UNIT - III

STABILITY

A linear-time invariant system is stable, if the following two notions of system stability are satisfied

(1) when the system is excited by a bounded input, the output is bounded.

(2) In the absence of the input, the output tends towards zero irrespective of initial conditions. This stability concept is known as asymptotic stability.

Let us observe the implication of the two notions of stability defined, by considering a single-input, single-output system with transfer function

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n} \quad ; \quad m < n$$

with zero initial conditions, the output of the system is given by

$$c(t) = L^{-1}[G(s)R(s)] = \int_0^\infty g(\tau) r(t-\tau) d\tau$$

where $g(t) = L^{-1}[G(s)]$ is the impulse response of the system

Taking the absolute value on both sides, we get

$$|c(t)| = \left| \int_0^\infty g(\tau) r(t-\tau) d\tau \right|$$

Since the absolute value of integral is not greater than the integral of the absolute value of the integrand

$$|c(t)| \leq \int_0^\infty |g(\tau)| |r(t-\tau)| d\tau$$

Since, the first notion of stability is satisfied if for every bounded input ($|r(t)| \leq M_1 < \infty$), the output is bounded ($|c(t)| \leq M_2 < \infty$); thus for bounded input, the bounded output condition is

$$|c(t)| \leq M_1 \int_0^\infty |g(\tau)| d\tau \leq M_2 \longrightarrow (1)$$

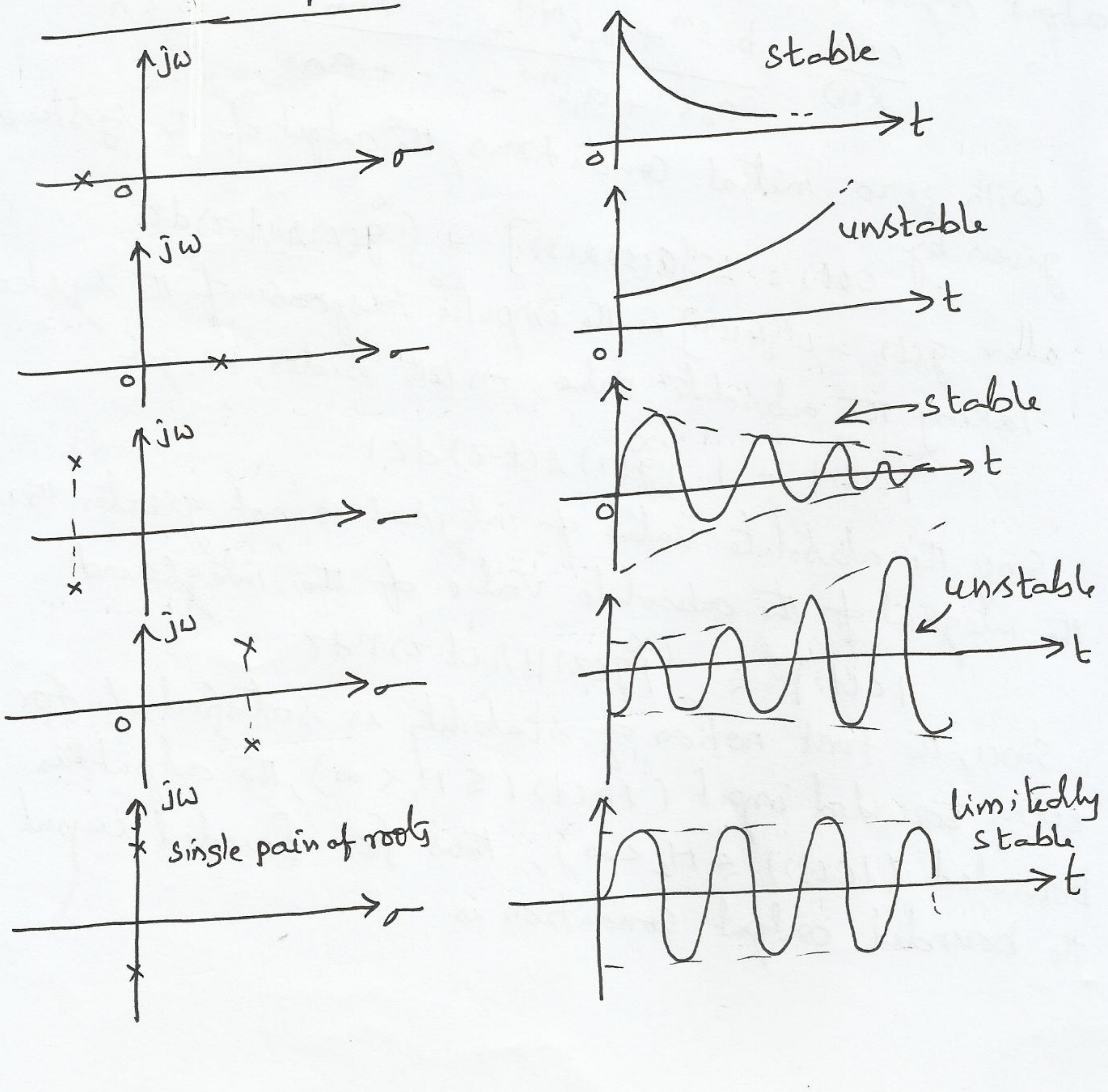
Thus the first notion of stability is satisfied if the impulse response $g(t)$ is absolutely integrable.

ie $\int_0^\infty |g(\tau)| d\tau$ is finite.

The nature of $g(t)$ depends upon the poles of the transfer function, which are the roots of the characteristic equation.

The nature of response terms contributed by all possible types of roots are shown in figure below.

Roots in the s-plane | Corresponding Impulse Response



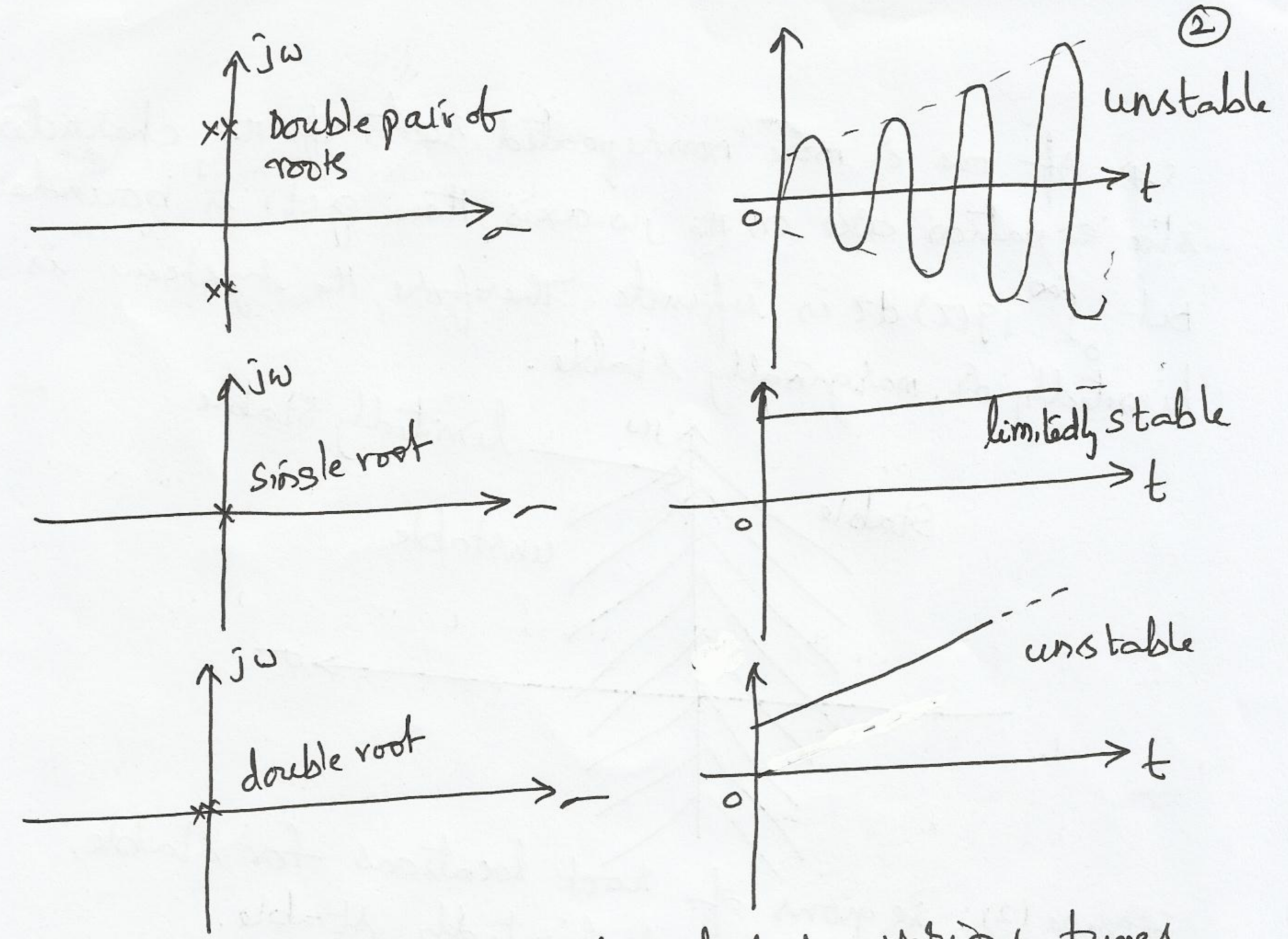


Figure: Response terms Contributed by various types of roots

The above observations lead to the following general conclusions regarding system stability

(1) If all the roots of characteristic equation have negative real parts, then the impulse response $g(t)$ is bounded and $\int_0^\infty |g(\tau)| d\tau$ is finite. Therefore the system is stable.

(2) If any root of the characteristic equation has a positive real part, $g(t)$ is unbounded and $\int_0^\infty |g(\tau)| d\tau$ is infinite. Therefore, the system is unstable.

(3) If the characteristic equation has repeated roots on the $j\omega$-axis, $g(t)$ is unbounded and $\int_0^\infty |g(\tau)| d\tau$ is infinite. Therefore, the systems is unstable.

(4) If one or more nonrepeated roots of the characteristic equation are on the jω-axis, then g(t) is bounded but $\int_0^\infty |g(\tau)|\,d\tau$ is infinite. Therefore, the system is limitedly or marginally stable.

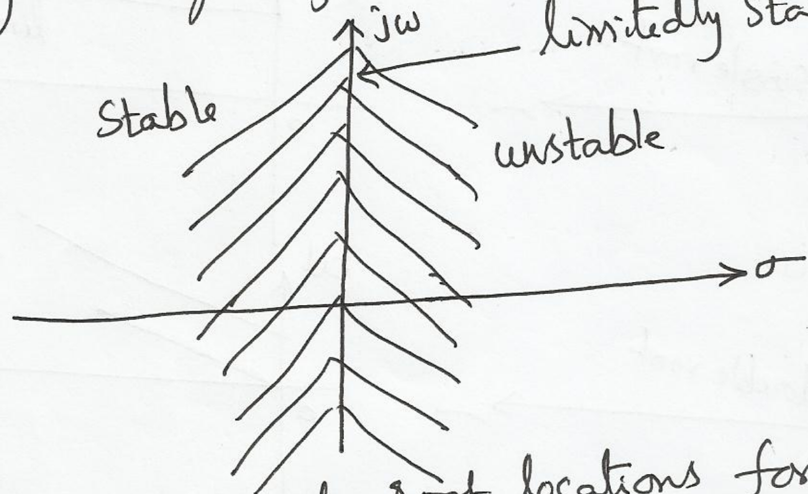


Figure (2): Regions of root locations for stable, unstable and limitedly stable.

In a vast majority of practical systems, the following statements on stability are quite useful.

(1) If all the roots of the characteristic equation have negative real parts, the system is stable.

(2) If any root of the characteristic equation has a positive real part or if there is a repeated root on the jω-axis, the system is unstable.

(3) If the condition (1) is satisfied except for the presence of one or more nonrepeated roots on the jω-axis, the system is limitedly or marginally stable.

A linear system is characterized as

(1) Absolutely stable with respect to a parameter of the system if it is stable for all values of this parameter.

(2) Conditionally stable with respect to a parameter, if the system is stable for only certain bounded ranges of values of this parameter.

Routh Stability criterion: This criterion is based on ordering the coefficients of the characteristic equation into an array, called Routh array as given below

Let us consider a characteristic equation given by

$$q(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0$$

The Routh array of $q(s)$ is as follows.

$$\begin{array}{lllll}
s^n & a_0 & a_2 & a_4 & a_6 - \\
s^{n-1} & a_1 & a_3 & a_5 & - - - \\
s^{n-2} & b_1 & b_2 & b_3 & - - - \\
s^{n-3} & c_1 & c_2 & c_3 & - - - \\
s^{n-4} & d_1 & d_2 & \cdots & \\
\vdots & \vdots & \vdots & & \\
s^2 & e_1 & a_n & & \\
s^1 & f_1 & & & \\
s^0 & a_n & & &
\end{array}$$

The coefficients $b_1, b_2 - - -$ are evaluated as follows

$$b_1 = \frac{(a_1 a_2 - a_0 a_3)}{a_1}; \quad b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

This process is continued till we get a zero as the last coefficient in the third row. In a similar way, the coefficients of $4^{th}, 5^{th} - - n^{th}$ and $(n+1)^{th}$ rows are evaluated.

$$c_1 = \frac{b_1 a_3 - b_2 a_1}{b_1}; \quad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}; - - -$$

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}; \quad d_2 = \frac{c_1 b_3 - b_1 c_3}{c_1}; - - -$$

In the process of generating routh array the missing terms are regarded as zero. Also all the elements of any row can be divided by a positive Constant during the process to simplify the computational work.

The Routh stability criterion is stated as below

" For a system to be stable, it is necessary and sufficient that each term of first column of Routh array of its characteristic equation be positive if $a_0 > 0$. If this condition is not met, the system is unstable and number of sign changes of the terms of the first column of Routh array corresponds to the number of roots of the characteristic equation in the right half of the s-plane".

① The characteristic equation of a system is given by $s^4 + 8s^3 + 18s^2 + 16s + 5 = 0$; check, whether the system is stable or not.

(Sol)

$$\begin{array}{llll} s^4 & 1 & 18 & 5 \\ s^3 & 8 & 16 & 0 \\ s^2 & \frac{8\times18-16\times1}{8} = 16 & \frac{8\times5-1\times0}{8} = 5 & \\ s^1 & \frac{16\times16-8\times5}{16} = 135 & 0 & \\ s^0 & 5 & & \end{array}$$

Since all the terms in the first column are positive hence the system is stable.

② The CE of a system is $3s^4 + 10s^3 + 5s^2 + 5s + 2 = 0$. check, whether the system is stable or not.

(Sol) The Routh array is

$$\begin{array}{llll} s^4 & 3 & 5 & 2 \\ s^3 & 10 & 5 & 0 \\ s^3 & 2 & 1 & 0 \end{array}$$

$s^2 \quad \frac{7}{2} \quad 2$

$s^1 \quad -1/7$

$s^0 \quad 2$

It may be noted that in order to simplify computational work, the s^3-row is modified by dividing it by 5. Examining the first column, there are two sign changes. Therefore, the system is <u>unstable</u> having two poles in the right-half s-plane.

(3) The characteristic equation of a system in differential equation form is $\ddot{x} - (K+2)\dot{x} + (2K+5)x = 0$.

(a) Find the value of 'K' for which the system is
(i) stable (ii) limitedly stable (iii) unstable

(b) For the stable case for what values of K, The system is
(i) critically damped (ii) under damped (iii) over damped

(Sol) Given that $\ddot{x} - (K+2)\dot{x} + (2K+5)x = 0$

taking Laplace transform with zero initial conditions

$s^2 X(s) - (K+2) s X(s) + (2K+5) X(s) = 0$

$X(s)[s^2 - (K+2)s + (2K+5)] = 0$

or $s^2 - (K+2)s + (2K+5) = 0$

The Routh array is

$s^2 \quad 1 \quad (2K+5)$

$s^1 \quad -(K+2) \quad 0$

$s^0 \quad (2K+5)$

(a)(i) For the system to be stable, all the terms must be +ve

$-(K+2) > 0$ and $2K+5 > 0$

or $K+2 < 0$ & $2K > -5$

$K < -2$ and $K > -2.5$

or $\boxed{-2.5 < K < -2}$

$\boxed{-2 > K > -2.5}$

(ii) For limitedly stable system

$K = -2$ and $K = -2.5$

(iii) For the system to be unstable

$K > -2$ and $K < -2.5$

(b) The roots of characteristic equation are

$$s_1, s_2 = \frac{1}{2}\left\{ K+2 \pm \sqrt{(K+2)^2 - 4(2K+5)} \right\}$$

(i) For critically damped system, the imaginary part is zero.

$$\therefore (K+2)^2 - 4(2K+5) = 0$$

$$K = 6.47, -2.47$$

For $K = 6.47$, the system is unstable. Hence, for the stable critically damped system $K = -2.47$

(ii) For under damped case (Larger than critically damped) $-2 > K > -2.47$

(iii) For over damped case (Smaller than critically damped) $-2.47 > K > -2.5$

Specical Cases: The following difficulties arise in Routh array formation.

Difficulty (1): when the first term in any row of Routh array is zero while rest of the row has at least one nonzero term. Because of this zero term, the terms in the next row become infinite and the Routh's test breaks down. The following methods can be used to overcome this difficulty.

(a) Substitute a small positive number 'ϵ' for the zero and proceed to evaluate the rest of the Routh array. Then examine the signs of the first column of Routh array by letting $\epsilon \to 0$

or

(b) Modify the original characteristic equation by replacing 's' by $\frac{1}{z}$. Apply the routh's test on the modified equations interms of 'z'. This transformation maps the left half of the s-plane into the left half of the z-plane and the right half of the s-plane into right half of the z-plane. The number of z-roots with positive real parts are the same as the number of s-roots with positive real parts. This method works in most but not all cases.

(3) The characteristic equation of a system is given by $s^5+s^4+2s^3+2s^2+3s+5=0$. Check the stability of the system.

(Sol) The Routh array is

$$\begin{array}{llll}
s^5 & 1 & 2 & 3 \\
s^4 & 1 & 2 & 5 \\
\boxed{s^3} & \boxed{0} & \boxed{-2} & \\
s^3 & \epsilon & -2 & \\
s^2 & \frac{2\epsilon+2}{\epsilon} & 5 & \\
s^1 & \frac{-4\epsilon-4-5\epsilon^2}{2\epsilon+2} & & \\
s^0 & 2 & &
\end{array}$$

when $\epsilon \to 0$ $\frac{-4\epsilon-4-5\epsilon^2}{2\epsilon+2} \to -2$

The first element in the third row is zero. It is replaced by ϵ, a small positive number. The first element in the 4th row is now $\frac{2\epsilon+2}{\epsilon}$ which has a positive sign as $\epsilon \to 0$. The first term in 5th row is −ve as $\epsilon \to 0$. Examining the terms in the first Column of Routh array, it is found that there are two changes in sign and hence the system is unstable having two poles in the right half s-plane.

II Method: Replacing 's' by $\frac{1}{z}$ in the characteristic equation, we will get

$$\left(\frac{1}{z}\right)^5+\left(\frac{1}{z}\right)^4+2\left(\frac{1}{z}\right)^3+2\left(\frac{1}{z}\right)^2+3\left(\frac{1}{z}\right)+5=0$$

$$\text{or } 5z^5+3z^4+2z^3+2z^2+z+1=0$$

The Routh array for this equation is

z^5	5	2	1
z^4	3	2	1
z^3	$-4/3$	$-2/3$	
z^2	$1/2$	1	
z^1	2		
z^0	1		

There are two changes of signs in the first column of Routh array, which indicates that there are 2 z-roots in RHS plane of 'z'. Therefore, the number of 's' roots in RHS plane of 's' is also 2.

Difficulty 2: when all the elements in any one row of the Routh array are zero. This condition indicates that there are symmetrically located roots in the s-plane. The polynomial whose coefficients are the elements of the row just above the row of zeros in the Routh array is called an auxiliary polynomial. This polynomial gives the number and location of root pairs of the characteristic equation which are symmetrically located in the s-plane. The order of the auxiliary polynomial is always even.

Because of a zero row in the array, the Routh's test breaks down. This situation is overcome by replacing the row of zeros in the Routh array by a row of coefficients of the polynomial generated by taking the first derivative of the auxiliary polynomial.

(1) check the system with CE $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$ is stable or not.

(Sol) Routh array is

s^6	1	8	20	16
s^5	2	12	16	
s^4	2	12	16	
s^3	0	0	0	
s^3	8	24		
s^3	1	3		
s^2	6	16		
s^2	3	8		
s^1	1/3			
s^0	8			

The auxiliary polynomial is

$2s^4+12s^2+16 = A(s)$

$\frac{d}{ds}A(s) = 8s^3+24s$

There are no sign changes in the first column. Therefore, the number of roots in RHS of s-plane are zero.

The roots of auxiliary equation are

$2s^4+12s^2+16=0$ or $s^4+6s^2+8=0$

$\therefore s^2 = \frac{-6\pm\sqrt{36-32}}{2} = \frac{-6\pm 2}{2} = -4, -2$

if $s^2=-4$; then $s=\pm\sqrt{-4} = \pm j2$

$s^2=-2$; then $s=\pm\sqrt{-2} = \pm j\sqrt{2}$

There are also the roots of CE, and the system is limitedly stable.

① The open loop transfer function of a unity feedback system is given by $G(s) = \frac{K}{s(s^2+s+1)(s+4)}$. Determine the value of for the system to be stable.

(Sol) $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{K}{s(s^2+s+1)(s+4)+K}$

$$= \frac{K}{s^4+5s^3+5s^2+4s+K}$$

The characteristic equation is $s^4+5s^3+5s^2+4s+K=0$

The Routh array is:

s^4	1	5	K
s^3	5	4	
s^2	$\frac{21}{5}$	K	
s^1	$\frac{\frac{84}{5}-5K}{(21/5)}$		
s^0	K		

For the system to be stable, all the terms in the first Column must be greater than zero

ie $\frac{\frac{84}{5}-5K}{(21/5)} > 0$ or $\frac{84}{5} - 5K > 0$ or $84 > 25K$

or $K < \frac{84}{25}$ and $K > 0$ $\boxed{0 < K < \frac{84}{25}}$

For the system to be stable

$$\boxed{0 < K < \frac{84}{25}}$$

① The open loop transfer function of a unity feedback control system is given by $G(s) = \dfrac{K}{(s+2)(s+4)(s^2+6s+25)}$. By applying Routh criterion, discuss the stability of the closed loop system as a function of 'K'. Determine the value of 'K' which will cause sustained oscillations in the closed loop system. What are the corresponding oscillation frequencies.

Sol) The characteristic equation of given control system is $1+G(s)H(s) = 0$

$$\Rightarrow 1 + \frac{K}{(s+2)(s+4)(s^2+6s+25)} = 0$$

$$\Rightarrow (s+2)(s+4)(s^2+6s+25) + K = 0$$

$$(s^2+6s+8)(s^2+6s+25) + K = 0$$

or $s^4 + 12s^3 + 69s^2 + 198s + (200+K) = 0$

The Routh array of given CE is

s^4	1	69	$(200+K)$
s^3	12	198	0
s^2	52.5	$(200+K)$	
s^1	$\dfrac{7995-12K}{52.5}$		
s^0	$(200+K)$		

(i) For the system to be stable, all the terms in the first column of Routh array of CE must be positive

ie $\dfrac{7995-12K}{52.5} > 0$ or $K < \dfrac{7995}{12}$

or $K < 666.25$

also $(200+K) > 0$ or $K > -200$

$\therefore$ For the system to be stable $\underline{-200 < K < 666.25}$

(ii) To sustain oscillations $\dfrac{7995-K}{52.5} = 0$

or $K = 666.25$

$\therefore$ The auxiliary equation is

$$(52.5)s^2 + (200+K) = 0$$

or $(52.5)s^2 + (200 + 666.25) = 0$

or $52.5s^2 = -866.25$

or $s^2 = -16.5$

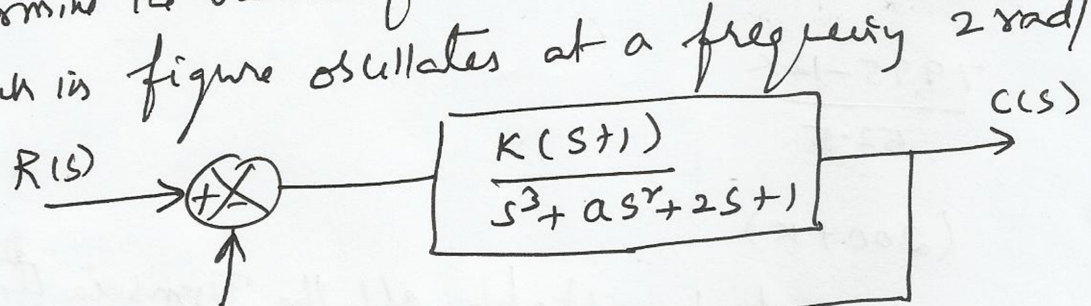
where $s = j\omega$

$\therefore (j\omega)^2 = -16.5$

$\therefore \omega = \sqrt{16.5} = 4.06$ rad/sec

The frequency of oscillations is $\omega = 4.06$ rad/sec

(2) A system oscillates with frequency 'ω' if it has poles at $s = \pm j\omega$ and no poles in the right half s-plane. Determine the values of 'K' and 'a' so that the system shown in figure oscillates at a frequency 2 rad/sec.



(Sol) The characteristic equation of given system is

$$1+G(s)H(s) = 0$$

$$\Rightarrow 1 + \frac{K(s+1)}{s^3+as^2+2s+1} = 0$$

$$s^3+as^2+2s+1 + K(s+1) = 0$$

or $s^3+as^2+s(2+K)+(K+1) = 0$

The Roreths array of given CE is

$$\begin{array}{lll} s^3 & 1 & (2+K) \\ s^2 & a & (K+1) \\ s^1 & \frac{a(2+K)-(K+1)}{a} & \\ s^0 & (K+1) & \end{array}$$

To sustain oscillations $\frac{a(2+K)-(K+1)}{a} = 0$

$$\Rightarrow a = \frac{K+1}{2+K}$$

The auxiliary polynomial is

$$as^2 + (K+1) = 0$$

$$\left(\frac{K+1}{K+2}\right)s^2 + (K+1) = 0$$

$$\Rightarrow s^2 = -(K+2)$$

Given that the freqency of oscillations is

$\omega = 2$ rad/sec.

$$\therefore (j\omega)^2 = -(K+2)$$

$$(j2)^2 = -(K+2)$$

or $K+2 = 4$

$$\therefore K = 2$$

and $a = \frac{K+1}{K+2} = \frac{2+1}{2+2} = \frac{3}{4} = 0.75$

(3) A feedback system has an open-loop TF

$$G(s)H(s) = \frac{Ke^{-s}}{s(s^2+5s+9)}$$. Determine the maximum value of 'K' for the system to be stable.

(Sol) Note: For low frequencies $e^{-s} = 1-s$

∴ The characteristic equation of the system is given by $1+G(s)H(s) = 0$

ie $$1 + \frac{K(1-s)}{s(s^2+5s+9)} = 0$$

$$s(s^2+5s+9) + K(1-s) = 0$$

$$s^3+5s^2+9s+K-Ks = 0$$

$$s^3+5s^2+s(9-K)+K = 0$$

The Routh array of CE is

s^3	1	$9-K$
s^2	5	K
s^1	$\frac{5(9-K)-K}{5}$	0
s^0	K	

For the system to be stable, all the terms in the first column of Routh array must be positive.

ie $\frac{45-6K}{5} > 0$ or $45 > 6K$ or $K < \frac{45}{6}$

also $K > 0$ ie $0 < K < \frac{45}{6}$

∴ The maximum value of 'K' for the system to be stable is $K = \frac{45}{6}$

Relative stability: once the system is said to be stable, the relative stability quantitatively determined by finding the settling time of the dominant roots of the characteristic equation. The roots nearer to imaginary axis in left hand side of the s-plane are known as dominant roots. The settling time is inversely proportional to the real part of the dominant roots.

The relative stability can be specified by requiring that all the roots of the characteristic equation are more negative than a certain value. That is, all the roots must lie to the left of the lines $s = -s_1$ $(s_1 > 0)$. Then the characteristic equation is modified by shifting the origin of the plane to $s = -s_1$, ie, by substituting $s = z - s_1$ as shown in figure.

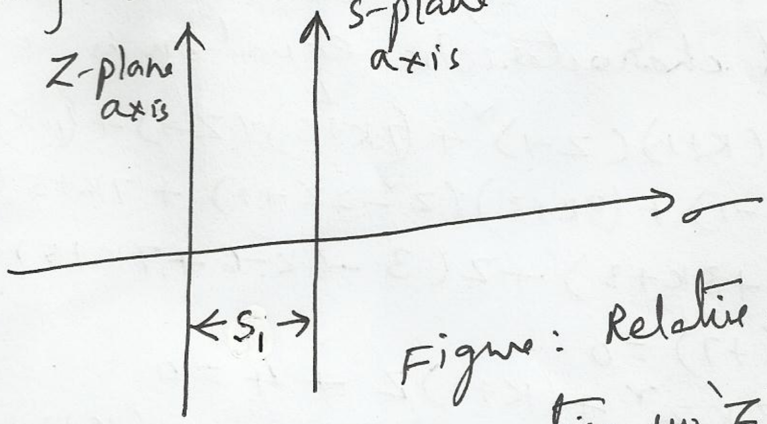


Figure: Relative stability

If the new characteristic equation in 'z' satisfies the Routh criterion, it implies that all the roots of the original characteristic equation are more negative than '$-s_1$'

① Show that the roots of given CE are more negative than −1. $s^3+7s^2+25s+39=0$

(Sol) put $s=z-1$ in the given CE

$\therefore (z-1)^3+7(z-1)^2+25(z-1)+39=0$

$z^3+4z^2+14z+20=0$

The Routh array is

$$\begin{array}{llll} z^3 & 1 & 14 \\ z^2 & 4 & 20 \\ z^1 & 9 & 0 \\ z^0 & 20 & \end{array}$$

Since all the terms in the first column are positive, hence all the roots have negative real parts more than −1.

② Determine the range of values of $K (K>0)$ such that the characteristic equation $s^3+3(K+1)s^2+(7K+5)s+(4K+7)=0$ has roots more negative than $s=-1$.

(Sol) If the roots are more negative than $s=-1$, shift the origin to $s=-1$ by substituting $s=z-1$. Therefore, the modified characteristic equation is

$(z-1)^3+3(K+1)(z-1)^2+(7K+5)\cdot(z-1)+(4K+7)=0$

$(z^3-3z^2+3z-1)+(3K+3)(z^2-2z+1)+7K+5(z-1)+4K+7=0$

$z^3+z^2(-3+3K+3)+z(3-6K-6+7K+5)+(-1+\cancel{3K}+3$
$-\cancel{7K}-5+4\cancel{K}+7)=0$

or $z^3+3Kz^2+(K+2)z+4=0$

The Routh array is

$$\begin{array}{lll} z^3 & 1 & (K+2) \\ z^2 & 3K & 4 \\ z^1 & \dfrac{3K(K+2)-4}{3K} & 0 \\ z^0 & 4 & \end{array}$$

For the roots to have −ve real parts more than −1

$\dfrac{3K(K+2)-4}{3K}>0$ or $3K^2+6K-4>0$; $K=\dfrac{-6\pm 9.17}{6}$

$\therefore K>-2.53$ & $K>\underline{0.53}$

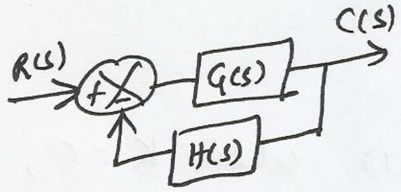
UNIT - III PART-B

THE ROOT LOCUS CONCEPT

The root locus concept introduced by W.R. Evans, provide a graphical method of plotting the locus of the roots in the s-plane as a given system parameter (open loop gain K) is varied over the complete range of values (from '0' to ∞). The root locus technique is a powerful tool for adjusting the location of closed loop poles to achieve the desired system performance by varying one or more system parameters.

From the Mason's gain formula, the transfer function of a system is

$$\frac{C(s)}{R(s)} = \frac{1}{\Delta} \sum_K P_K \Delta_K \longrightarrow (1)$$



Also

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} \longrightarrow (2)$$

The characteristic equation of the system is

$$1+G(s)H(s) = 0 \quad \text{or} \quad \Delta(s) = 0$$

let $G(s)H(s) = P(s)$

$\therefore 1+P(s) = 0$ is the CE of the system

$$\therefore P(s) = -1 \longrightarrow (3)$$

Since, 's' is a complex variable, P(s) has magnitude and phase angle given by

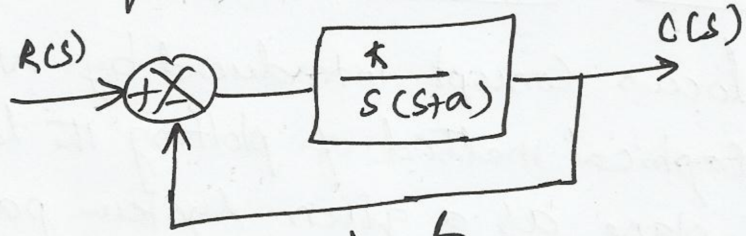
$$\longrightarrow (I)$$

$$|P(s)| = 1$$

$$\angle P(s) = \pm(2q+1)180^\circ, \quad q = 0, 1, 2 \ldots \longrightarrow (II)$$

Therefore, a plot of the points satisfying the angle criterion equation II in the s-plane is the root locus. A point on the root locus can be determined from magnitude equation.

① For the System shown below, Sketch the locus of the roots, when open loop gain 'K' is varied from 0 to ∞.



where 'K' and 'a' are Constants.

(Sol) The closed loop transfer function $\frac{C(s)}{R(s)} = \frac{K}{s(s+a)+K}$

$$\frac{C(s)}{R(s)} = \frac{K}{s^2+as+K}$$

The characteristic equation is $s^2+as+K=0$

The roots of CE are $s_1, s_2 = \frac{-a \pm \sqrt{a^2-4K}}{2}$

$$= -\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - K}$$

(1) when $K=0$, the two roots are $s_1, s_2 = -a/2 \pm \frac{a}{2}$
$= 0, -a$
which are same as the open loop poles.

(2) If $K = \left(\frac{a}{2}\right)^2$, the roots of CE are $s_1, s_2 = -\frac{a}{2}$

(3) For $K > \frac{a^2}{4}$, the roots are imaginary with real part equal to $-\frac{a}{2}$.

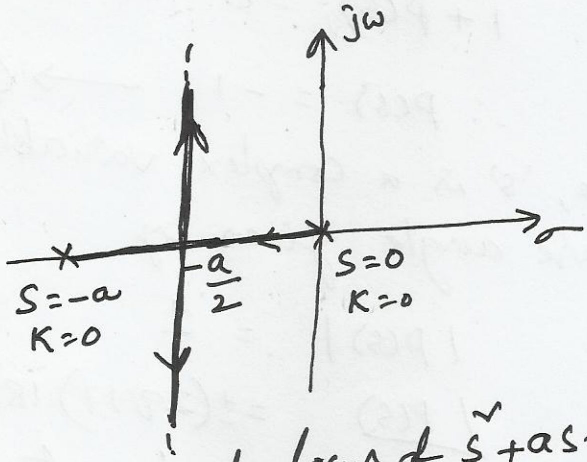


Figure: Root locus of $s^2+as+K=0$ as a function of 'K'.

Rules to Construct Root locus:

A set of rules have been developed to reduce the task involved in sketching root locus and to develope a quick approximate sketch. To develope root locus, the open loop transfer function is required.

Rule 1: The root locus is symmetrical about the real axis (σ-axis).

Since, the roots of the characteristic equation are either real or complex conjugate or combinations of both. Therefore, their locus must be symmetrical about the σ-axis of the s-plane.

Rule 2: As 'K' increases from zero to infinity, each branch of the root locus originates from an open loop pole with $K=0$ and terminates either on an open-loop zero or on an infinity with $K=\infty$. The number of branches terminating on infinity equals the number of open-loop poles minus zeros.

In general, the characteristic equation in pole-zero form can be represented as

$$1+G(s)H(s) = 1 + \frac{K\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = 0 \longrightarrow ①$$

where m = number of zeros; n = number of poles (open loop)

equation ①, can also be represented as

$$\prod_{j=1}^{n}(s+p_j) + K\prod_{i=1}^{m}(s+z_i) = 0 \longrightarrow ②$$

if $K=0$, the roots of CE are $-p_j$, which are same as open loop poles.

equation ① can also be represented as

$$\frac{1}{K}\prod_{j=1}^{n}(S+P_j) + \prod_{i=1}^{m}(S+Z_i) = 0$$

if $K \to \infty$, $\frac{1}{K} = 0$; therefore, the roots of CE are same as open loop zeros, $-Z_i$.

<u>Rule 3</u>: A point on the real axis lies on the locus if the number of open-loop poles plus zeros on the real axis to the right of this points is odd.

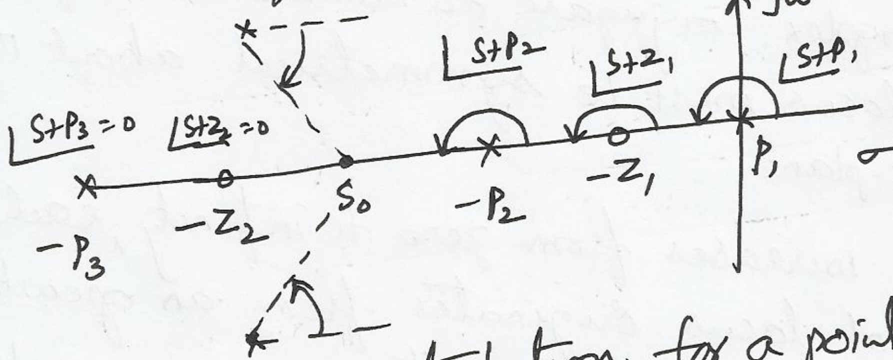


Figure: Angle <u>contribution</u> for a point <u>on the real axis</u>

As shown in the figure, (i) the poles and zeros on the real axis to the right of point So contribute an angle of $180°$ each (ii) The poles and zeros to the left of this point contribute an angle of $0°$ each
(iii) The net angle contribution of a complex conjugate pole or zero pair is always zero.

∴ The angle criterion equation becomes

$$\angle G(s)H(s) = (m_r - n_r)\,180° = \pm(2q+1)180° \;;\; q = 0,1,2,\ldots$$

where m_r = right side zeros
n_r = right side poles

Therefore for a point S_0 on the real axis, the angle criterion is only met if $(m_r - n_r)$ or $(m_r + n_r)$ is odd, hence the rule.

Rule 4: The (n-m) branches of the root locus which tend to infinity, do so along straight line asymptotes whose angles are given by

$$\phi_A = \frac{(2q+1)180^\circ}{(n-m)} \; ; \quad q = 0, 1, 2 \text{ -- } (n-m-1)$$

Rule 5: The asymptotes cross the real axis at a point known as centroid, determined by

$$-\sigma_A = \frac{\text{Sum of real parts of poles} - \text{Sum of real part of zeros}}{(\text{number of poles} - \text{number of zeros})}$$

$$= \frac{\Sigma \text{ real parts of poles} - \Sigma \text{ real parts of zeros}}{\text{number of poles} - \text{number of zeros}}.$$

Rule 6: The breakaway points (points at which multiple roots of the characteristic equation occur) of the root locus are the solutions of $\frac{dK}{ds} = 0$.

Rule 7: The angle of departure from an open-loop pole is given by $\phi_p = \pm 180^\circ(2q+1) + \phi$; $q = 0, 1, 2, \text{--}$

where 'ϕ' is the net angle contribution of all other open-loop poles and zeros at this point.

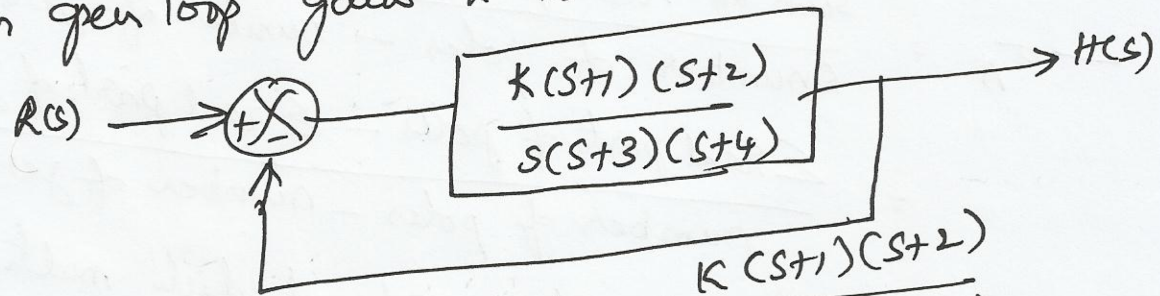
Similarly the angle of arrival at an open-loop zeros is given by $\phi_z = \pm 180^\circ(2q+1) - \phi$; $q = 0, 1, 2 \text{ ---}$

Rule 8: The intersection of root locus branches with the imaginary axis can be determined by use of Routh criterion

(9) The open-loop gain 'K' in pole-zero form at any point 'S_0' on the root locus is given by

$$K = \frac{\prod_{j=1}^{n} (S_0 + P_j)}{\prod_{i=1}^{m} (S_0 + Z_i)} = \frac{\text{product of phasor lengths from } S_0 \text{ to open loop poles}}{\text{product of phasor lengths from } S_0 \text{ to open loop zeros.}}$$

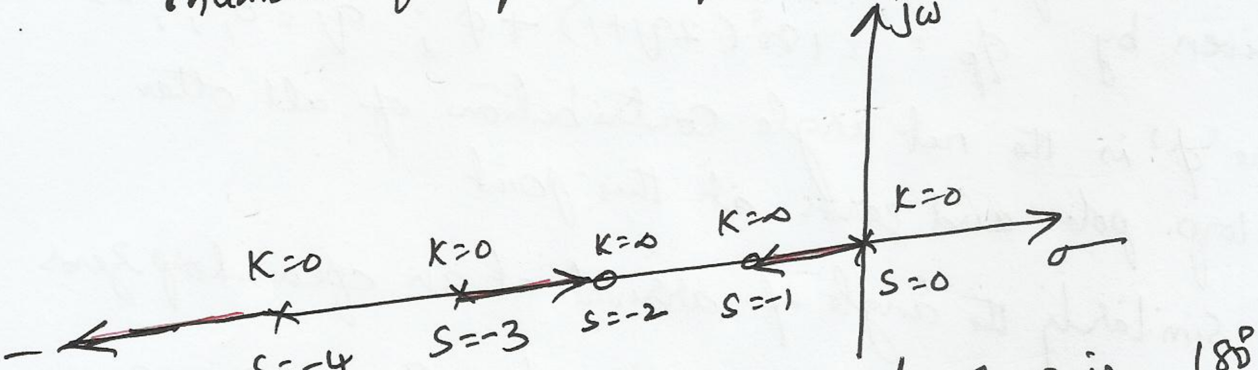
(1) For the system shown below, sketch the root locus when open loop gain 'K' is varied from 0 to ∞.



(Sol) The open loop TF $G(s) = \dfrac{K(S+1)(S+2)}{S(S+3)(S+4)}$

(i) system has open loop poles at $s = 0, -3, -4$

(ii) The number of root locus branches are equal to number of open loop poles.



(iii) The angle of departure at $S=0$ is $180°$

The angle of departure at $S=-3$ is

$180 - 180 - 180 + 180 = 0$

The angle of departure at $S=-4$ is

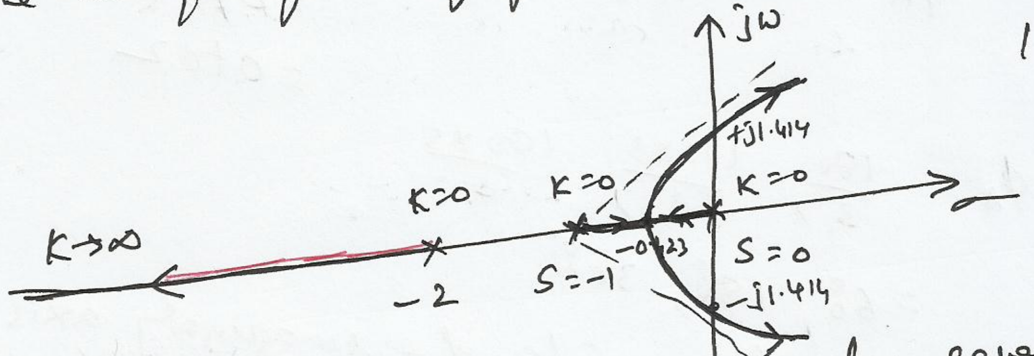
$180 - 180 + 180 + 180 - 180 = 180°$

(2) Consider a feedback system with characteristic equation $1 + \frac{K}{S(S+1)(S+2)} = 0$. Sketch the root locus when open loop gain 'K' is varied from 0 to ∞.

(Sol) The open loop TF $G(s)H(s) = \frac{K}{S(S+1)(S+2)}$

(i) The no of open loop poles are '3' at $S = 0, -1, -2$

180° right side
0° left side.



(ii) The number of root locus branches equal to '3'

(iii) (a) The angle of departure at $S=0$ is given by $180 + 0 = 180°$

(b) The angle of departure at $S=-1$ is $180° - 180° = 0$

(c) The angle of departure at $S=-2$ is $180 - 180 - 180° = -180°$

(iv) The root locus branches from $S=0$ and $S=-1$ are moving in opposite direction, therefore the break away points are the solutions of $\frac{dK}{ds} = 0$ from the characteristic equation

From the CE, $K = -S(S+1)(S+2)$

$$= -(S^2+S)(S+2)$$

$$= -(S^3+2S^2+S^2+2S)$$

$$= -(S^3+3S^2+2S)$$

$$\frac{dK}{ds} = -3S^2+6S+2 = 0$$

$$\therefore S_1, S_2 = \frac{-6 \pm \sqrt{36-24}}{6} = -0.423, -1.577$$

Since the break away point must be in between 0 and -1, $S = -0.423$ is the actual breakaway point

(V) The centroid $-\sigma_A = \frac{S.R.P - S.R.Z}{n-m} = \frac{-1-2-0}{3-0} = -1$

(Vi) The angles of asymptotes are given by

$$\phi_A = \frac{(2q+1)180^\circ}{n-m}; \quad q = 0, 1, 2 \ldots (n-m-1)$$

$$= 0 \text{ to } (3-0-1)$$

$$= 0 \text{ to } 2$$

$$\phi_A = \frac{180}{3}, \frac{180 \times 3}{3}, \frac{180 \times 5}{3}$$

$$= 60^\circ, 180^\circ, 300^\circ$$

(Vii) The intersection points of imaginary axis and root locus can be determined from R-H criterion

The CE of the systems is $\Rightarrow$ $s^3 + 3s^2 + 2s + K = 0$

$$1 + \frac{K}{s(s+1)(s+2)} = 0$$

s^3	1	2
s^2	3	K
s^1	$\frac{6-K}{3}$	0
0	K	

To have roots on the imaginary axis

$$\frac{6-K}{3} = 0 \text{ or } K = 6$$

The Auxiliary equation is

$$3s^2 + K = 0 \text{ or } 3s^2 + 6 = 0$$

$$s^2 = -2$$

$$s = \pm j\sqrt{2} = \pm j1.414$$

(3) The open loop transfer function of a system is given by $G(s)H(s) = \dfrac{K}{s(s+4)(s^2+4s+20)}$

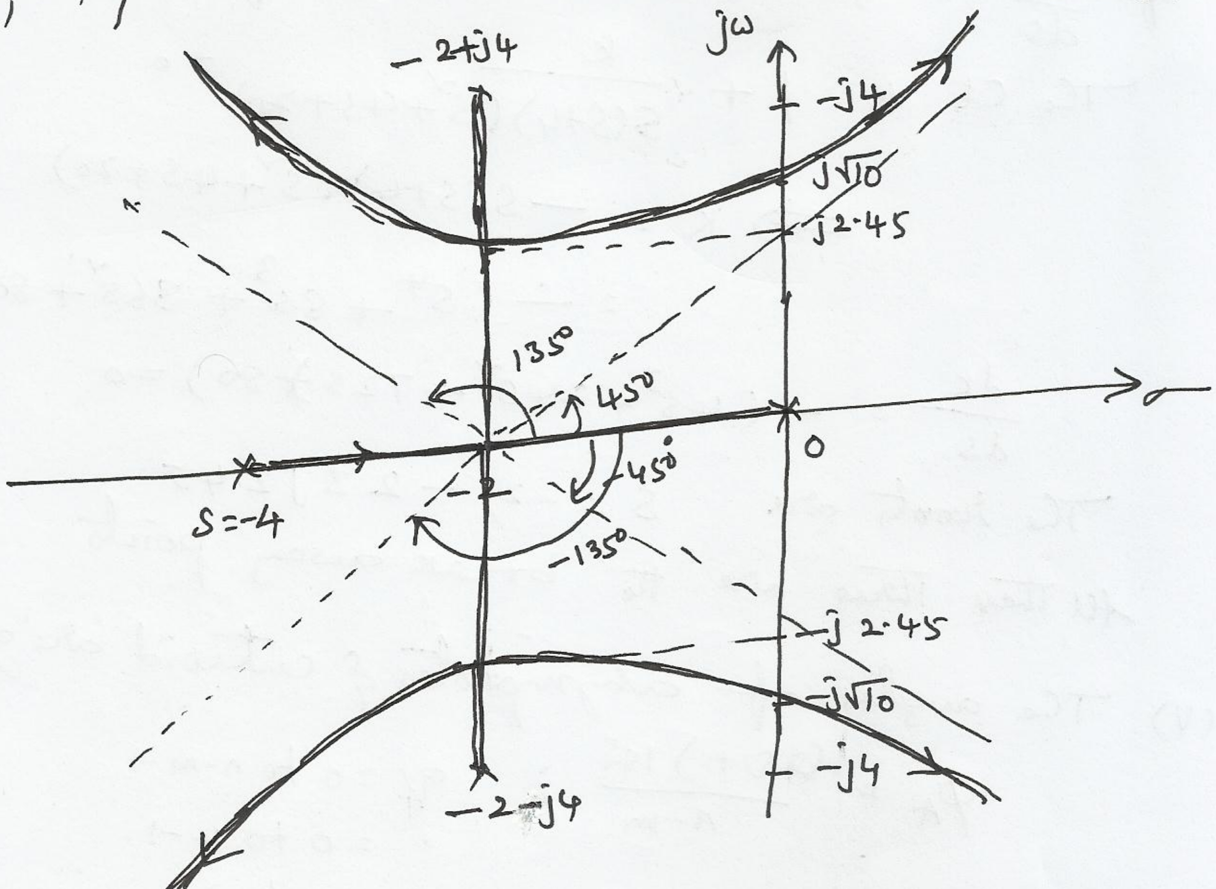
(Sol) The roots of $s^2+4s+20$ are $s = \dfrac{-4 \pm \sqrt{16-80}}{2}$

$= \dfrac{-4 \pm \sqrt{-64}}{2}$

$= \dfrac{-4 \pm j8}{2} = -2+j4$ & $-2-j4$

(i) The open loop poles are at

$s = 0, -4, -2+j4$ and $-2-j4$



(ii) The root locus branches are '4'

(iii)(a) The angle of departure at $s=0$ is given by

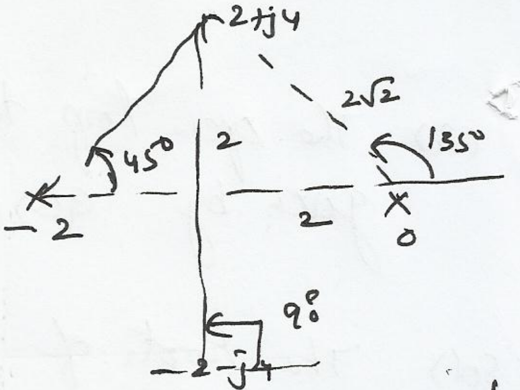
$180° + 0 = 180°$

(b) The angle of departure at $s=-4$ is given by

$180 - 180° = 0°$

(c) The angle of departure at $s = -2+j4$ is

$$\phi_P = 180^\circ - 135^\circ + 90 - 45^\circ$$
$$= 180^\circ - 270^\circ = -90^\circ$$



∴ The angle of departure at $s = -2-j4$ is $+90^\circ$

(IV) All the root locus branches are moving in opposite direction, therefore the break away points are the solutions of $\frac{dK}{ds} = 0$ from the characteristic equation

The CE is $1 + \dfrac{K}{s(s+4)(s^2+4s+20)} = 0$

$$\Rightarrow K = -s(s+4)(s^2+4s+20)$$
$$= -(s^4 + 8s^3 + 36s^2 + 80s)$$

$$\frac{dK}{ds} = -(4s^3 + 24s^2 + 72s + 80) = 0$$

The roots are $s = -2, \; -2 \pm j2.45$

All these three are the break away points

(V) The angles of asymptotes & centroid are given by

$$\phi_A = \frac{(2q+1)180^\circ}{n-m} \; ; \quad q = 0 \text{ to } n-m-1$$
$$= 0 \text{ to } 4-1$$
$$= 0 \text{ to } 3$$

$$\phi_A = \frac{180^\circ}{4}, \; \frac{3 \times 180^\circ}{4}, \; \frac{5 \times 180^\circ}{4}, \; \frac{7 \times 180^\circ}{4} = 45^\circ, 135^\circ, 225^\circ, 315^\circ$$

centroid $\sigma_A = \dfrac{-4-2-2}{4-0} = -\dfrac{8}{4} = -2$

(Vi) The intersection points of root locus & imaginary axis are the solutions of $\frac{dK}{ds} = 0$

The characteristic equation of the system is given by

$$1+\frac{K}{s(s+4)(s^2+4s+20)}=0$$

$$\Rightarrow \quad s^4+8s^3+36s^2+80s+K=0$$

$$\begin{array}{c|ccc}
s^4 & 1 & 36 & K \\
s^3 & 8 & 80 & \\
s^3 & 1 & 10 & \\
s^2 & 26 & K & \\
s & \frac{260-K}{26} & 0 & \\
s^0 & K & &
\end{array}
\qquad \frac{36-10}{1}=26$$

To have roots on the imaginary axis, $\frac{260-K}{26}=0$

$$\therefore K = 260$$

The AE is $26s^2+K=0$ or $26s^2+260=0$

$$s^2+10=0$$

$$s^2=-10 \text{ or } s=\pm j\sqrt{10}$$

(4) Sketch the root locus plot of the system with the characteristic equation $1+\frac{K(s+2)}{(s^2+2s+2)}=0$

(Sol) The open loop TF is $G(s)H(s)=\frac{K(s+2)}{(s^2+2s+2)}$

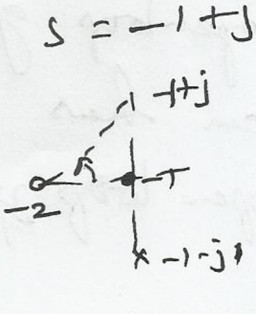
(i) The system has open loop zero at $s=-2$

(ii) The open loop poles are $s=\frac{-2\pm\sqrt{2^2-4\times2}}{2}$

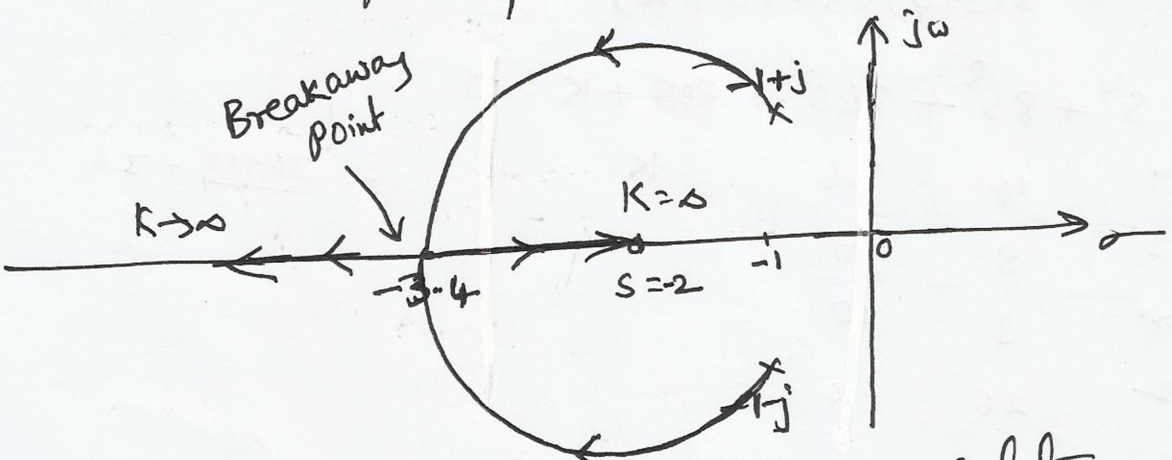
$$=-\frac{2\pm j2}{2}=-1\pm j1$$

(III) The angle of departure at $s=-1+j$ is

$$\phi_p=180-90^\circ+45^\circ=135^\circ$$



$\therefore$ The angle of departure at $s = -1-j$ is -135°



(iii) The break away points are the solutions of $\frac{dK}{ds} = 0$

From the characteristic equation

$$K = \frac{-(s+2)}{s^2+2s+2}$$

$$\frac{dK}{ds} = \frac{-(s^2+2s+2) + (s+2)(2s+2)}{(s^2+2s+2)^2} = 0$$

$$\Rightarrow -s^2 - \cancel{2s} - 2 + 2s^2 + \cancel{2s} + 4s + 4 = 0$$

$$s^2 + 4s + 2 = 0$$

$$s = \frac{-4 \pm \sqrt{4^2 - 4(2)}}{2} = -\frac{4 \pm \sqrt{16-8}}{2}$$

$$= \frac{-4 \pm \sqrt{8}}{2} = -\frac{4 \pm 2\sqrt{2}}{2}$$

$= -3.4$ is the break away point

(5) The open loop transfer function of a unity feedback system is given by

$G(s)H(s) = \frac{K(s+1)(s+2)}{(s+0.1)(s-1)}$. Sketch the root locus when the open loop gain 'K' is varied from 0 to ∞.

(Sol) i) System has open loop zeros at $s = -1$ and -2 and open loop poles at $s = -0.1$ and $s = 1$

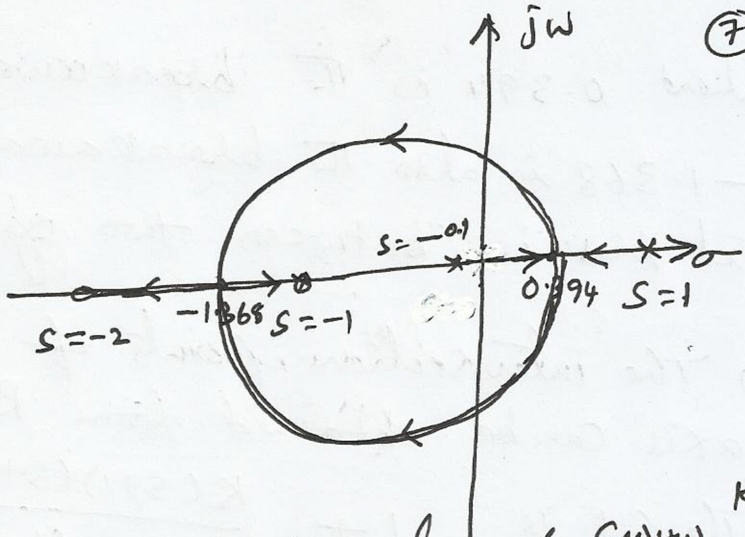


Figure: Root locus of $G(s)H(s) = \frac{K(s+1)(s+2)}{(s+0.1)(s-1)}$

(ii) (a) The angle of departure at pole $s = -0.1$ is given by

$$\phi_p = 180 - 180^\circ = 0$$

(b) The angle of departure at pole $s = +1$ is given by

$$\phi_p = 180 - 0 = 180$$

∴ Both the branches move in opposite direction.

(iii) The break away points are the solutions of $\frac{dK}{ds} = 0$. From the characteristic equation

$$1 + \frac{K(s+1)(s+2)}{(s+0.1)(s-1)} = 0$$

$$\therefore K = \frac{-(s+0.1)(s-1)}{(s+1)(s+2)} = \frac{-s^2 + 0.9s + 0.1}{s^2 + 3s + 2}$$

$$\frac{dK}{ds} = \frac{(s^2+3s+2)(-2s+0.9) - (-s^2+0.9s+0.1)(s^2+3)}{(s^2+3s+2)^2} = 0$$

$$\Rightarrow 3.9s^2 + 3.8s - 2.1 = 0$$

$$s = \frac{-3.8 \pm \sqrt{(3.8)^2 - 4(3.9)(-2.1)}}{2(3.9)}$$

$$= -1.368, 0.394$$

where 0.394 is the breakaway point and -1.368 is also the breakaway point because it lies in between two open loop zeros.

(iv) The intersection points of root locus & imaginary axis can be obtained from Routh criterion

The CE is $$1 + \frac{K(s+1)(s+2)}{(s+0.1)(s-1)} = 0$$

$$\Rightarrow (s^2 - 0.9s - 0.1) + K(s^2 + 3s + 2) = 0$$

$$s^2(1+K) + s(3K - 0.9) + (2K - 0.1) = 0$$

s^2	$(1+K)$	$(2K-0.1)$
s	$(3K-0.9)$	
s^0	$(2K-0.1)$	

To have roots on the imaginary axis $3K - 0.9 = 0$

$\therefore K = 0.3$

The Auxiliary equation is

$$(1+K)s^2 + (2K - 0.1) = 0 \qquad (K = 0.3)$$

$$1.3s^2 + 0.5 = 0 \Rightarrow s^2 = \frac{-0.5}{1.3} = -0.3846$$

$$\therefore s = \pm j\sqrt{0.3846}$$

1. The characteristic equation of a system is given by $1+\frac{Ke^{-s}}{s(s+2)}=0$. Sketch the root locus, when the open loop gain 'K' is varied from 0 to ∞.

(Sol) The O.L.T.F $G(s)H(s) = \frac{Ke^{-s}}{s(s+2)}$

For small values of frequency $G(s) = \frac{K(1-s)}{s(s+2)}$

i) The system has open loop poles at $s=0$ and $s=-2$
The system has open loop zero at $s=1$

$\therefore$ one branch of root locus terminates $s=1$ and the other branch terminates on infinity as $K \to \infty$.

(ii) Note: In this case the CE is $1+\frac{K(1-s)}{s(s+2)}=0$

or $1-\frac{K(s-1)}{s(s+2)}=0$

or $1-P(s)=0$

$\therefore$ The angle of departure at open loop poles is given by $\phi_p = \pm 180(2q)$; $q=0,1,2$

(a) The angle of departure at $s=0$ is given by
$\phi_p = 0+(+180) = 180^\circ$

(b) The angle of departure $s=-2$ is given by
$\phi_p = 0+(-180+180) = 0^\circ$

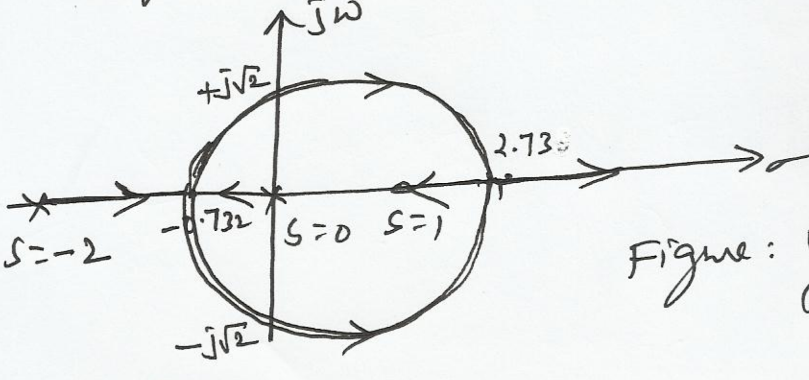


Figure: Root locus of $G(s)H(s) = \frac{Ke^{-s}}{s(s+2)}$

∴ Both the branches of root locus move in opposite direction.

(iii) The breakaway points are the solutions of $\frac{dK}{ds}=0$

From CE, $1+\frac{K(1-s)}{s(s+2)}$, $K=\frac{-s(s+2)}{1-s}$

$$\therefore \frac{dK}{ds}=\frac{(1-s)(-2s-2)+s(s+2)(-1)}{(1-s)^2}=0$$

or $-\cancel{2s}+2s^2+\cancel{2s}-2-s^2-2s=0$

$s^2-2s-2=0$

$$\therefore s=\frac{2\pm\sqrt{4+8}}{2}=\frac{2\pm\sqrt{12}}{2}=1\pm\sqrt{3}$$

$$=1\pm1.732=2.732 \ \& \ -0.732$$

Both are the breakaway points

(IV) The intersection points of root locus and imaginary axis can be obtained from Routh criterion.

The CE is $s(s+2)+K(1-s)=0$

$s^2+2s+K-Ks=0$ or $s^2+s(2-K)+K=0$

s^2	1	K
s	2−K	
s^0	K	

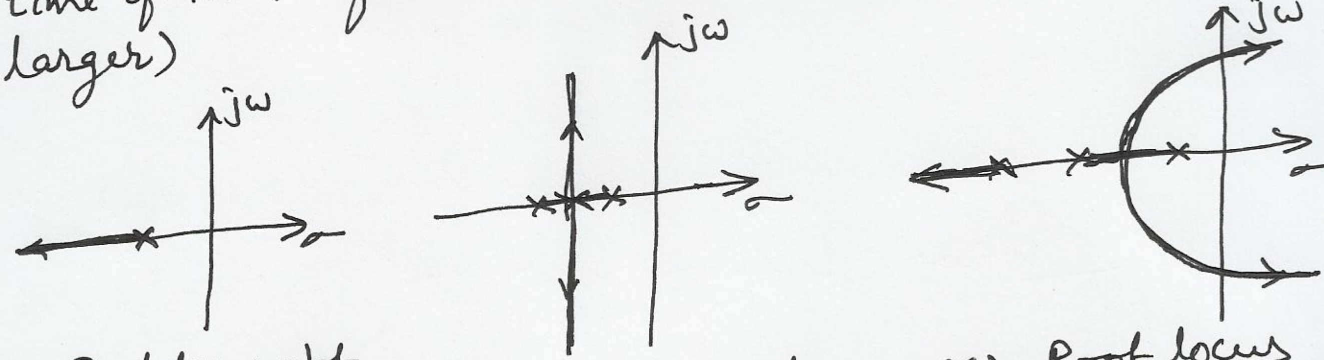
To have roots on imaginary axis, $2-K=0$ or $K=2$

∴ The Auxiliary equation is $s^2+K=0$, where $K=2$

$\therefore s^2+2=0$ or $s^2=-2$ or $s=\pm j\sqrt{2}$

Effect of Adding poles and Zeros to G(s)H(s) on the root loci:

(1) **Effects of Addition of poles**: The addition of pole to the open loop transfer function G(s)H(s) has the effect of pulling the root locus to the right, tending to lower the system relative stability and to slow down the settling time of the response (ie the value of settling time becomes larger)



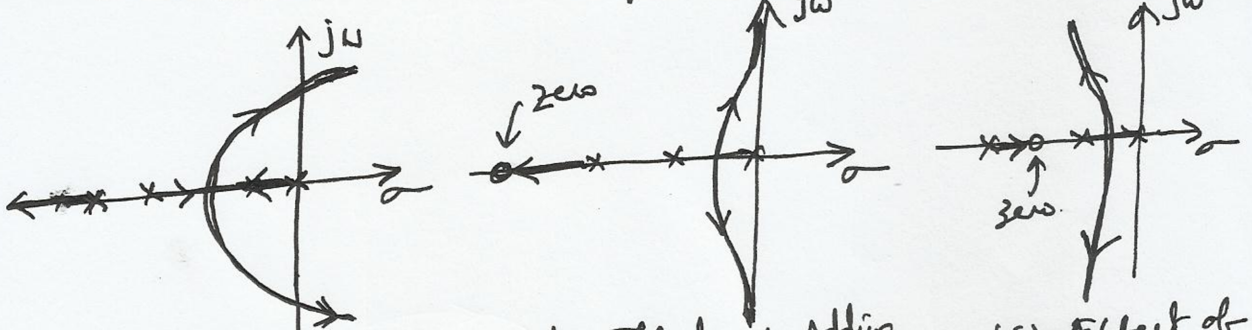
(a) Root locus plot for single pole system

(b) Root locus plot of two-pole system

(c) Root locus plot of three-pole system.

Figure (1): Effect of adding poles to G(s)H(s) on root locus

(2) **Effects of Addion of Zeros**: The addition of zero to the open loop transfer functions has the effect of pulling the root locus to the left, tending to make the system more stable and to speed up the settling of the response



(a) Root Locus plot of three-pole system.

(b) Effect of Adding zero to the three-pole root locus

(c) Effect of adding zero between two poles.

Procedure to Construct Root locus:

(1) Locate open loop poles and zeros in the s-plane (Note: we need the open loop transfer function to construct the root locus.

(2) Determine the angle of departure at each open loop pole.

(3) If the root locus branches are moving in opposite direction, determine the break away points.

(4) If the number of open loop zeros are less than the number of open loop poles, determine the centroid.

(5) Find the angles of asymptotes.

(6) If the asymptotes cross the imaginary axis, find the intersection points of imaginary axis and root locus.

Note (1) poles are in the denominator, hence the angle contributed by poles at a particular point is -ve of angle contributed at that point.

(2) If the angle of departure is $\pm 180°$, the root locus branch moves towards left on the real axis.

(3) If the angle of departure is 0 or $360°$, the root locus branch moves towards right on the real axis.