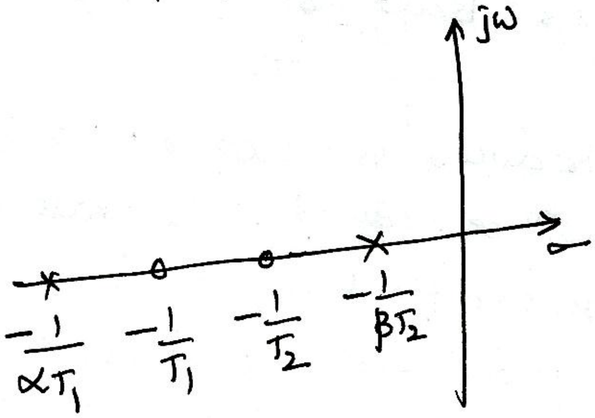


Figure: pole-zero pattern of Lead-lag Compensator

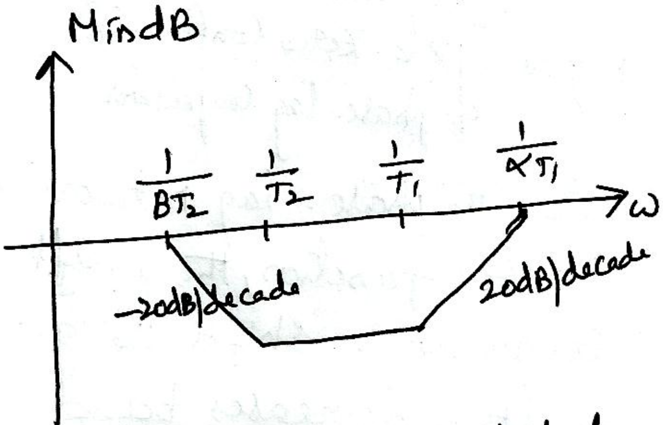


Figure: Magnitude plot of Lead-lag Compensator.