JAWAHARLAL NEHRU TECHNOLOGICAL UNIVERSITY ANANTAPUR

II B.Tech II-Sem (E.C.E)	**T**	**Tu**	**C**
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(15A02303) CONTROL SYSTEMS ENGINEERING

http://nptel.ac.in/courses/108101037/15

For all the 5 Units

OBJECTIVES:

To make the students learn about:

- Merits and demerits of open loop and closed loop systems; the effects of feedback
- The use of block diagram algebra and Mason's gain formula to find the effective transfer function between two nodes
- Transient and steady state responses , time domain specifications
- The concept of Root loci
- Frequency domain specifications, Bode diagrams and Nyquist plots
- The fundamental aspects of modern control

UNIT – I INTRODUCTION

Open Loop and closed loop control systems and their differences- Examples of control systems- Classification of control systems, Feedback Characteristics, Effects of positive and negative feedback. Mathematical models – Differential equations of Translational and Rotational mechanical systems, and Electrical Systems, Block diagram reduction methods – Signal flow graph - Reduction using Mason's gain formula. Transfer Function of DC Servo motor - AC Servo motor - Synchro transmitter and Receiver

UNIT-II TIME RESPONSE ANALYSIS

Step Response - Impulse Response - Time response of first order systems – Characteristic Equation of Feedback control systems, Transient response of second order systems - Time domain specifications – Steady state response - Steady state errors and error constants

UNIT – III STABILITY

The concept of stability – Routh's stability criterion – Stability and conditional stability – limitations of Routh's stability. The root locus concept - construction of root loci-effects of adding poles and zeros to G(s)H(s) on the root loci.

UNIT – IV FREQUENCY RESPONSE ANALYSIS

Introduction, Frequency domain specifications-Bode diagrams-Determination of Frequency domain specifications and transfer function from the Bode Diagram-Stability Analysis from Bode Plots. Polar Plots-Nyquist Plots- Phase margin and Gain margin-Stability Analysis.

Compensation techniques – Lag, Lead, Lag-Lead Compensator design in frequency Domain.

UNIT – V STATE SPACE ANALYSIS

Concepts of state, state variables and state model, derivation of state models from differential equations. Transfer function models. Block diagrams. Diagonalization. Solving the Time invariant state Equations- State Transition Matrix and it's Properties. System response through State Space models. The concepts of controllability and observability.

OUTCOMES:

After completing the course, the student should be able to do the following:

- Evaluate the effective transfer function of a system from input to output using (i) block diagram reduction techniques (ii) Mason's gain formula
- Compute the steady state errors and transient response characteristics for a given system and excitation
- Determine the absolute stability and relative stability of a system
- Draw root loci
- Design a compensator to accomplish desired performance
- Derive state space model of a given physical system and solve the state equation

TEXT BOOKS:

1. Modern Control Engineering, Katsuhiko Ogata, PEARSON, 1st Impression 2015.
2. Control Systems Engineering, I. J. Nagrath and M. Gopal, New Age International Publishers, 5th edition, 2007, Reprint 2012.

REFERENCE BOOKS:

1. Automatic Control Systems, Farid Golnaraghi and Benjamin. C. Kuo, WILEY, 9th Edition, 2010.
2. Control Systems, Dhanesh N. Manik, CENGAGE Learning, 2012.
3. John J D'Azzo and C. H. Houpis , "Linear Control System Analysis and Design: Conventional and Modern", McGraw - Hill Book Company, 1988.

UNIT-I

INTRODUCTION

The Control System is that means by which any quantity of interest in a machine, mechanism or other equipment is maintained or altered in accordance with a desired manner.

When a number of elements or components are connected in a sequence to perform a specific function, that group of elements is called a system. In a system, when the output quantity is controlled by varying the input quantity, the system is called controlled system. The output quantity is called controlled variable or response and input quantity is called command signal or excitation.

Basically, there are two types of control systems, namely open loop and closed loop control systems.

Open-loop System: Any physical system which does not automatically correct the variation in its output, is called open loop system. This means that the output is not fedback to the input for correction.

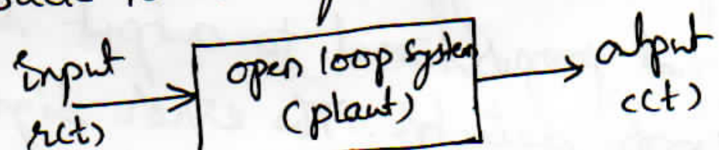


Figure: open-loop systems

In open-loop systems the output is varied by varying the input, but due to external disturbances the system output may change. When the output changes due to disturbances, it is not followed by changes in input to correct

the output. In open loop systems, the changes in output are corrected by changing the input manually.
Ex: Traffic light controller, Combinational circuits etc.

<u>Closed-loop System</u>: Control systems in which the output has an effect upon the input quantity inorder to maintain the desired output are called closed loop systems.

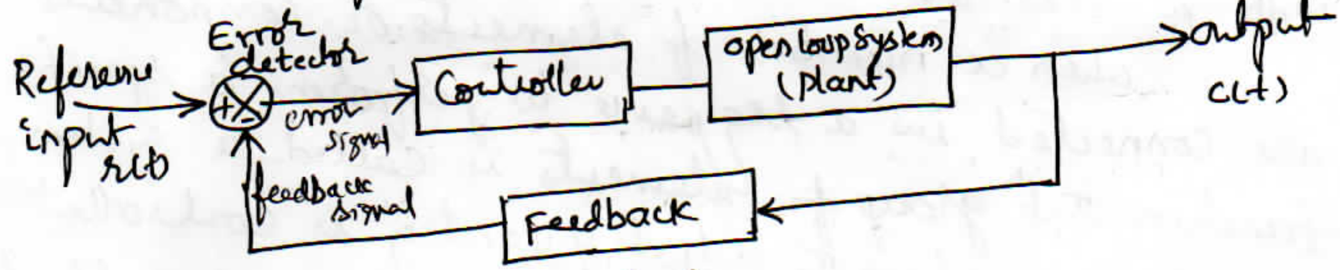


Figure: closed loop System

The open loop system can be modified as closed loop system by providing a feedback. The provision of feedback automatically corrects the changes in output due to disturbances. Hence the closed loop system is also called automatic control system.

The reference signal corresponds to desired output. The feedback path elements samples the output and converts it to a signal of same type as that of reference signal. The feedback signal proportional to output signal and it is fed to the error detector. The error signal generated by the error detector is the difference between reference signal and feedback signal. The controller modifies and amplifies the error signal to produce better control action. The modified error signal is fed to the plant to correct its output. Ex: Sequential circuits, Driving of automobile

Advantages of open loop systems:

(1) The open loop systems are simple and economical
(2) The open loop systems are easier to correct
(3) Generally the open loop systems are stable.

Disadvantages of open loop systems:

(1) The open loop systems are inaccurate and unreliable
(2) The changes in the output due to external disturbances are not corrected automatically.

Advantages of closed loop systems:

(1) The closed loop systems are accurate
(2) The closed loop systems are accurate even in the presence of non-linearities.
(3) The sensitivity of the system may be made small to make the systems more stable.
(4) The closed loop systems are less affected by noise

Disadvantages of closed loop systems:

(1) The closed loop systems are complex and costly.
(2) The feedback in closed loop system may lead to oscillatory response.
(3) The feedback reduces the over all gain of the system
(4) stability is a major problem in closed loop system and more care is needed to design a stable closed loop system.

Examples of Control systems :

(1) Driving of Automobile :

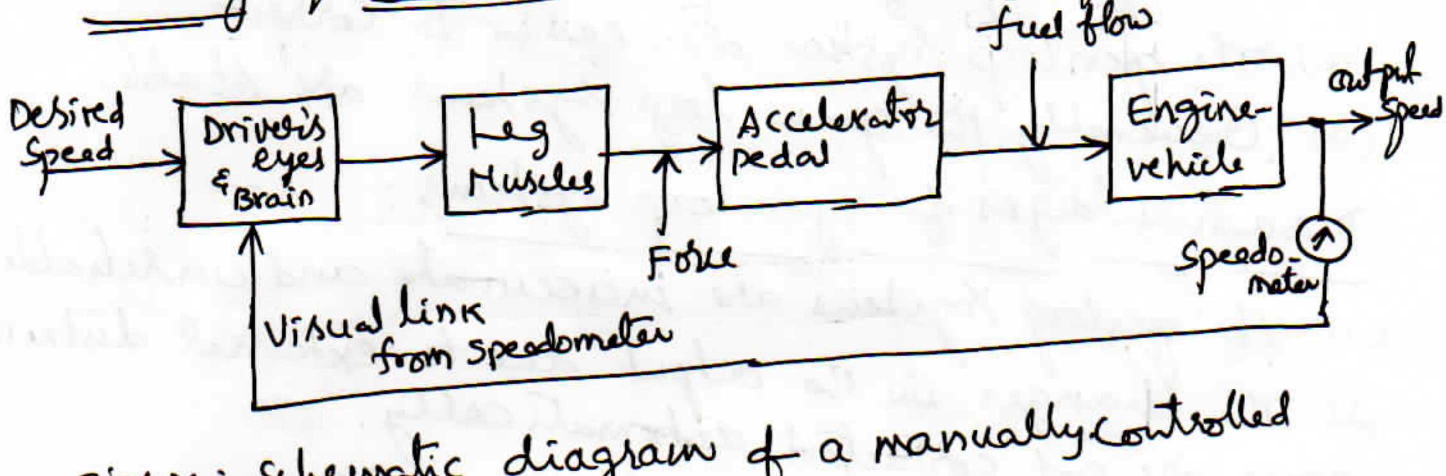


Figure : schematic diagram of a manually controlled closed-loop system.

The automobile driving system (accelerator, carburator, and engine-vehicle) constitutes a control system. The speed of the automobile is a function of the position of its accelerator. The desired speed can be maintained by controlling pressure on the accelerator pedal.

The route, speed and acceleration of the automobile are determined and controlled by the driver by observing traffic and road conditions and by properly manipulating the accelerator, clutch, gear-lever, brakes and steering wheel etc. Suppose the driver wants to maintain a speed of 50 km, the actual speed of the automobile is measured by the speedometer and indicated on its dial. The driver reads the speed dial visually and compares the actual speed with the desired speed mentally. If there is a deviation of speed from the desired speed, the driver takes the decision to increase or decrease the speed. The decision is executed by change in pressure of foot on the accelerator pedal.

(2) Temperature Control System:

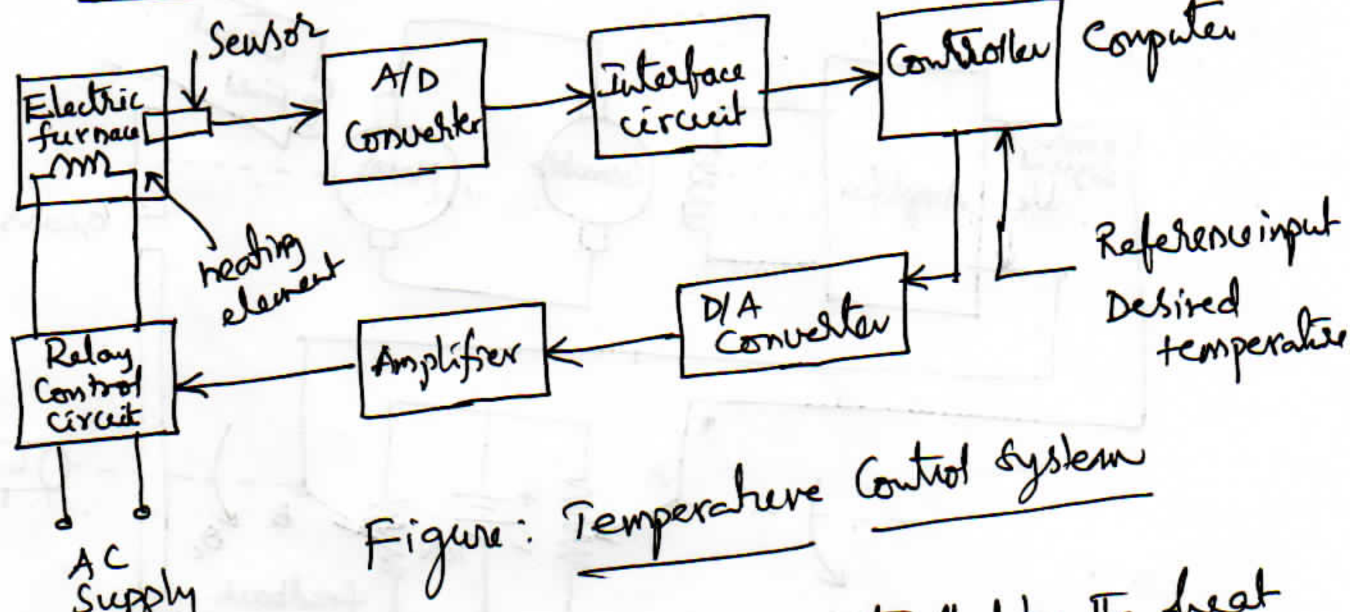


Figure: Temperature Control System

The temperature of the system is controlled by the heat generated by the heating element. The furnace output temperature depends on the time during which the supply to heater remains ON.

The ON and OFF of supply is governed by the time setting of the relay. The temperature of the furnace is measured by sensor and is converted to digital signal by A/D converter.

The switching ON and OFF of the relay is controlled by a controller which is a digital system or computer. The computer reads the actual temperature and compares with desired temperature. If it finds any difference then it sends signal to switch ON or OFF the relay through D/A converter and amplifier. Thus the system automatically corrects any changes in output.

(3) position control System using Servomotor:

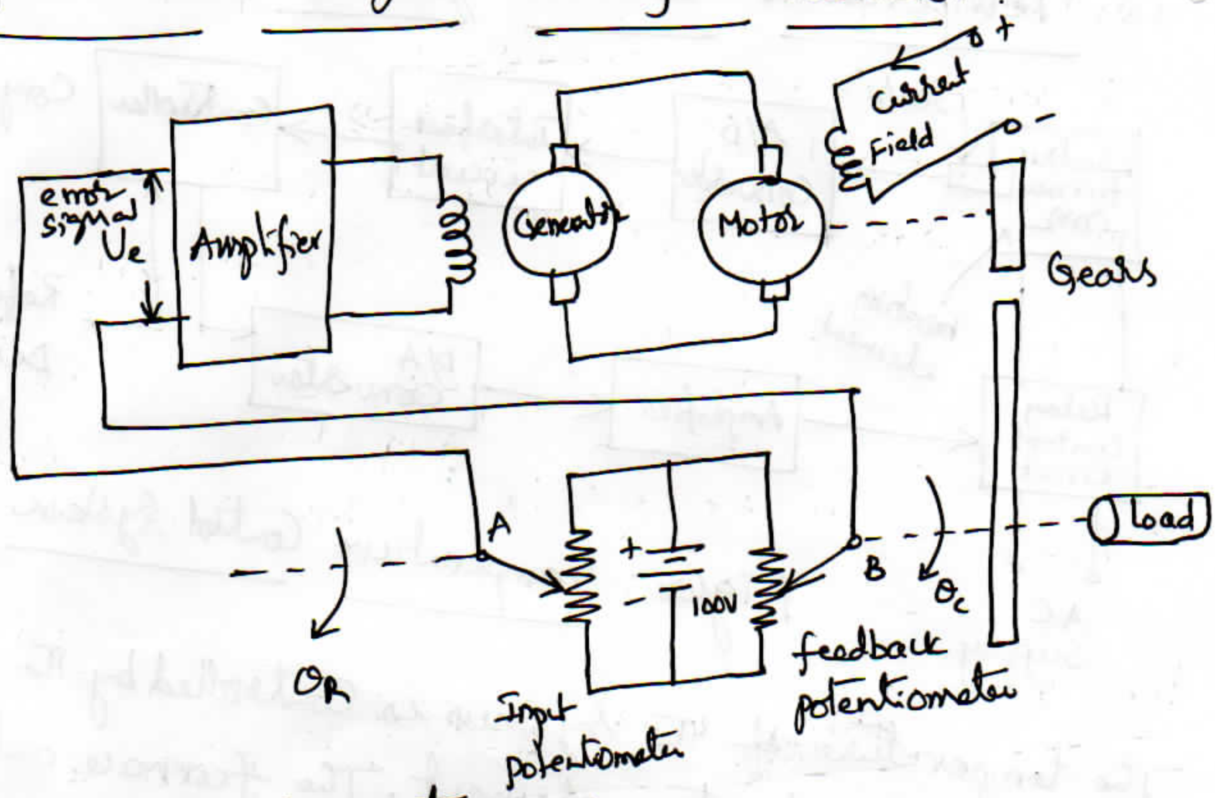


Figure: position Control System

The position Control System is a closed loop System. The Systems consists of a Servo motor powered by a generator. The load whose position has to be controlled is connected to motor shaft through gear wheels. potentiometers are used to convert the mechanical motion to electrical Signals. The desired position θ_R is set on the input potentiometer and the actual load position θ_c is fed to feedback potentiometer. The difference between two angular positions generates an error Signal V_e, which is amplified and fed to generator field circuit. The induced emf in the generator drives the motor in such a way that to get $\theta_c = \theta_R$.

If $\theta_c = \theta_R$; then $V_e = 0$ and the motion of the motor is stopped.

The feedback Control Systems in which the controlled variable is position or time derivatives of position (velocity and acceleration) are called Servo mechanisms. (Servo Mechanism)

Classification of Control Systems: Basically, feedback control systems are classified as
(1) Linear or Non-linear systems
(2) Time-varying or Time-invariant systems

(1) Linear Versus Non-linear Systems: If the system satisfies the homogeneous and super position principles, then the system is linear otherwise non-linear. Most real-life control systems have non-linear characteristics to some extent.

(2) Time-Invariant Versus Time-varying Systems: If the parameters of the control system do not change with time, the system is called time-invariant, otherwise time-varying systems. In practice, most of the physical systems contain elements that drift or vary with time. These systems are further classified as continuous-data and Discrete-data control systems.

(i) Continuous-data Control Systems: The signals at various parts of the system are all functions of time 't', the system is said to be Continuous-data control system. These continuous-data control systems are further classified as ac or dc control systems. If the signals in the system are modulated by some form of modulation scheme, then the systems are said to be ac or modulated control systems. On the other hand, if the ac signals are unmodulated, the system is said to be dc or un-modunmodulated Control System.

(ii) Discrete-data control systems: If the signals at one or more points of the systems are in the form of either a pulse-train or a digital code. These systems are further classified into sampled data and digital control systems.

In sampled data control systems, the signals are in the form of pulse train.

In digital control systems, the signals are digitally coded such as binary code to use digital computer.

Feedback characteristics, Effects of positive and negative

Feedback:

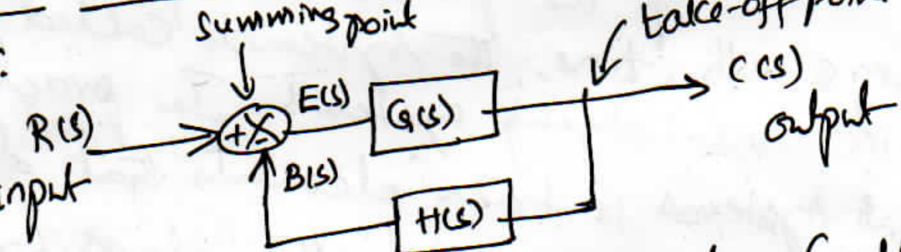


Figure: Negative or Degenerative feedback system

where $G(s)$ = Forward path gain
$H(s)$ = Feedback path gain
$E(s)$ = Error signal
$B(s)$ = Feedback signal

where the output $C(s) = E(s)\,G(s)$

$$= [R(s) - B(s)]\,G(s)$$

$$= [R(s) - C(s)H(s)]\,G(s)$$

$$\therefore C(s)[1 + G(s)H(s)] = R(s)G(s)$$

$\therefore$ The system Transfer function $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)H(s)}$

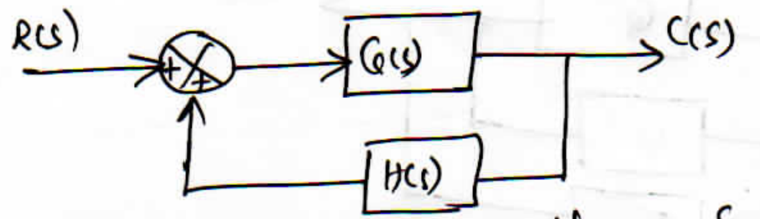


Figure: Positive Feedback System

where $\frac{C(s)}{R(s)} = \frac{G(s)}{1-G(s)H(s)}$

The feedback has effects on stability, bandwidth, over all gain, impedance and sensitivity

(i) Effect of feedback on over all gain: Let us assume that the system function $M = \frac{G}{1+GH}$, for convenience. In practical control systems, both G and H are functions of frequency, so the magnitude of $1+GH$ may be greater than 1 in one frequency range but less than 1 in other. Therefore, feedback could increase the gain of the system in one frequency range but decrease it in another.

(ii) Effect of feedback on stability: A system is said to be unstable, if its output is out of control.
we have system gain $M = \frac{G}{GH+1}$; if $GH=-1$ the output of the system is infinite for any finite output, and the system is said to be unstable. Therefore, we may state that feedback can cause a system that is originally stable to become unstable.

Now, let us consider a system with two feedbacks shown in figure, where the output is C(s) and

$$\frac{C(s)}{R(s)} = \frac{G}{1+GH+GF}$$

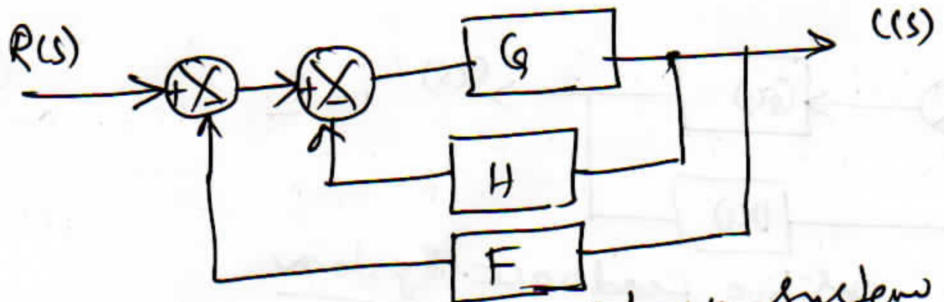


Figure: Negative feed back system

$$\frac{C(s)}{R(s)} = \frac{G}{1+G(H+F)} = \frac{G}{1+GH+GF}$$

If the above system is unstable, because of feed back G, then if $GH = -1$, we will get

$$\frac{C(s)}{R(s)} = \frac{G}{1-1+GF} = \frac{G}{GF} = \frac{1}{F}$$

Now, the over all system can be made stable by properly selecting 'F'.

In practice, GH is a function of frequency, and the stability condition of the closed-loop system depends on the magnitude and phase of GH. Thus, the feedback can improve stability or harmful to stability if is not properly applied.

(iii) Effect of feedback on Sensitivity: In general, a good control systems should be very insensitive to parameter variations but sensitive to the input.

The sensitivity of the gain of the over all system M to the variation in 'G' is defined as

$$S_G^M = \frac{\partial M/M}{\partial G/G} = \frac{\text{percentage change in } M}{\text{percentage change in } G}$$

$$= \frac{\partial M}{\partial G}\cdot\frac{G}{M} = \frac{\partial}{\partial G}\left[\frac{G}{1+GH}\right]\cdot\frac{G}{\left(\frac{G}{1+GH}\right)}$$

$$= \frac{1}{1+GH}$$

Thus, the Sensitivity of a closed loop System with respect to variation in G is reduced by a factor 1+GH as compared to that of an open-loop system. (6)

The sensitivity of output M w.r.t feedback 'H' is given by

$$S_H^M = \frac{\partial M/M}{\partial H/H} = \frac{\partial M}{\partial H}\cdot\frac{H}{M} = \frac{\partial}{\partial H}\left(\frac{G}{1+GH}\right)\cdot\frac{H}{\left(\frac{G}{1+GH}\right)}$$

$$= \frac{-GH}{1+GH}$$

In practice, GH is a function of frequency, the magnitude of 1+GH may be less than unity in one frequency range and greater than unity in another. Hence the feedback may increase or decrease Sensitivity of the System.

Differential Equations of Translational and Rotational Systems & Electrical Systems:

Mathematical Models of physical Systems:

A physical System is a collection of physical objects connected together to serve an objective. Mathematical representation of the physical model through use of appropriate physical laws is known as mathematical model.

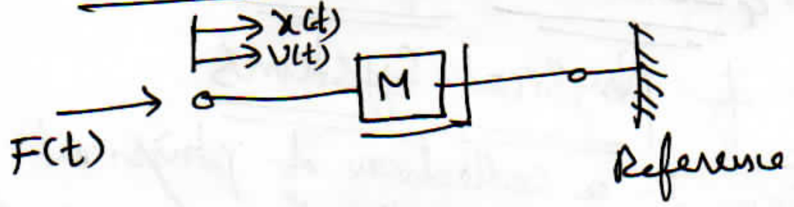
Mathematical models of most physical systems are characterised by differential equations. If the mathematical model obeys superposition and homogeneity principles, then the model is said to be linear. If the coefficients of differential equations are independent of time 't', then the physical model is said to be linear-time invariant.

Mechanical Systems: Mechanical systems are analysed by three idealised elements namely the mass, the spring and the damper, using Newton's law of motion. The motion of mechanical elements can be translatory, rotational or combination of both.

Translational Systems: The motion along a straight line is called the translatory motion. The variables which describe the translatory motion of mechanical systems are velocity, acceleration and displacement. The elements involved in the translatory motion are

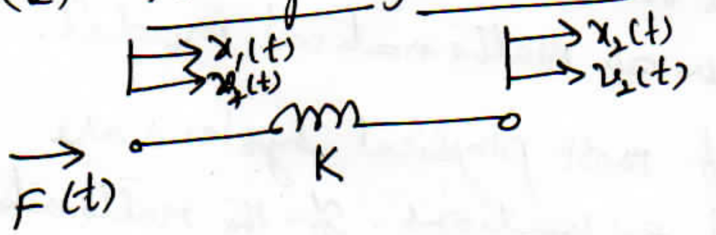
(1) The Mass element:

$$F = M\frac{dv}{dt} = M\frac{d^2x}{dt^2}$$



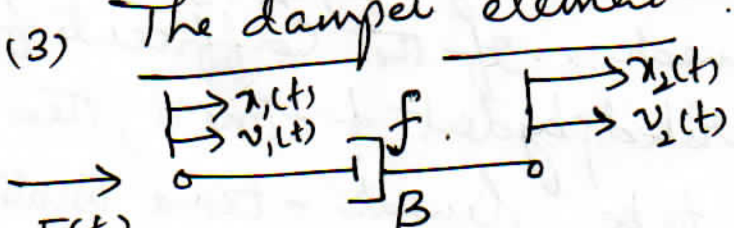
(2) The Spring element

$$F = K(x_1 - x_2) = Kx$$
$$= K\int_{-\infty}^{t}(v_1 - v_2)\,dt$$
$$= K\int_{-\infty}^{t} v\,dt$$



(3) The damper element:

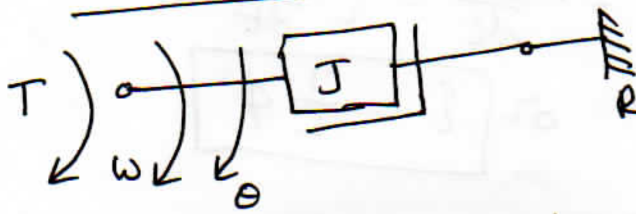
$$F = f(v_1 - v_2)$$
$$= f\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right)$$
$$= f\frac{dx}{dt}$$



where x(m), v(m/sec), M(Kg), F(Newton), K(N/m), f(N/m/sec)
B(N/m/sec)

Rotational Systems: The movement of a body around its fixed axis is called the rotational motion. The basic elements of rotational motion are moment of inertia (J), spring stiffness (K) and viscous friction coefficient (f or B).

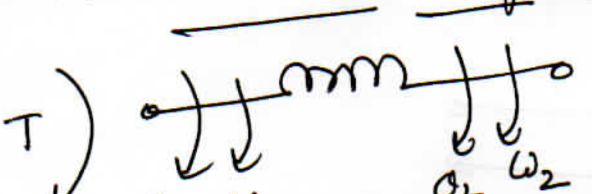
(1) The Moment of Inertia (J)



$$T = J\frac{d\omega}{dt} = J\frac{d^2\theta}{dt^2}$$

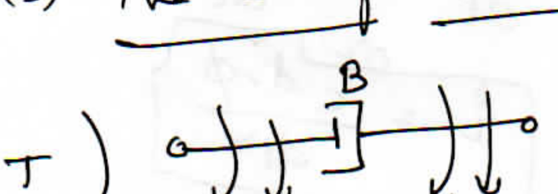
where T is torque in Nm
J is inertia in Kgm^2
ω is angular velocity in rad/sec
θ is angular displacement in rad.

(2) The Torsional Spring element (K)



$$T = K(\theta_1-\theta_2) = K\theta$$
$$= K\int_{-\infty}^{t}(\omega_1-\omega_2)\,dt$$
$$= \int_{\infty}^{t}\omega\,dt$$

(3) The damper element (f or B):

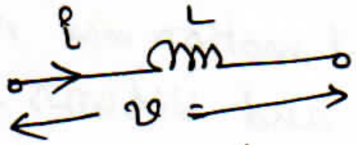


$$T = B(\omega_1-\omega_2) = f(\omega_1-\omega_2)$$
$$= B\left(\frac{d\theta_1}{dt}-\frac{d\theta_2}{dt}\right)$$
$$= B\dot{\theta}$$

where K is in Nm/rad, viscous friction coefficient f or B in (Nm/rad/sec).

Electrical Systems: The passive electric elements are inductor, resistor and Capacitor.

(1) Inductor:



$v = L\frac{di}{dt}$; $i = \frac{dq}{dt}$

$\therefore \boxed{v = L\frac{d^2q}{dt^2}}$

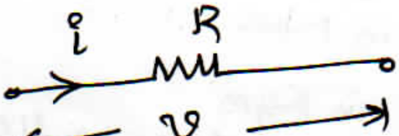
Also $\frac{di}{dt} = \frac{1}{L}v$,

where $v = \frac{d\phi}{dt}$

$\therefore \frac{di}{dt} = \frac{1}{L}\frac{d\phi}{dt}$

or $\boxed{i = \frac{1}{L}\phi}$

(2) Resistor:



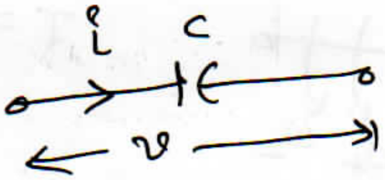
where $v = iR$

or $\boxed{v = R\frac{dq}{dt}}$

Also $i = \frac{1}{R}v = \frac{1}{R}\frac{d\phi}{dt}$

$\therefore \boxed{i = \frac{1}{R}\frac{d\phi}{dt}}$

(3) Capacitor:



where $v = \frac{1}{C}\int i\,dt$

$= \frac{1}{C}\int \frac{dq}{dt}dt$

$\boxed{v = \frac{1}{C}(q)}$

Also $i = C\frac{dv}{dt}$

$\frac{dq}{dt} = C\frac{d^2\phi}{dt^2}$

or $\boxed{i = C\frac{d^2\phi}{dt^2}}$

$v = \frac{1}{C}q$

$i = C\frac{d^2\phi}{dt^2}$

Analogous Systems: Systems with identical differential equations are called analogous systems. There are two types of analogy namely

(1) Force (Torque) – voltage analogy.
(2) Force (Torque) – current analogy:

(1) Force (Torque) – voltage Analogy:

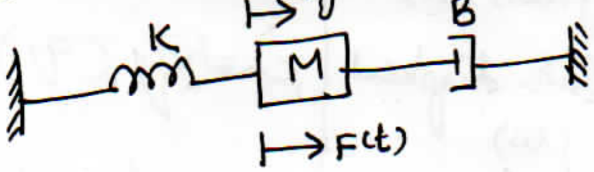
Mechanical system		Electrical system
Translational system	Rotational system	
Force (F)	Torque (T)	voltage (V)
Mass (M)	Inertia (J)	Inductance (L)
Viscous friction coefficient (B)	Viscous friction coefficient (B)	Resistance (R)
Spring stiffness (K)	Torsional spring stiffness (K)	Reciprocal of Capacitance (1/c)
Displacement (x)	Angular displacement (ω)	charge (q)
velocity (v)	Angular velocity (ω	current (i)

Table: Analogous quantities in Force (Torque) – voltage Analogy:

(2) Force (Torque) – Current Analogy:

Mechanical System		Electrical Systems
Translational	Rotational	
Force (F)	Torque (T)	Current (i)
Mass (M)	Moment of Inertia (J)	Capacitance (C)
Viscous friction Coefficient (B)	Viscous friction Coefficient (B)	Reciprocal of Resistance (1/R)
Spring stiffness (K)	Torsional spring stiffness (K)	Reciprocal of Inductance (1/L)
Displacement (x)	Angular displacement (θ)	flux linkages (ϕ or λ)
velocity 'v'	Angular velocity (ω)	voltage (v)

① Draw the mechanical network, node equations and electrical analogous circuits of the system shown in figure.



$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2+Bs+K}$$

(Sol)

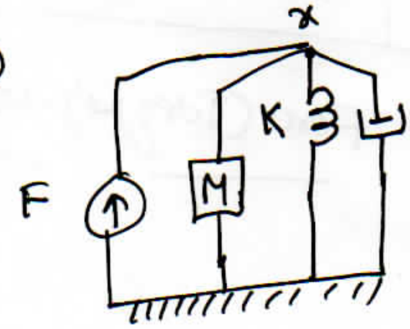


Figure: Mechanical Network

At the node 'x'

$$F = F_M + F_K + F_B$$

$$= M\frac{d^2x}{dt^2} + Kx + B\frac{dx}{dt}$$

$$v = L\frac{d^2q}{dt^2} + \frac{1}{C}q + R\frac{dq}{dt}$$

$$= L\frac{di}{dt} + \frac{1}{C}\int i\,dt + Ri$$

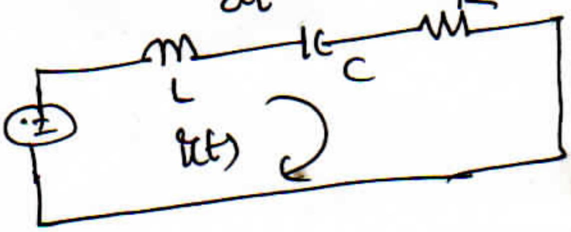
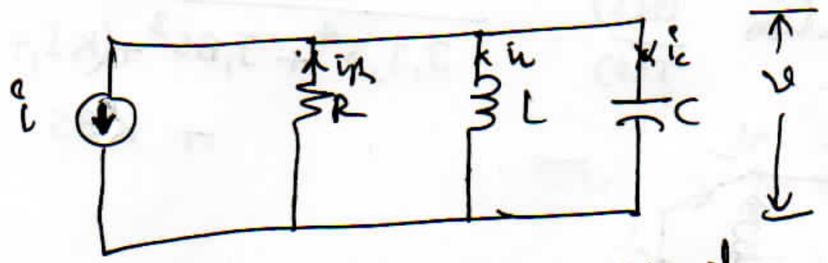


Figure: Force-voltage analogous circuit

In force-current analogy

$$i = C\frac{d^2\phi}{dt^2} + \frac{1}{R}\frac{d\phi}{dt} + \frac{1}{C}\phi$$

$$= C\frac{dv}{dt} + \frac{1}{R}v + \frac{1}{C}\int v\,dt$$

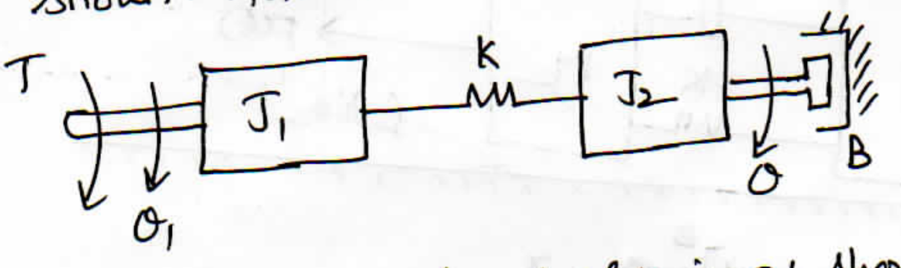


Force-current analogus circuit

Note (1) The force-current analogous circuit has same structure as that of mechanical network

(2) In force-voltage analogous circuit, the parallel elements may appear in series and vice-versa.

(2) obtain the transfer function of the mechanical system shown. Also draw the electrical analogous circuit.



(Sol) The mechanical network is as shown in figure

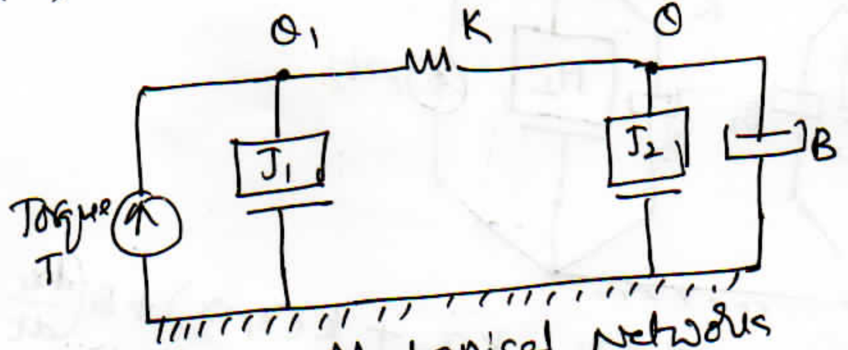


Figure : Mechanical Networks

At node θ_1, $J_1\frac{d^2\theta_1}{dt^2} + K(\theta_1 - \theta) = T \longrightarrow (1)$

At node θ, $J_2\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K(\theta - \theta_1) = 0 \longrightarrow (2)$

Applying Laplace transform

$$(J_1 s^2 + K)\,\theta_1(s) + K\theta(s) = T(s) \longrightarrow (3)$$

and $$(J_2 s^2 + Bs + K)\,\theta(s) = K\theta_1(s) \longrightarrow (4)$$

∴ The Transfer function $$\frac{\theta(s)}{T(s)} = \frac{K}{J_1 J_2 s^4 + J_1 B s^3 + (KJ_1 + KJ_2)s^2 + KBS}$$

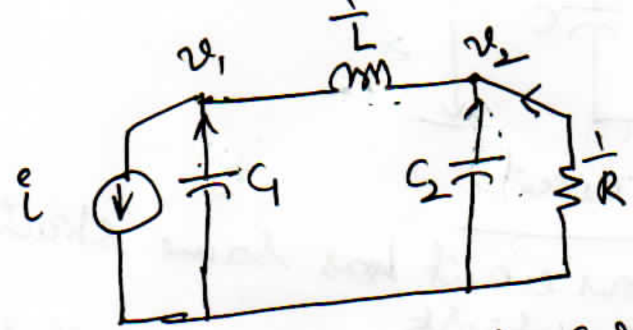
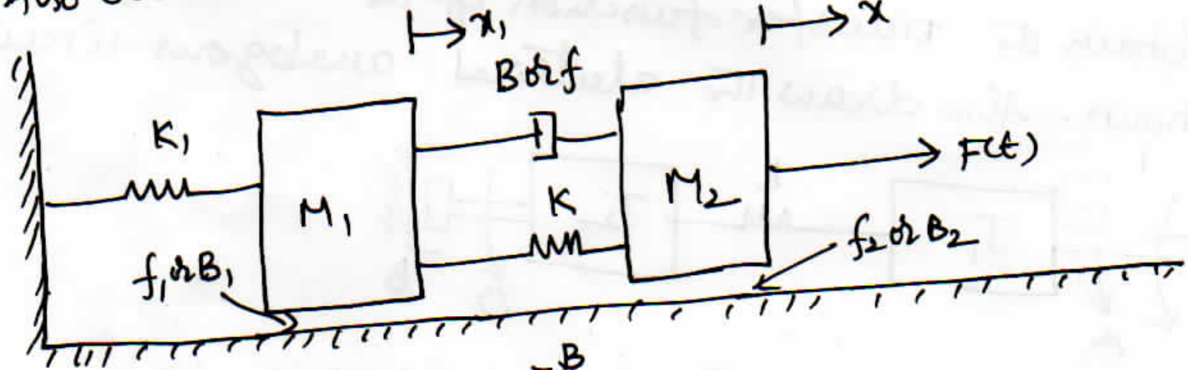
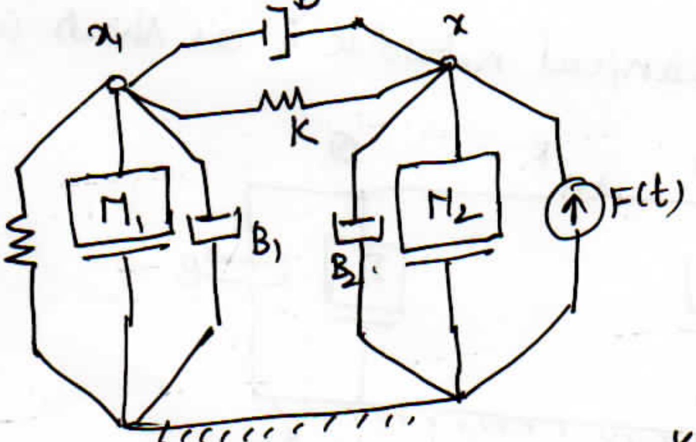


Figure: Force(Torque) - current analogous circuit

(3) Draw the mechanical network and write the node equations. Also draw the electrical analog circuit.



(Sol)



At node x_1, $$M_1 \frac{d^2x_1}{dt^2} + B_1\frac{dx_1}{dt} + K_1 x_1 + K(x_1 - x) + B\left(\frac{dx_1}{dt} - \frac{dx}{dt}\right) = 0$$

At node x, $$\left.\begin{aligned} M_2 \frac{d^2x}{dt^2} + K(x - x_1) + B\left(\frac{dx}{dt} - \frac{dx_1}{dt}\right) \\ + B_2 \frac{dx}{dt} \end{aligned}\right\} = F(t)$$

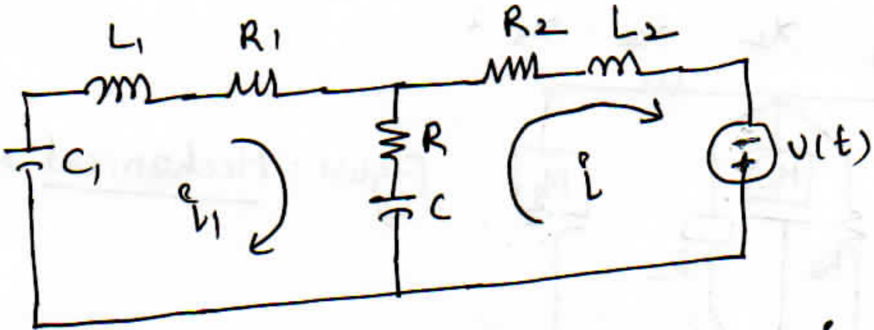


Figure: Force-voltage analogous circuit.

for 1st mesh

$$\frac{1}{C}\int(i_1-i)\,dt + L_1\frac{di}{dt^2} + R_1 i_1 + R(i_1-i) + \frac{1}{C_1}\int i_1\,dt = 0$$

for 2nd mesh

$$\frac{1}{C}\int(i-i_1)\,dt + R(i-i_1) + R_2 i + L_2\frac{di}{dt} - v(t) = 0$$

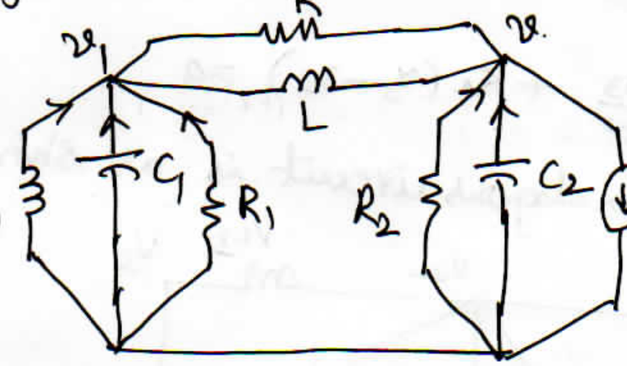


Figure: Force-Current analogous circuit

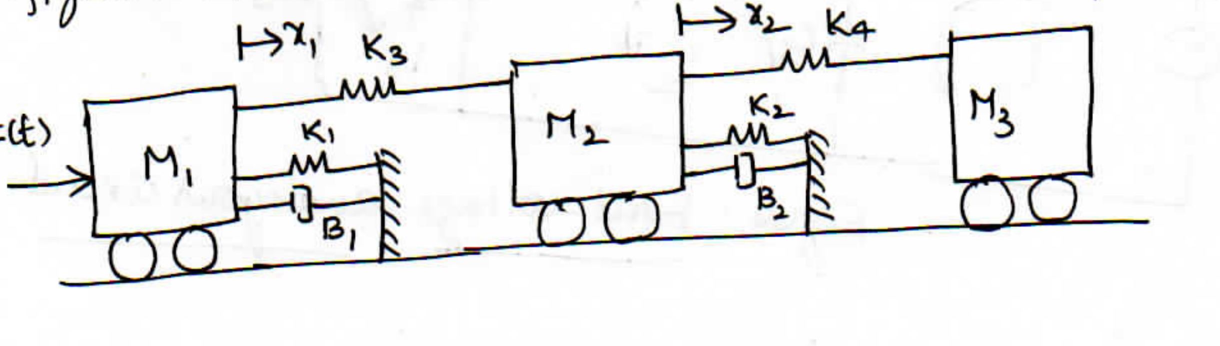
at node v_1,

$$\frac{1}{L_1}\int v_1\,dt + C_1\frac{dv_1}{dt} + \frac{v_1}{R_1} + \frac{v_1-v}{R} + \frac{1}{L}\int(v_1-v)\,dt = 0$$

at node v;

$$C_2\frac{dv}{dt} + \frac{v}{R_2} + \frac{v-v_1}{R} + \frac{1}{L}\int(v-v_1)\,dt = i$$

(3) Draw the mechanical network of the system shown in figure. Also draw the analogous circuit



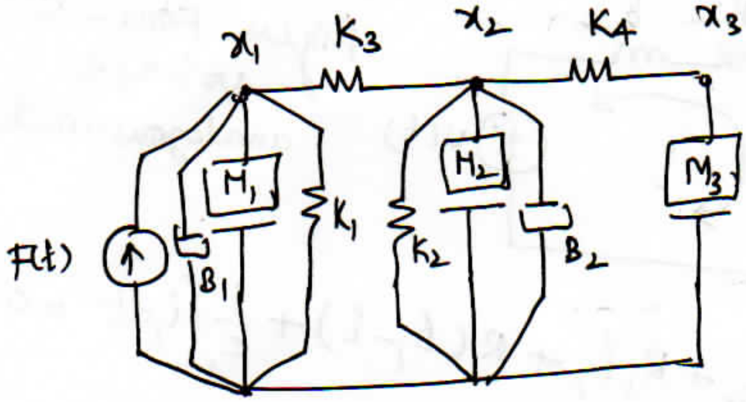


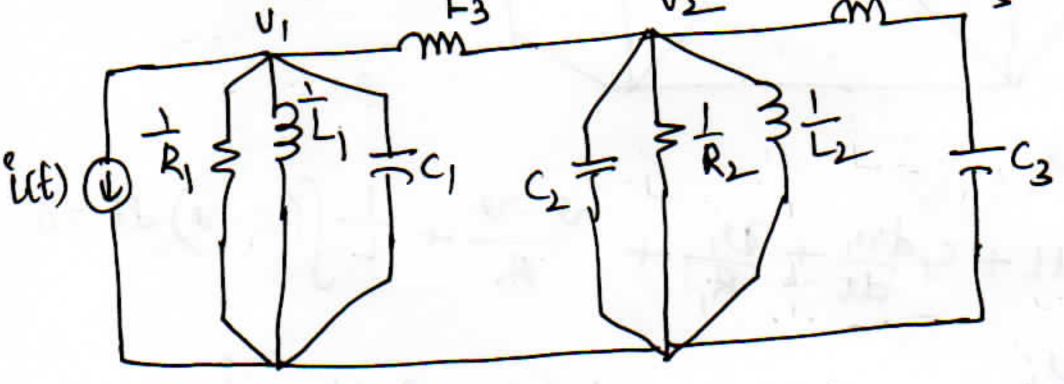
Figure : Mechanical Network

at node x_1, $M_1\frac{d^2x_1}{dt^2} + B_1\frac{dx_1}{dt} + K_1x_1 + K_3(x_1 - x_2) = F(t)$

at node x_2, $M_2\frac{d^2x_2}{dt^2} + B_2\frac{dx_2}{dt} + K_2x_2 + K_3(x_2 - x_1) + K_4(x_2 - x_3) = 0$

at node x_3, $M_3\frac{d^2x_3}{dt^2} + K_4(x_3 - x_2) = 0$

The force-current analogous circuit is as shown in figure



Force-current Analogous circuit

The force-voltage analogous circuit is as follows

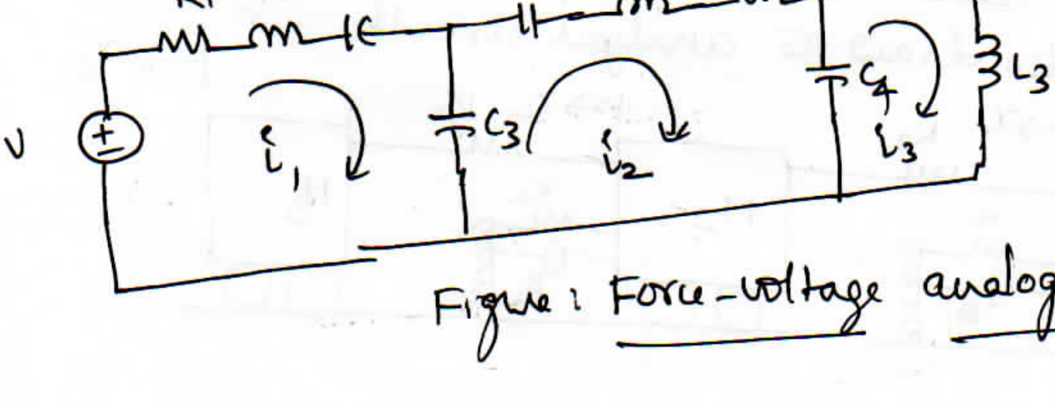
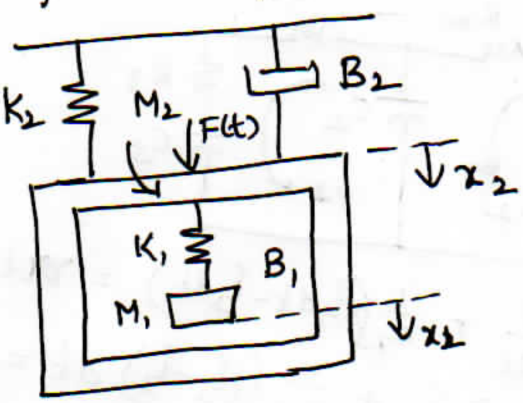
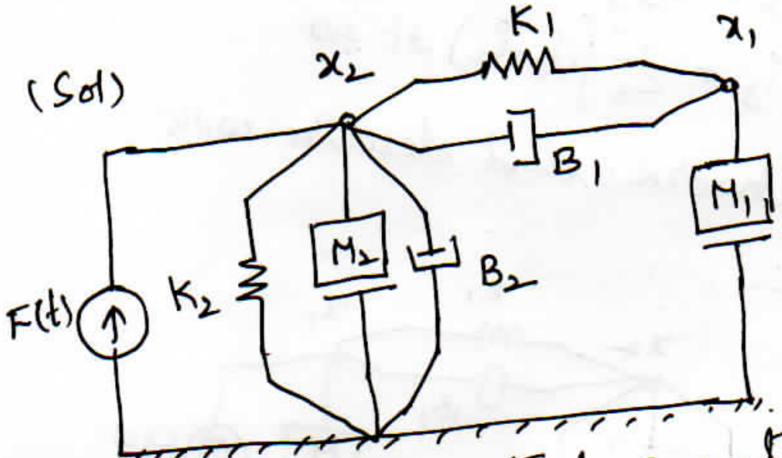


Figure : Force-voltage analogous circuit

(1) Draw the mechanical network and write the differential equations for the system shown. II



(Sol)



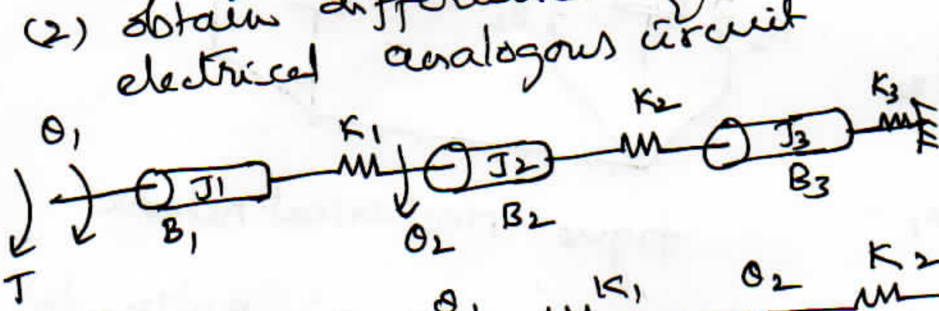
at 'x_2'

$$F(t) = M_2 \frac{d^2 x_2}{dt^2} + K_2 x_2 + B_2 x_2 + K_1(x_2 - x_1) + B_1\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right)$$

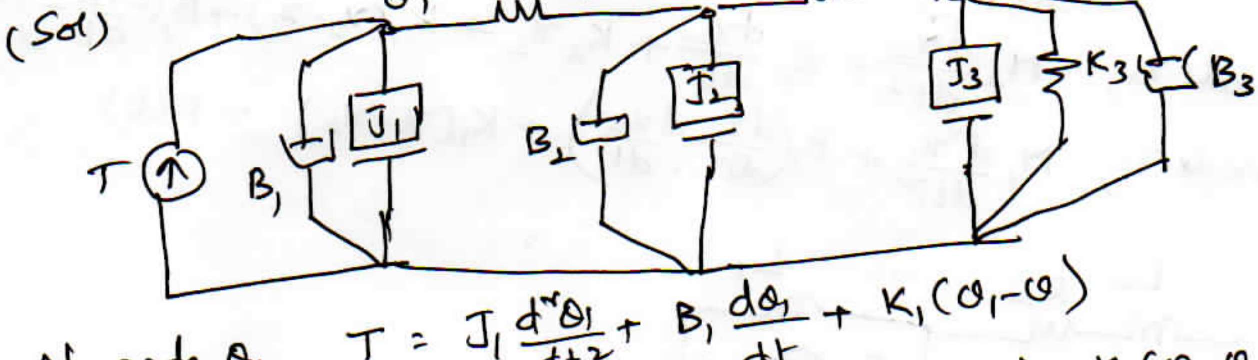
at 'x_1',

$$M_1 \frac{d^2 x_1}{dt^2} + K_1(x_1 - x_2) + B\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right) = 0$$

(2) Obtain differential equations and also draw the electrical analogous circuit



(Sol)

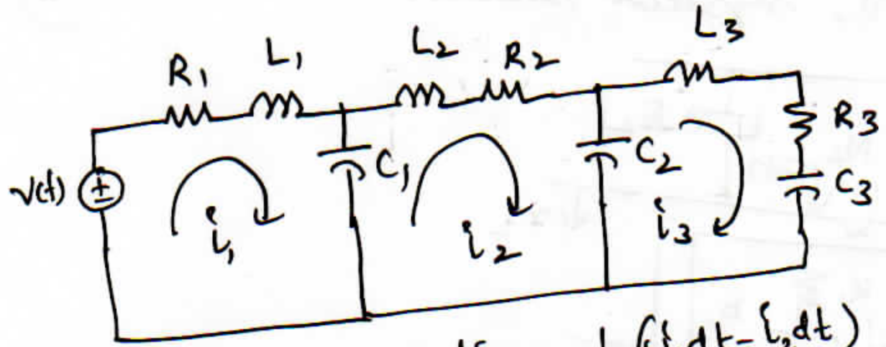


At node θ_1, $\quad T = J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d\theta_1}{dt} + K_1(\theta_1 - \theta)$

At node θ_2, $\quad J_2 \frac{d^2\theta_2}{dt^2} + B_2 \frac{d\theta_2}{dt} + K_1(\theta_2 - \theta_1) + K_2(\theta_2 - \theta_3) = 0$

At node θ_3, $\quad J_3 \frac{d^2\theta_3}{dt^2} + K_3\theta_3 + B_3 \frac{d\theta_3}{dt} + K_2(\theta_3 - \theta_2) = 0$

(Torque)
The force-voltage anagous circuit is as follows

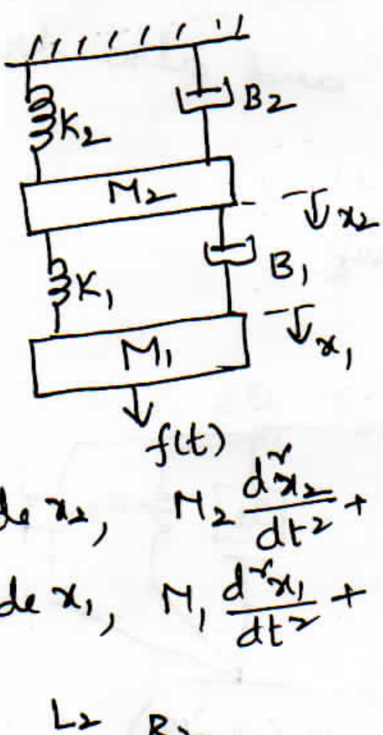


for mesh ① $R_1 i_1 + L_1 \frac{di}{dt} + \frac{1}{C_1}\int (i_1 dt - i_2 dt) = v(t)$

for mesh ② $L_2 \frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_2}\int (i_2 - i_3)\, dt = 0$

for mesh ③ $L_3 \frac{di_3}{dt} + R_3 i_3 + \frac{1}{C_3}\int (i_3 - i_2)\, dt = 0$

② Draw the mechanical network and describe with differential equations.



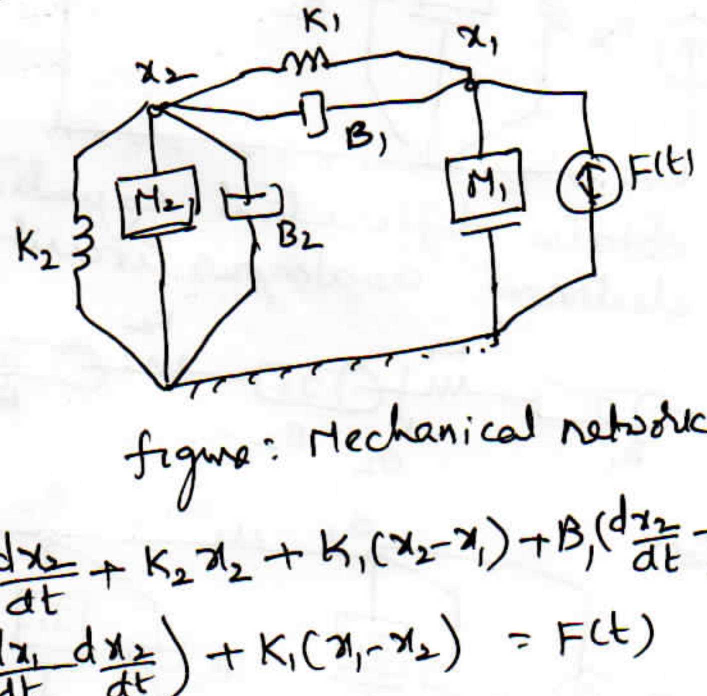


figure: Mechanical network

At node x_2, $M_2 \frac{d^2 x_2}{dt^2} + B_2 \frac{dx_2}{dt} + K_2 x_2 + K_1(x_2 - x_1) + B_1\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right) = 0$

at node x_1, $M_1 \frac{d^2 x_1}{dt^2} + B_1\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right) + K_1(x_1 - x_2) = F(t)$

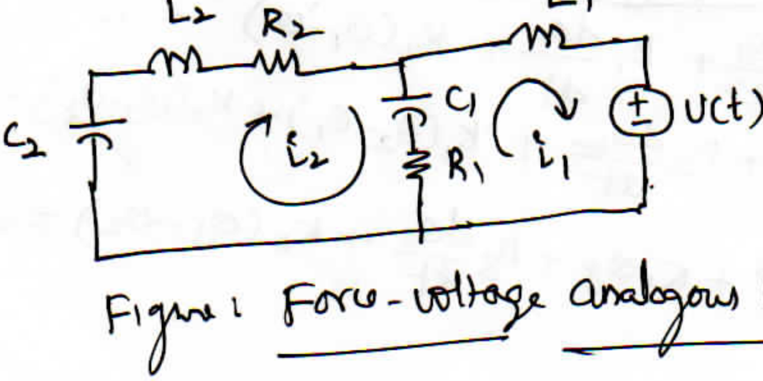
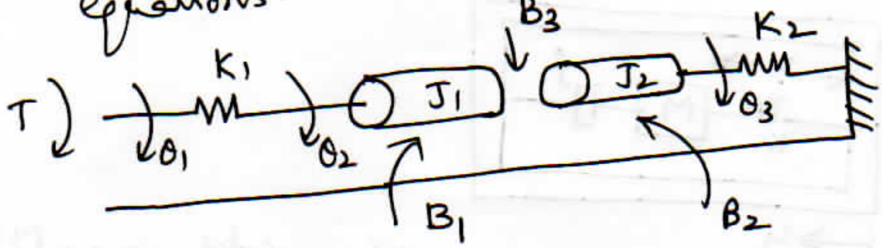
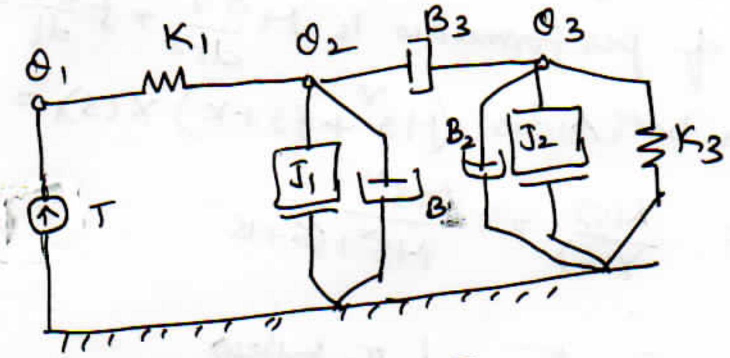


Figure: Force-voltage analogous circuit

① Draw the mechanical network and write the differential equations. (12)



(Sol)



At node θ_1, $\quad K_1(\theta_1-\theta_2) = T$

node θ_2, $\quad J_1 \frac{d^2\theta_2}{dt^2} + B_1 \frac{d\theta_2}{dt} + K_1(\theta_2-\theta_1) + B_3\left(\frac{d\theta_2}{dt} - \frac{d\theta_3}{dt}\right) = 0$

node θ_3, $\quad J_2 \frac{d^2\theta_3}{dt^2} + B_2 \frac{d\theta_3}{dt} + K_3\theta_3 + B_3\left(\frac{d\theta_3}{dt} - \frac{d\theta_2}{dt}\right) = 0$

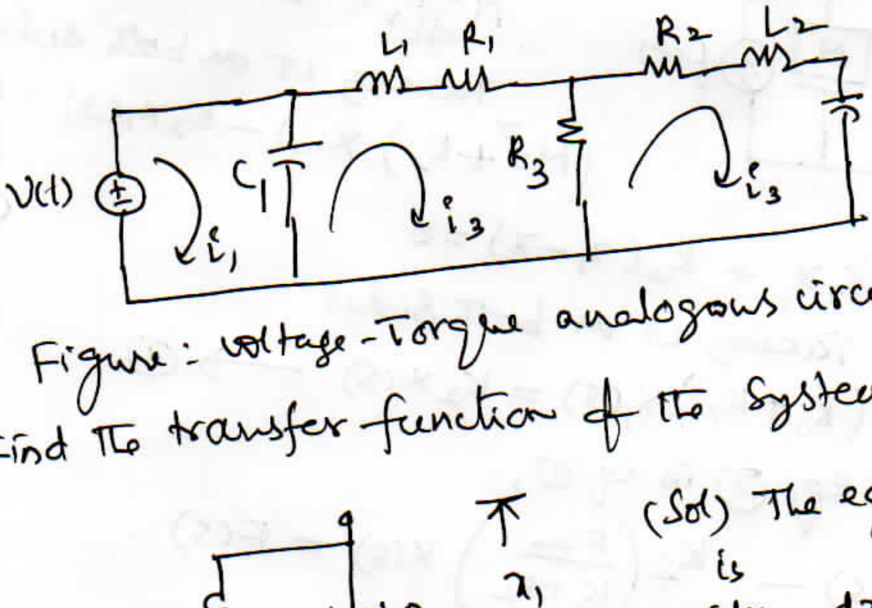
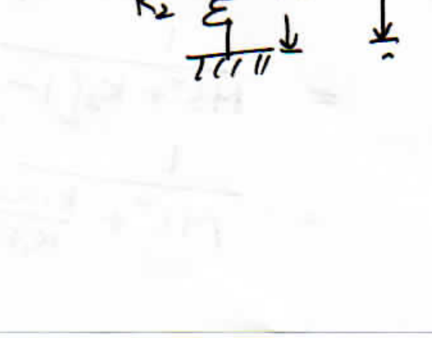


Figure: voltage-Torque analogous circuit

② Find the transfer function of the system.



(Sol) The equation of performance is

$$B_2\left(\frac{dx_1}{dt} - \frac{dx_0}{dt}\right) + K_1(x_1 - x_0) = K_2 x_0$$

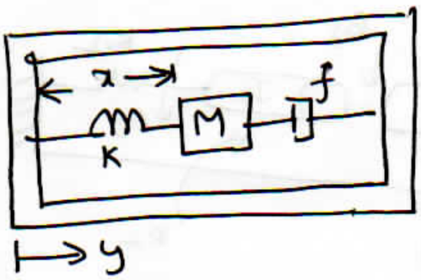
Taking LT on both sides

$$X_1(s)\left[B_2 s + K_1\right] = X_0(s)\left[B_2 s + K_1 + K_2\right]$$

∴ Transfer function

$$\frac{X_0(s)}{X_1(s)} = \frac{B_2 s + K_1}{B_2 s + K_2 + K_1}$$

② Findout the transfer function of the mechanical accelerator.

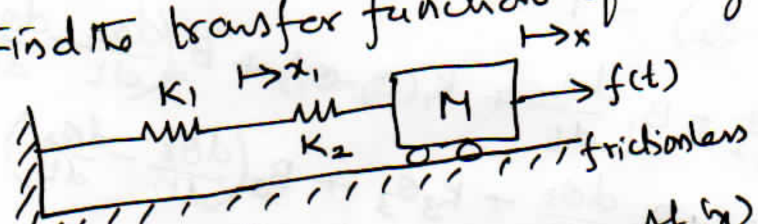


(Sol) The equation of performance is $M\frac{d^2x}{dt^2} + f\frac{dx}{dt} + Kx = M\frac{d^2y}{dt^2}$

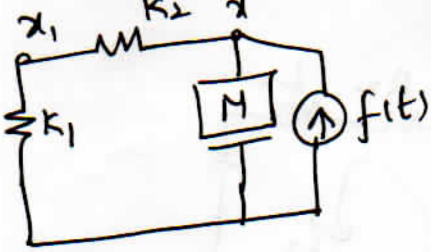
Taking LT on both sides $(Ms^2 + fs + K)\,X(s) = Ms^2 Y(s)$

$$\therefore \text{TF} \quad \frac{X(s)}{Y(s)} = \frac{Ms^2}{Ms^2 + fs + K}$$

(3) Find the transfer function of the system



(Sol)



At 'x'

$$M\frac{d^2x}{dt^2} + K_2(x - x_1) = f(t)$$

Taking LT on both sides

$$(Ms^2 + K_2)\,X(s) - K_2 X_1(s) = F(s) \quad \rightarrow ①$$

At node 'x_1', $K_1 x_1 + K_2(x_1 - x) = 0$

Taking LT on both sides

$$(K_1 + K_2)\,X_1(s) = K_2 X(s) \longrightarrow ②$$

Substituting eq ② in eq ①,

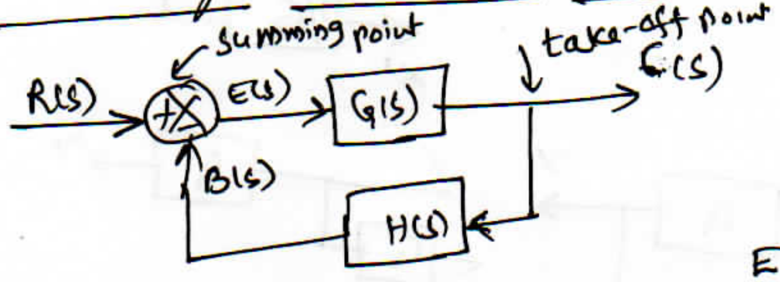
$$(Ms^2 + K_2)\,X(s) - K_2\left(\frac{K_2}{K_1 + K_2}\right) X(s) = F(s)$$

$$\therefore \text{Transfer function} \quad \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + K_2 - \frac{K_2^2}{K_1 + K_2}}$$

$$= \frac{1}{Ms^2 + K_2\left[1 - \frac{K_2}{K_1 + K_2}\right]}$$

$$= \frac{1}{Ms^2 + \frac{K_1 K_2}{K_1 + K_2}}$$

Block Diagram Reduction Techniques:



R(s) = Reference input

C(s) = output or Controlled variable

E(s) = Actuating signal or Error signal

B(s) = Feedback Signal

$C(s) = E(s)G(s)$

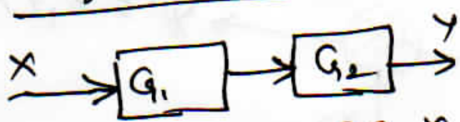
$\therefore \frac{C(s)}{E(s)} = G(s)$ is Forwardpath Transfer function

$B(s) = C(s)H(s) \Rightarrow \frac{B(s)}{C(s)} = H(s)$ = Feedback Transfer function

Block Diagram Reduction Albera:

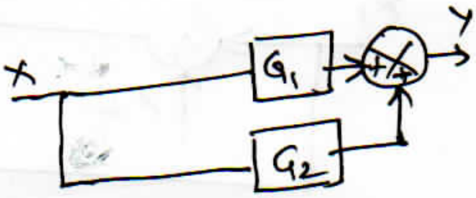
Rule | **Original Diagram** | **Equivalent diagram**

(1) Combing blocks in Cascade

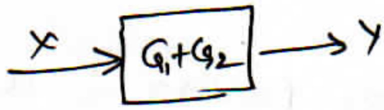


$y = G_1G_2X$

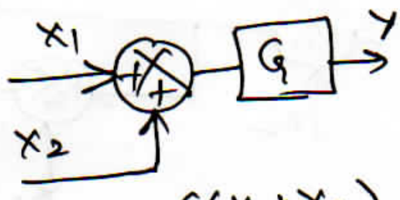
(2) Combing blocks in parallel



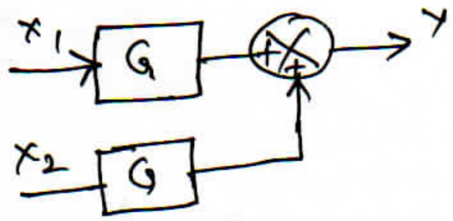
$Y = (G_1 + G_2)X$



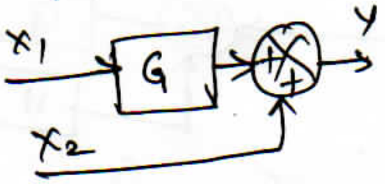
(3) Moving a Summing point after a block



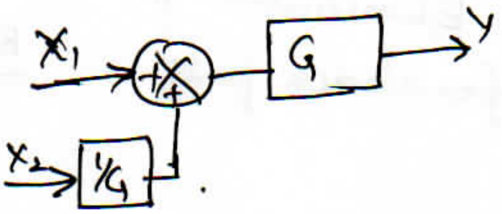
$Y = G(X_1 + X_2)$



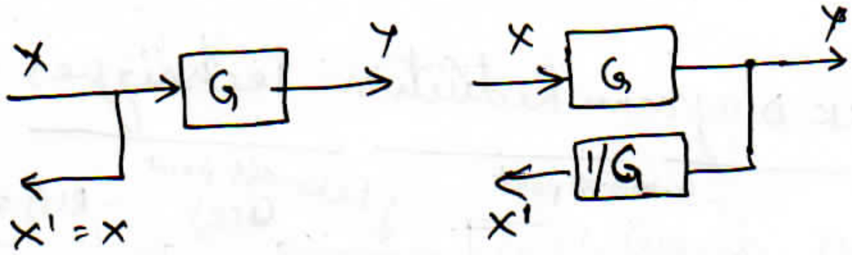
(4) Moving a Summing point ahead of a block



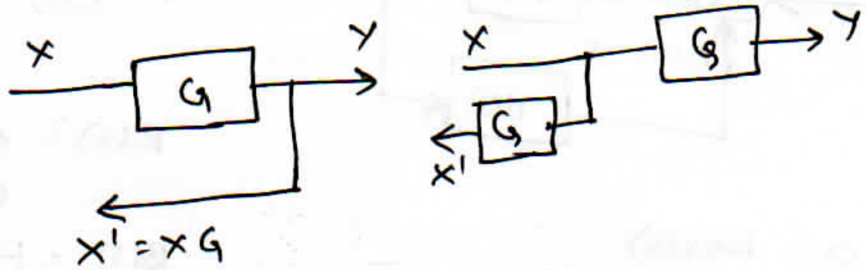
$Y = X_1G + X_2$



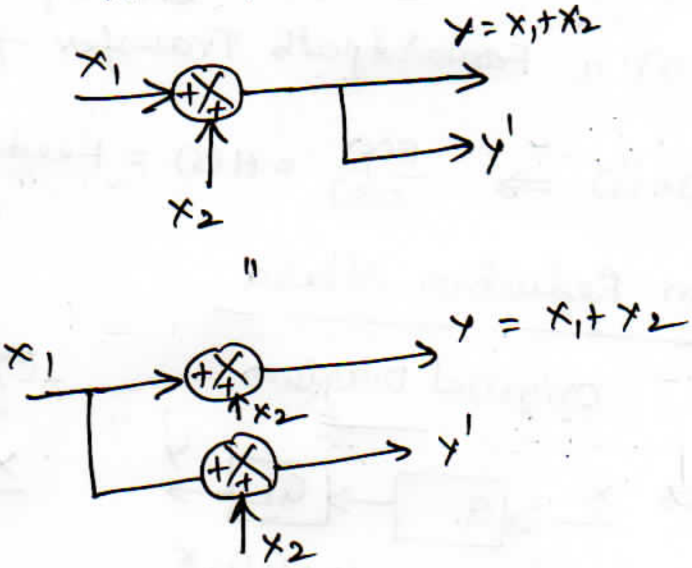
(5) Moving a take-off point after a block



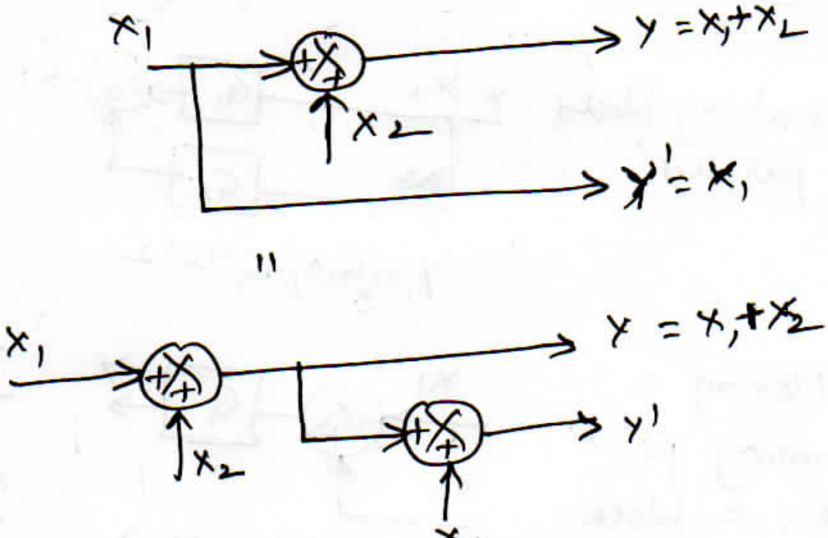
(6) Moving a take-off point ahead of a block



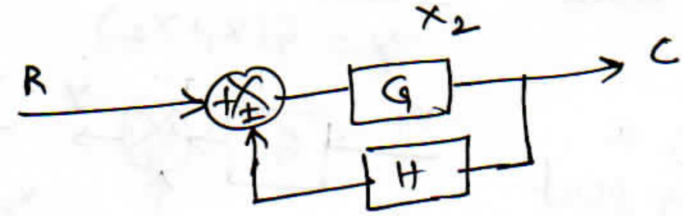
(7) Moving of a take-off point ahead of a summing point



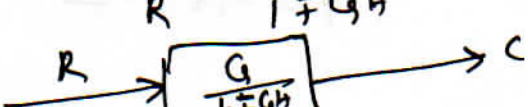
(8) Moving of a take-off point after a summing point



(9) Elimination of feedback path

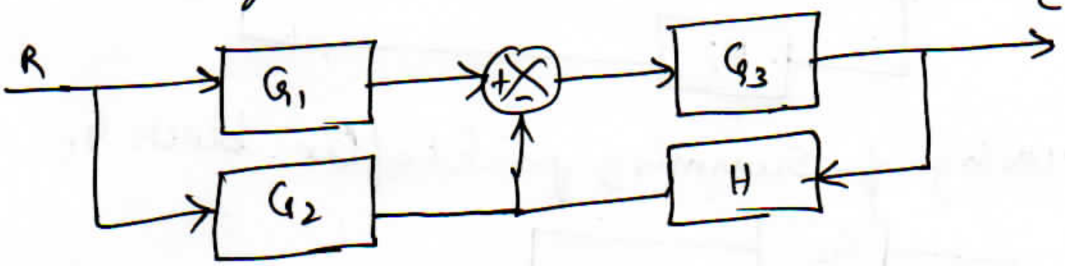


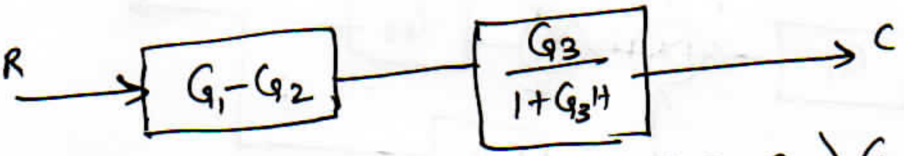
$$\frac{C}{R} = \frac{G}{1 \mp GH}$$



(1) Determine the transfer function of the block diagrams shown in figure (14)

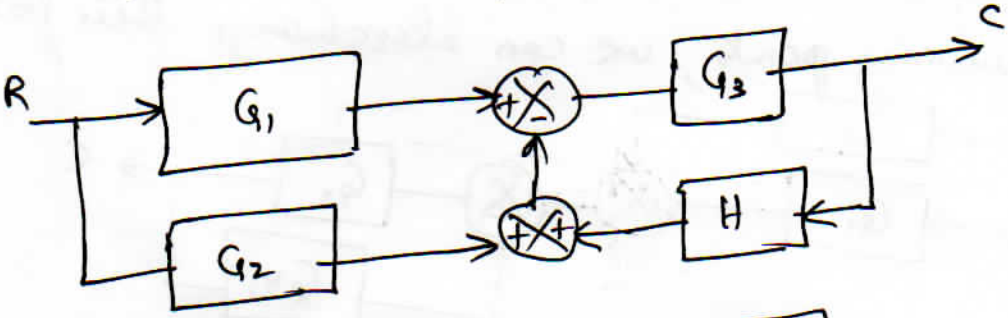
(i)

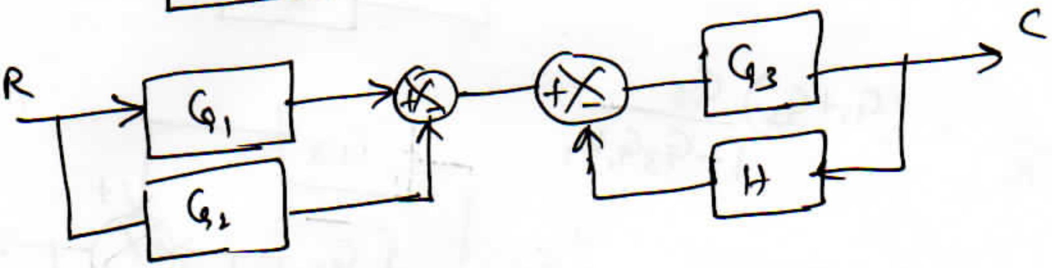




$\therefore$ Transfer function $\frac{C}{R} = \frac{(G_1 - G_2)\,G_3}{1+G_3H}$

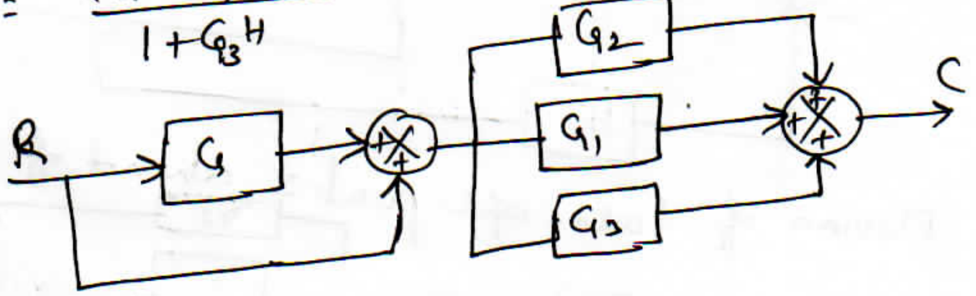
(ii)





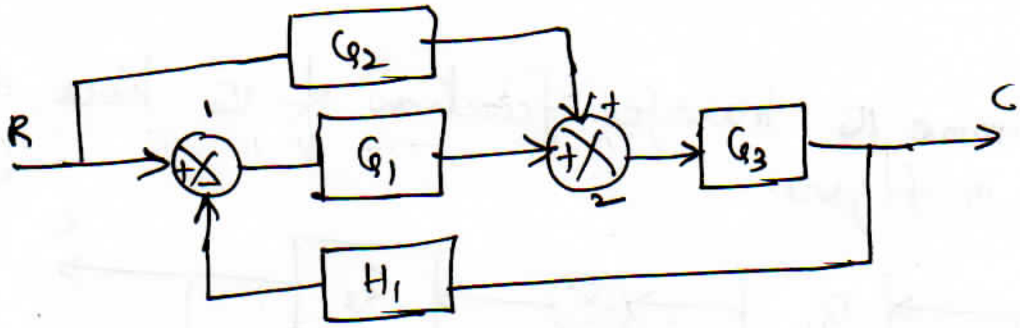
$$\frac{C}{R} = \frac{(G_1 - G_2)\,G_3}{1+G_3H}$$

(iii)

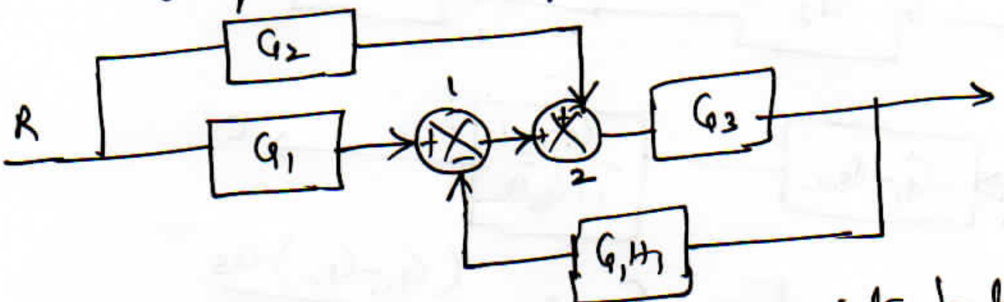


$$\frac{C}{R} = (1+G)(G_1+G_2+G_3)$$

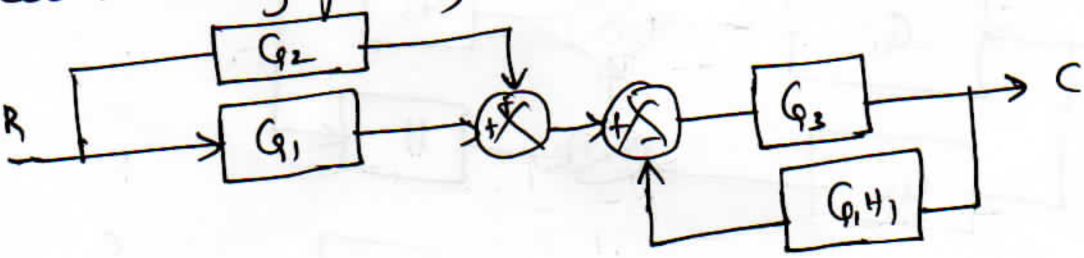
(iv)



(Sol) Moving of summing point 1 after block G_1

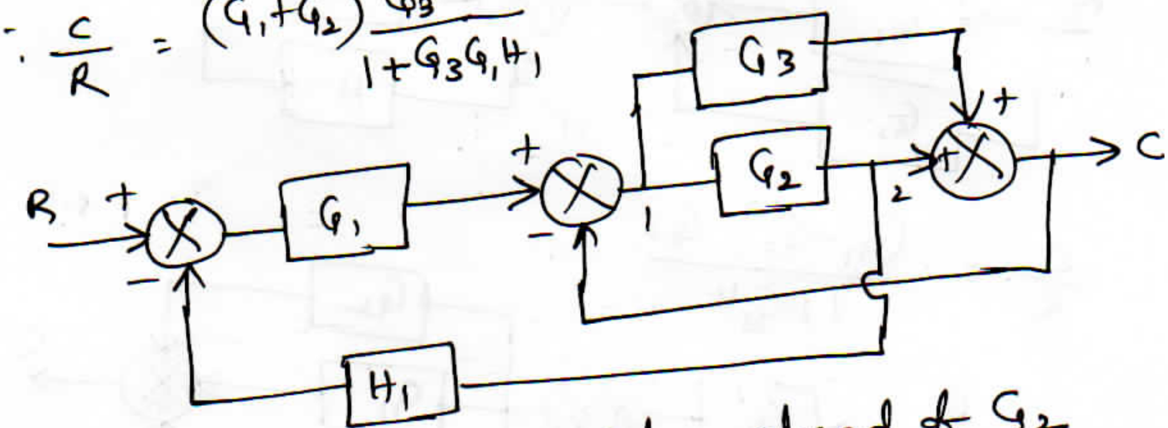


If there are no blocks & take-off points between two summing points, we can interchange their positions

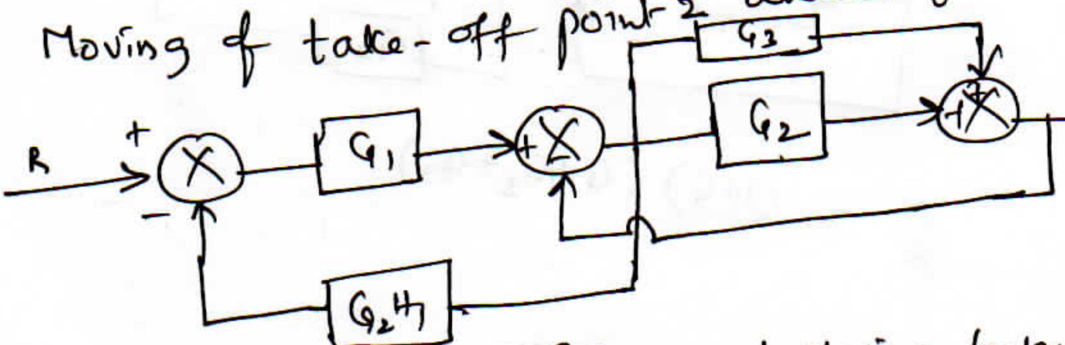


$$\therefore \frac{C}{R} = (G_1+G_2)\frac{G_3}{1+G_3G_1H_1}$$

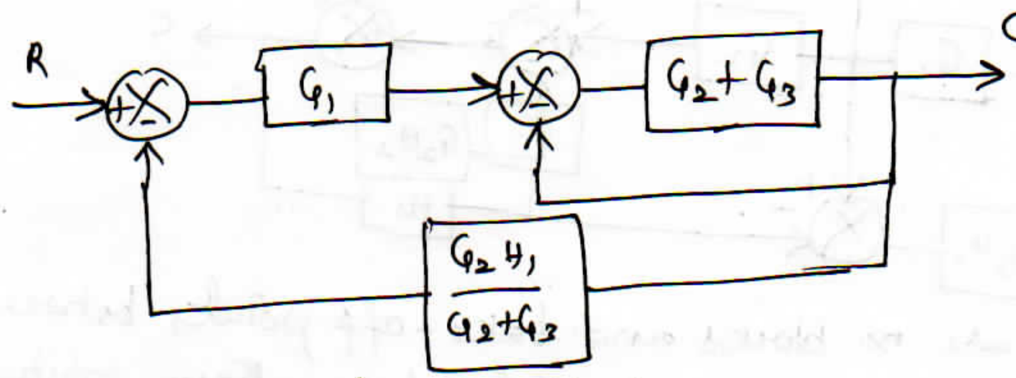
(v)



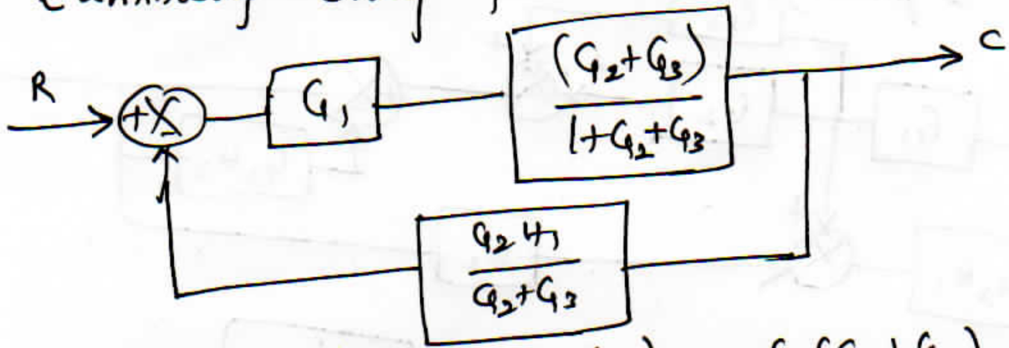
(Sol) Moving of take-off point 2 ahead of G_2



Combining blocks G_2 & G_3 and moving take-off point after the combination.

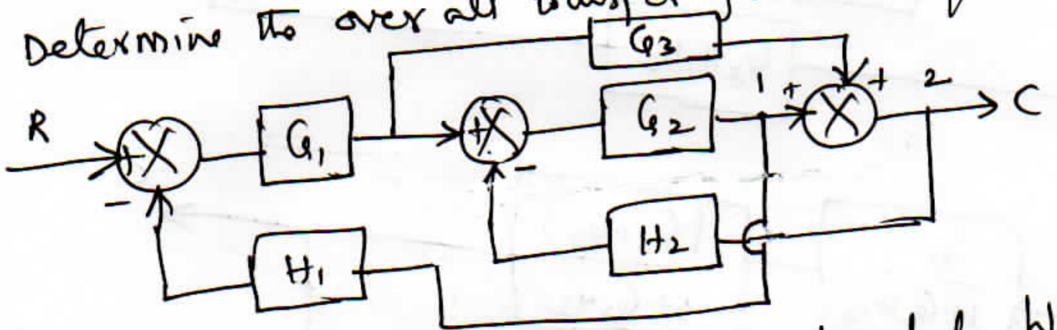


Eliminating unity feedback

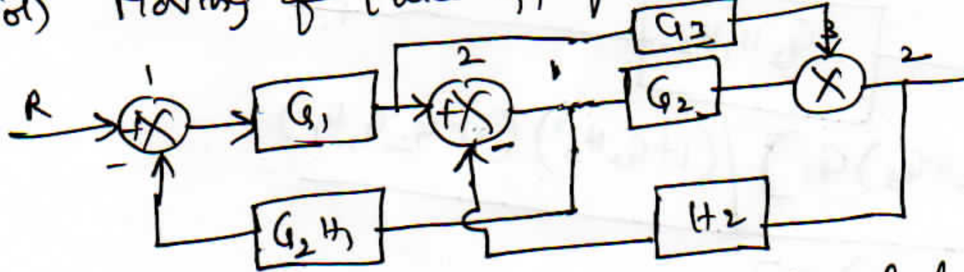


$$\frac{C}{R} = \frac{(G_2+G_3)\,G_1\,/(1+G_2+G_3)}{1+\frac{(G_2+G_3)}{(1+G_2+G_3)}\,G_1 \cdot \frac{G_2 H_1}{(G_2+G_3)}} = \frac{G_1(G_2+G_3)}{1+G_2+G_3+G_1G_2H_1}$$

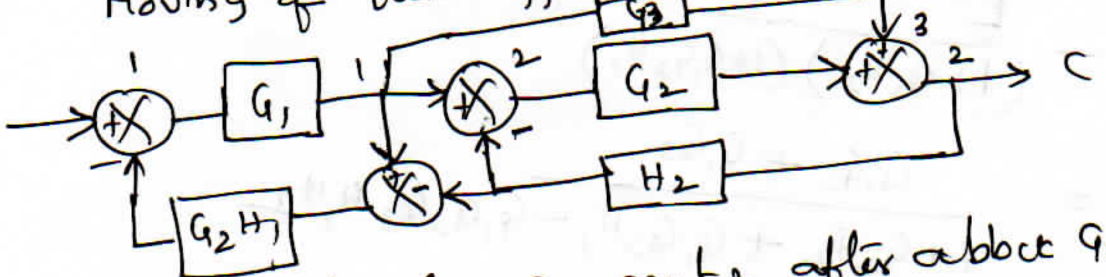
(vi) Determine the over all transfer function of the system



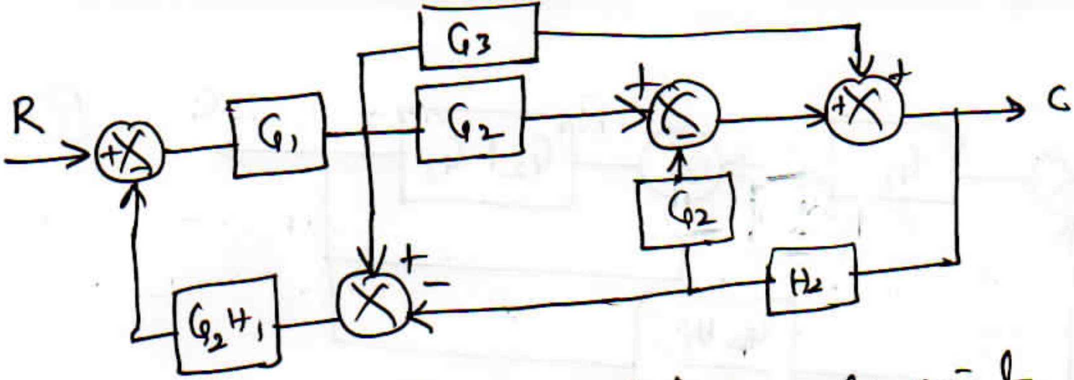
(Sol) Moving of take-off point 1 ahead of block G_2



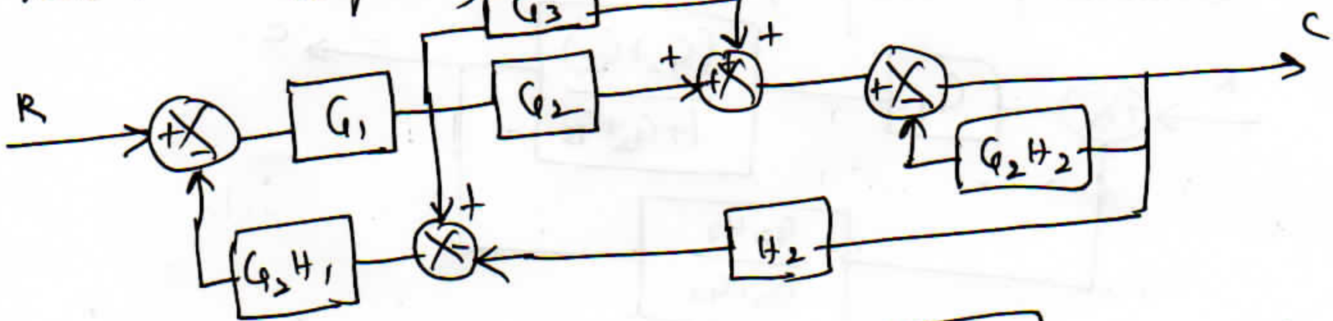
Moving of take-off point 1 ahead of summing point 2

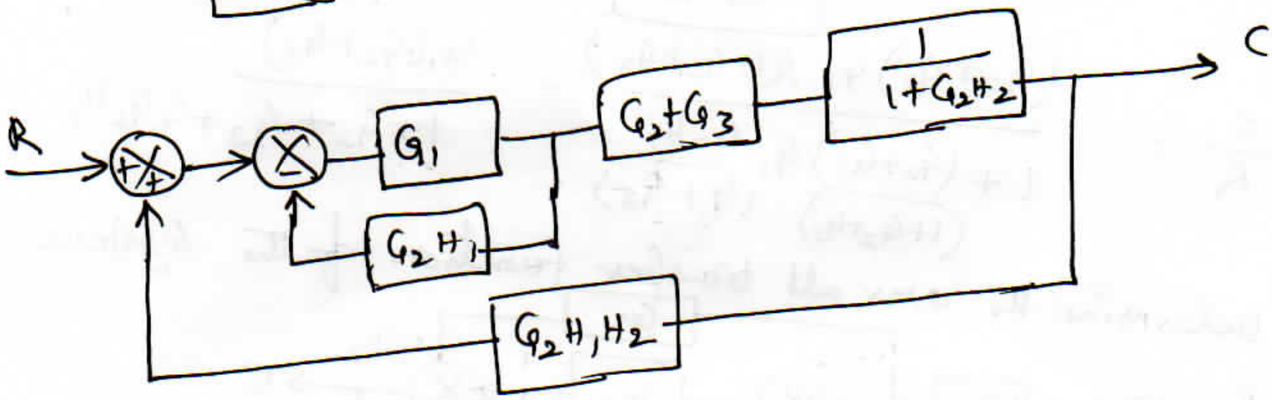


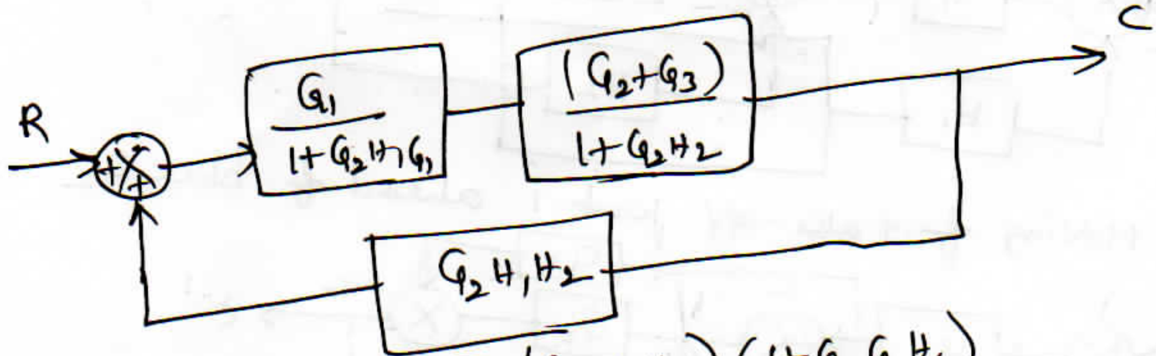
moving of summing point 2 after a block G_2



If there are no blocks and take-off points between two summing points, we can interchange their position.



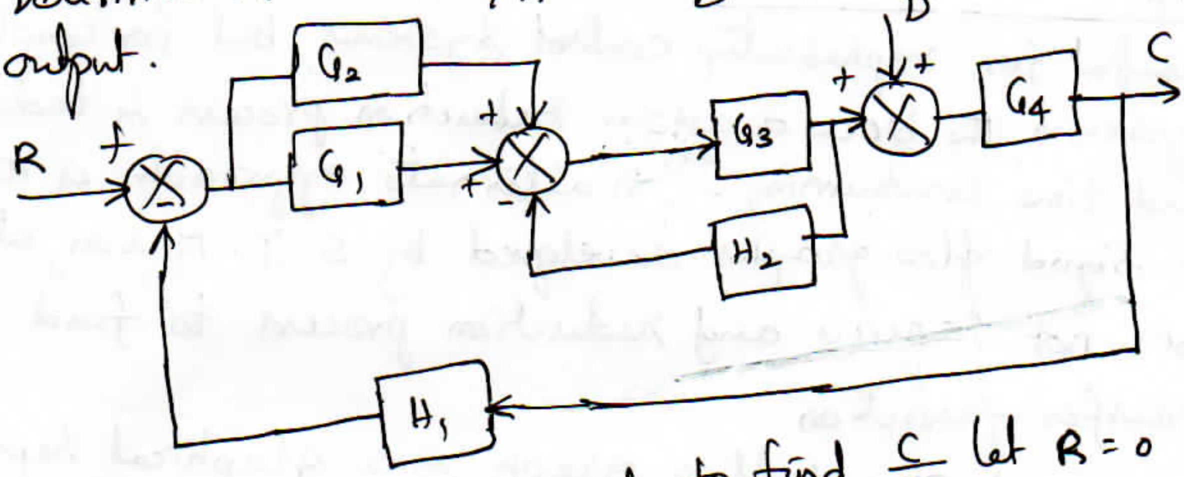




$$\frac{C}{R} = \frac{\left[(G_2+G_3)G_1\right] \Big/ (1+G_2H_2)(1+G_2G_1H_1)}{1 - \dfrac{\left[(G_2+G_3)G_1\right]}{(1+G_2H_2)(1+G_1G_2H_1)} \cdot G_2H_1H_2}$$

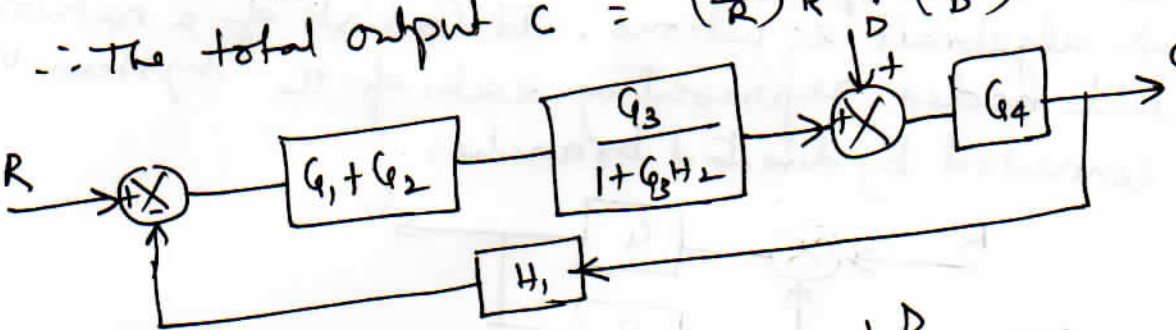
$$= \frac{G_1G_2 + G_1G_3}{1+G_2H_2+G_1G_2H_1 - G_1G_2G_3H_1H_2}$$

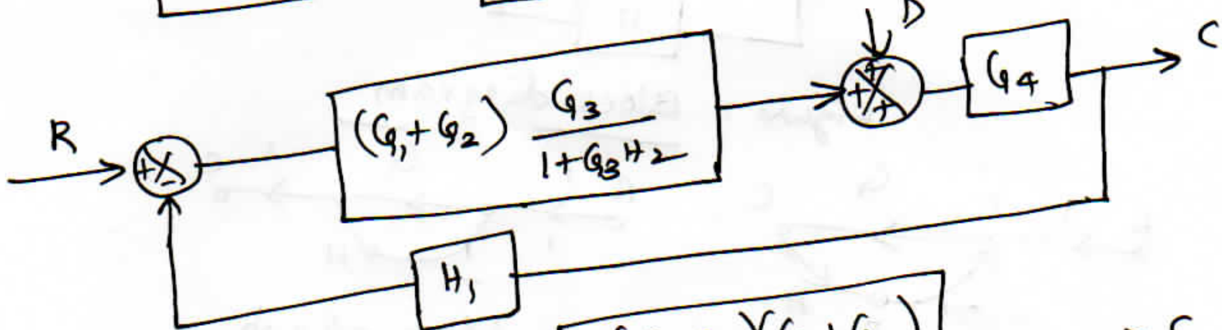
(4) Determine the ratios C/R and $\frac{C}{D}$ also find the total output. (16)



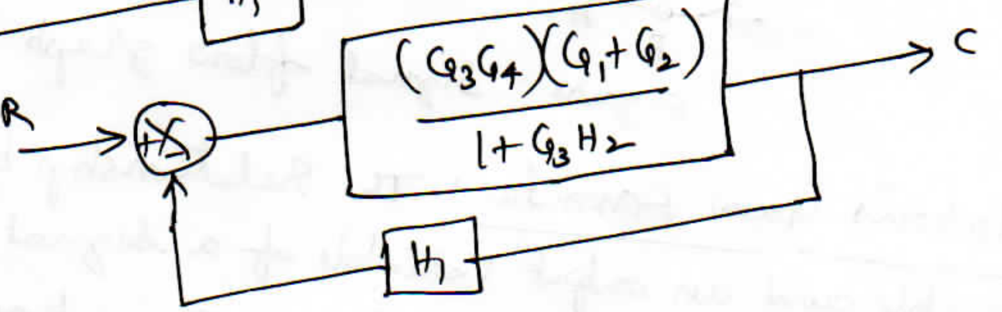
(Sol) To find $\frac{C}{R}$, let $D=0$ and to find $\frac{C}{D}$ let $R=0$

$\therefore$ The total output $C = \left(\frac{C}{R}\right)R + \left(\frac{C}{D}\right)D$



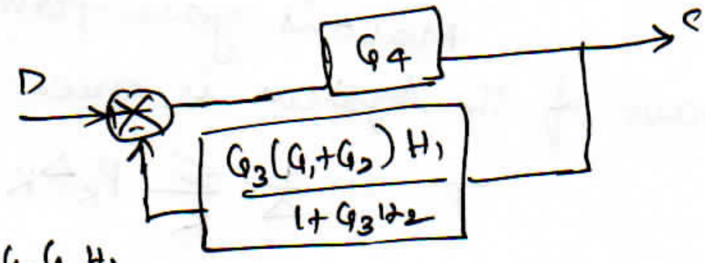


(i) $\frac{C}{R} = ?$



$$\frac{C}{R} = \frac{G_1G_3G_4 + G_2G_3G_4}{1+G_3H_2 + G_1G_3G_4H_1 + G_2G_3G_4H_1}$$

(ii) To find $\frac{C}{D}$ let $R=0$



$$\frac{C}{D} = \frac{G_4(1+G_3H_2)}{1+G_3H_2 + G_1G_3G_4H_1 + G_2G_3G_4H_1};$$

$$\text{Total o/p } C = \frac{(G_1G_3G_4 + G_2G_3G_4)R}{(\,''\,)} + \frac{G_4(1+G_3H_2)D}{(\,''\,)}$$

Signal Flow Graphs: Block diagrams are very useful for representing control systems, but for complicated systems, the block diagram reduction process is tedious and time consuming. An alternate approach is that of signal flow graphs developed by S. J. Mason, which does not require any reduction process to find the transfer function.

A signal flow graph is a graphical representation of the relationships between the variables of a set of linear algebraic equations. It consists of a network in which nodes representing each of the system variables are connected by directed branches.

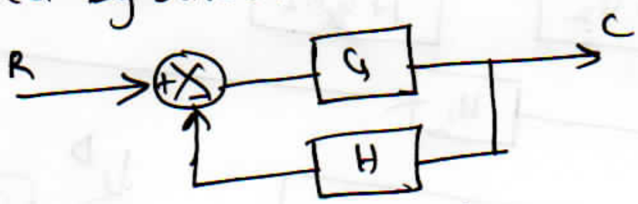


Figure: Block diagram

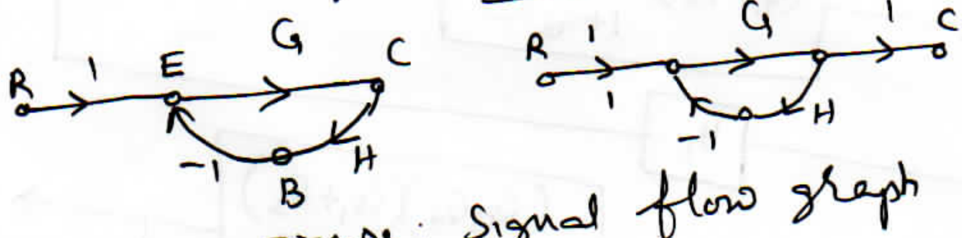


Figure: Signal flow graph

Mason's Gain Formula: The relationship between an input variable and an output variable of a signal flow graph is given by the net gain between the input and output nodes and is known as the overall gain of the system.

Mason's gain formula to determine the overall gain of the system is given by

$$T = \frac{1}{\Delta} \sum_{K} P_K \Delta_K$$

$$T = \frac{1}{\Delta} \sum_K P_K \Delta_K$$

where P_K = path gain of K^{th} forward path

Δ = Determinant of the graph

= 1 − (Sum of loop gains of all individual loops)
+ (Sum of gain products of all possible Combinations of two non-touching loops)
− (Sum of gain products of all possible Combinations of three non-touching loops)
+ - - - -

$$\therefore \Delta = 1 - \sum_m P_{m_1} + \sum_m P_{m_2} - \sum_m P_{m_3} + - - - -$$

where P_{mr} = gain product of m-th possible Combinations of 'r' non-Touching loops

Δ_K = The value of Δ for the part of the graph not-touching the K^{th} forward path.

T = over all gain of the system.

① Draw the signal flow graph and find the over all gain of the system equations given by

$$x_2 = a_{12} x_1 + a_{32} x_3 + a_{42} x_4 + a_{52} x_5$$

$$x_3 = a_{23} x_2$$

$$x_4 = a_{34} x_3 + a_{44} x_4$$

$$x_5 = a_{35} x_3 + a_{45} x_4$$

where x_1 is the input variable and x_5 is the output variable

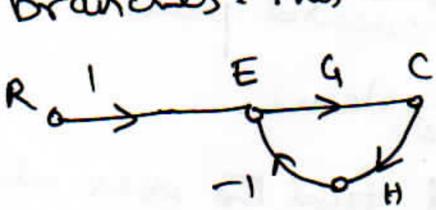
(1) Node: It represents a system variable which is equal to the sum of all incoming signals at the node

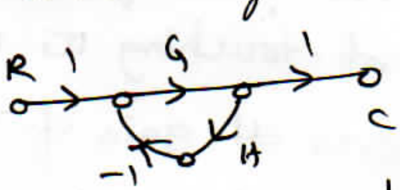
(2) Branch: A signal travels along a branch from one node to another in the direction indicated by the branch arrow and in the process gets multiplied by the gain or transmittance of the branch.

(3) Notation: a_{ij} is the transmittance of the branch directed from node x_i to node x_j.

(4) Input node or Source: It is a node with only outgoing branches.

(5) Output node or Sink: It is a node only with incoming branches. This does not meet always. In that case,





where 'c' is output node

an additional branch with unit gain may be introduced in order to meet the specified condition

(6) path: It is the traversal of connected branches in the direction of the branch arrows such that no node is traversed more than once.

(7) Forward path: It is a path from the input node to the output node.

(8) Loop: Loop is a path which originates and terminates at the same node.

(9) Non-touching loops: Loops are said to be non-touching if they do not possess any common node.

(10) Forward path gain: It is the product of the branch gains encountered in traversing a forward path.

(11) Loop gain: It is the product of branch gains encountered in traversing a loop.

Construction of Signal flow graph:

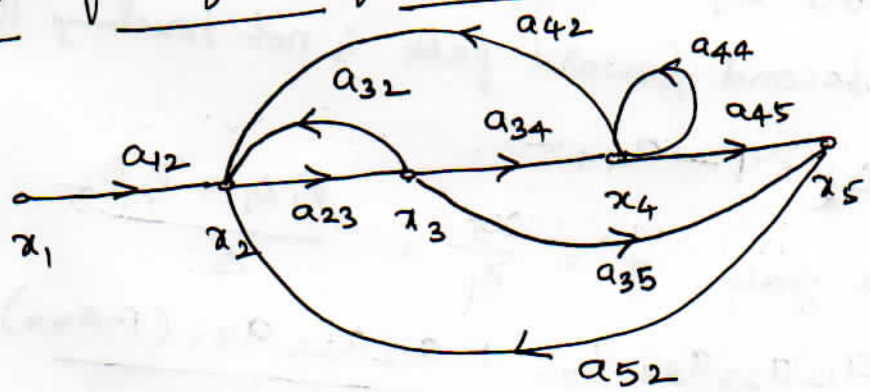


Figure: Signal flow graph

(1) There are two forward paths with path gains

$P_1 = a_{12}a_{23}a_{34}a_{45}$; $P_2 = a_{12}a_{23}a_{35}$

(2) There are five individual loops with loop gains

$P_{11} = a_{23}a_{32}$; $P_{21} = a_{23}a_{34}a_{42}$

$P_{31} = a_{44}$; $P_{41} = a_{23}a_{34}a_{45}a_{52}$

$P_{51} = a_{23}a_{35}a_{52}$

(3) There are two possible combinations of two non-touching loops with loop gain products

$P_{12} = a_{23}a_{32}a_{44}$

$P_{22} = a_{23}a_{35}a_{52}a_{44}$

4) There are no combinations of three-non-touching loops, four non-touching loops etc.

Therefore $P_{m3} = P_{m4} = \cdots = 0$

Hence $\Delta = 1-(a_{23}a_{32} + a_{23}a_{34}a_{42} + a_{44} + a_{23}a_{34}a_{45}a_{52} + a_{23}a_{35}a_{52}) + (a_{23}a_{32}a_{44} + a_{23}a_{35}a_{52}a_{44})$

(5) The first forward path is in touch with all the loops

Therefore $\Delta_1 = 1$

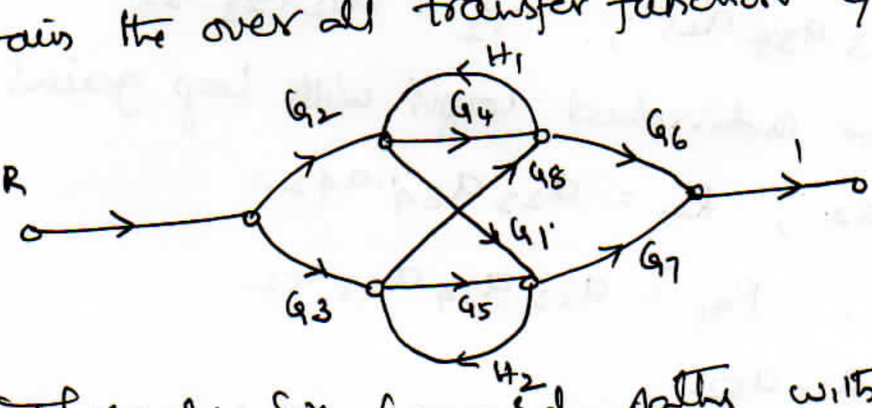
The second forward path is not touching the loop a_{44}

$\therefore \Delta_2 = 1 - a_{44}$

$\therefore$ The gain $T = \dfrac{x_5}{x_1} = \dfrac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$

$$= \frac{a_{12}a_{23}a_{34}a_{45} + a_{12}a_{23}a_{35}(1-a_{44})}{1-a_{23}a_{32}-a_{23}a_{34}a_{42}-a_{44}-a_{23}a_{34}a_{45}a_{52}+ a_{23}a_{32}a_{44}+a_{23}a_{35}a_{52}a_{44}}$$

(2) obtain the over all transfer function C/R.



(Sol) There are six forward paths with gains

$P_1 = G_2G_4G_6$; $P_2 = G_2G_1G_7$; $P_3 = G_2G_1H_2G_8G_6$

$P_4 = G_3G_5G_7$ $P_5 = G_3G_8G_6$; $P_6 = G_3G_8H_1G_1G_7$

(2) There are 3 individual loops.

$P_{11} = G_4 H_1$; $P_{21} = G_5 H_2$

$P_{31} = G_1 H_2 G_8 H_1$

(3) There is only one combination of non-touching loops

$P_{12} = P_{11} P_{21} = G_4 G_5 H_1 H_2$

(4) There are no combinations of three non-touching loops

$\therefore P_{m3} = P_{m4} = 0$

$\therefore \Delta = 1 - P_{m1} + P_{m2} - P_{m3} + \cdots$

$= 1 - (G_4 H_1 + G_5 H_2 + G_1 H_2 G_8 H_1) + G_4 G_5 H_1 H_2$

(5) Forward path P_1 is not touching $G_5 H_2$

$\therefore \Delta_1 = 1 - G_5 H_2$

Forward path P_4 is not touching the loop $G_4 H_1$

$\therefore \Delta_4 = 1 - G_4 H_1$

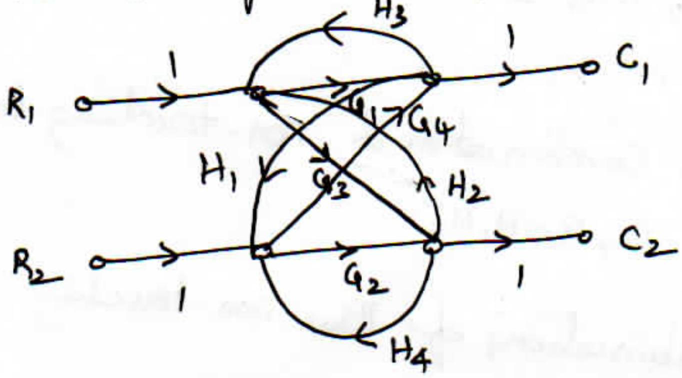
Remaining forward paths touching all the loops

$\therefore \Delta_2 = \Delta_3 = \Delta_5 = \Delta_6 = 1$

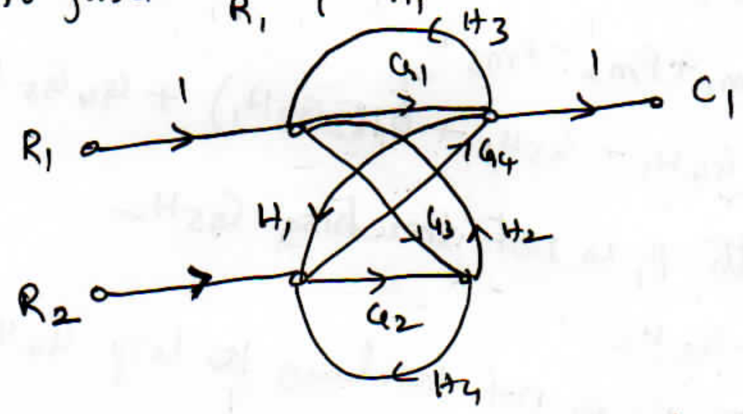
$\therefore$ The transfer function $\frac{C}{R} = \frac{P_1 \Delta_1 + P_2 \Delta_2 + P_3 \Delta_3 + P_4 \Delta_4 + P_5 \Delta_5 + P_6 \Delta_6}{1 - \sum_m P_{m1} + \sum_m P_{m2}}$

$$\frac{C}{R} = \frac{G_2 G_4 G_6 (1 - G_5 H_2) + G_2 G_1 G_7 + G_2 G_1 H_2 G_8 G_6 + G_3 G_5 G_7 (1 - G_4 H_1) + G_3 G_8 G_6 + G_3 G_8 H_1 G_1 G_7}{1 - (G_4 H_1 + G_5 H_2 + G_1 H_2 G_8 H_1) + G_4 G_5 H_1 H_2}$$

(2) Find the expressions for the outputs C_1 and C_2.



(Sol) To find $\frac{C_1}{R_1}$ & $\frac{C_2}{R_1}$ assume $R_2 = 0$



$\frac{C_1}{R_1} = ?$ (1) $P_1 = G_1$; $P_2 = G_3 H_4 G_4$

(2) $\sum P_{m1} = ?$ $P_{11} = G_2 H_4$; $P_{21} = G_1 H_3$

$P_{31} = G_3 H_2$ $P_{41} = G_4 H_1$

$P_{51} = G_1 H_1 G_2 H_2$

$P_{61} = G_3 H_4 G_4 H_3$

(3) There is only one Combination of non-touching Loops. $\sum P_{m2} = ?$

$P_{12} = P_{11} P_{21} = G_1 G_2 H_3 H_4$

$\sum_m P_m = \sum_m P_{m4} = 0$

$\therefore \Delta = 1 - \sum_m P_{m1} + \sum_m P_{m2}$

$= 1 - (G_2 H_4 + G_1 H_3 + G_3 H_2 + G_4 H_1 + G_1 H_1 G_2 H_2 + G_3 H_4 G_4 H_3) +$

$G_1 G_2 H_3 H_4$

(4) $\Delta_1 = 1 - G_2H_4$; $\Delta_2 = 1$

$$\therefore \frac{C_1}{R_1} = \frac{G_1(1-G_2H_4) + G_3G_4H_4}{\Delta} \longrightarrow (1)$$

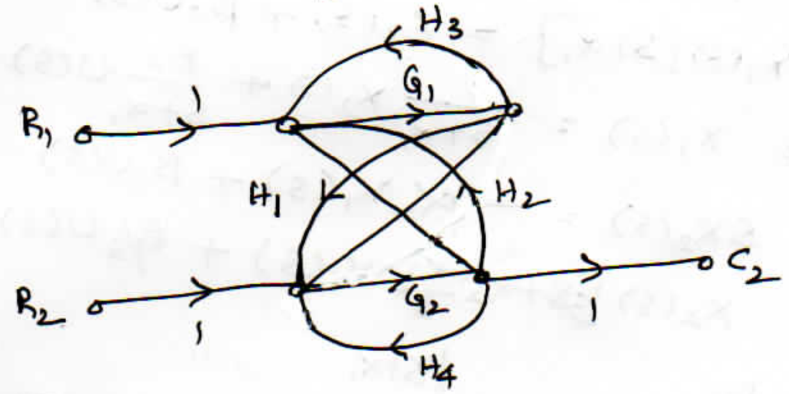
$\frac{C_1}{R_2} = ?$ $\quad P_1 = G_4$; $\quad P_2 = H_2G_1G_2$

$\Delta_1 = 1 - G_3H_2$; $\quad \Delta_2 = 1$

$$\therefore \frac{C_1}{R_2} = \frac{G_4 + G_1H_2}{\Delta}$$

$$\therefore \text{output } C_1 = \frac{[G_1(1-G_2H_4) + G_3G_4H_4]R_1 + [G_4(1-G_3H_2) + G_1G_2H_2]R_2}{\Delta}$$

To find C_2:



$\frac{C_2}{R_1} = ?$ $\quad P_1 = G_3$; $\quad P_2 = G_1H_1G_2$

$\Delta_1 = 1 - G_4H_1$; $\quad \Delta_2 = 1$

$$\therefore \frac{C_2}{R_1} = \frac{G_3(1-G_4H_1) + G_1G_2H_1}{\Delta}$$

$\frac{C_2}{R_2} = ?$ $\quad P_1 = G_2$; $\quad P_2 = G_4H_3G_3$

$\Delta_1 = 1 - G_1H_3 \quad \Delta_2 = 1$

$$\frac{C_2}{R_2} = \frac{G_2(1-G_1H_3) + G_4G_3H_3}{\Delta}$$

$$\therefore \text{output } C_2 = \frac{[G_3(1-G_4H_1) + G_1G_2H_1]R_1 + [G_2(1-G_1H_3) + G_4G_3H_3]R_2}{\Delta}$$

C_1 is independent of R_2 if $G_4(1-G_3H_2) + G_1G_2H_2 = 0$

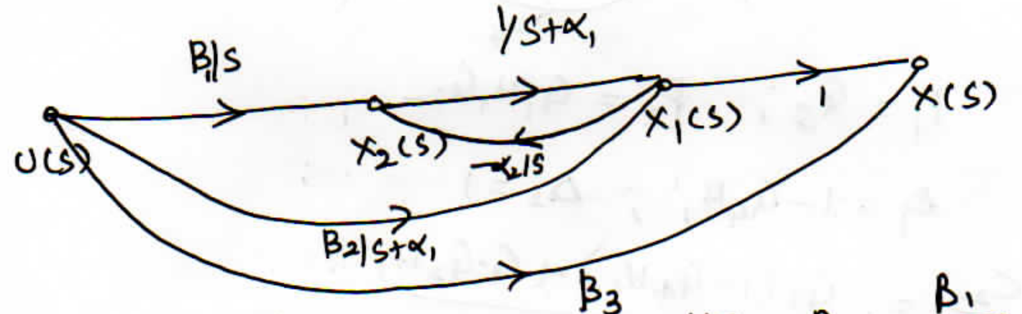
C_2 is independent of R_1 if $G_3(1-G_4H_1) + G_1G_2H_1 = 0$

(4) For the system represented by the following equations, find the transfer function $X(s)/U(s)$ by using signal flow graph technique.

$$x = x_1 + \beta_3 u$$
$$\dot{x}_1 = -\alpha_1 x_1 + x_2 + \beta_2 u$$
$$\dot{x}_2 = -\alpha_2 x_1 + \beta_1 u$$

Sol) Taking LT of the equations

$$X(s) = X_1(s) + \beta_3 U(s) \longrightarrow ①$$
$$sX_1(s) = -\alpha_1 X_1(s) + X_2(s) + \beta_2 U(s)$$
$$X_1(s)[s+\alpha_1] = X_2(s) + \beta_2 U(s)$$
$$\text{or } X_1(s) = \frac{1}{s+\alpha_1} X_2(s) + \frac{\beta_2}{s+\alpha_1} U(s) \longrightarrow ②$$
$$sX_2(s) = -\alpha_2 X_1(s) + \beta_1 U(s)$$
$$X_2(s) = -\frac{\alpha_2}{s} X_1(s) + \beta_1/s\, U(s) \longrightarrow ③$$



(1) Number of forward paths

$$P_1 = \frac{\beta_1}{s(s+\alpha_1)} \qquad \Delta_1 = 1$$
$$P_2 = \frac{\beta_2}{(s+\alpha_1)} \qquad \Delta_2 = 1$$
$$P_3 = \beta_3 \qquad \Delta_3 = 1$$

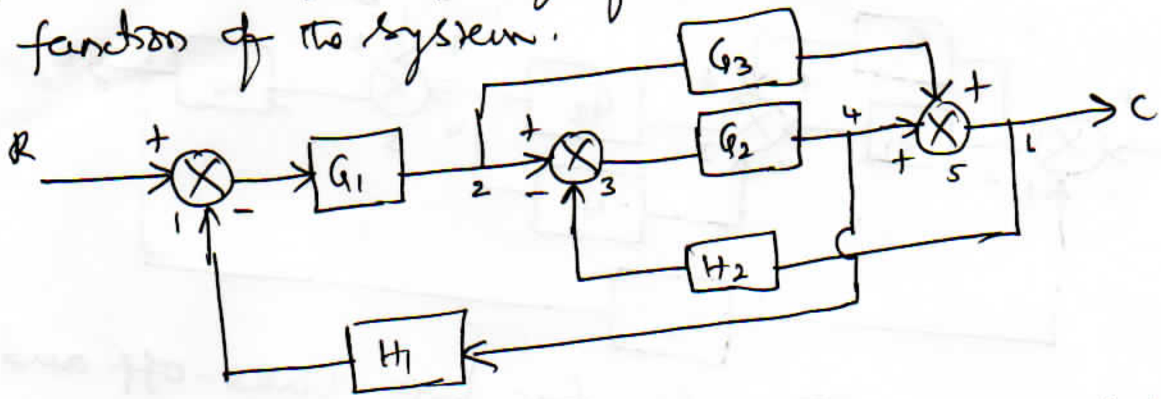
$$\sum_m P_{m1} = ?$$
$$P_{11} = -\frac{\alpha_2}{s(s+\alpha_1)}$$

$$\therefore \text{Transfer function } \frac{X(s)}{U(s)} = \frac{\frac{\beta_1}{s(s+\alpha_1)} + \frac{\beta_2}{(s+\alpha_1)} + \beta_3}{1 - (-\alpha_2/s(\alpha_1+s))}$$

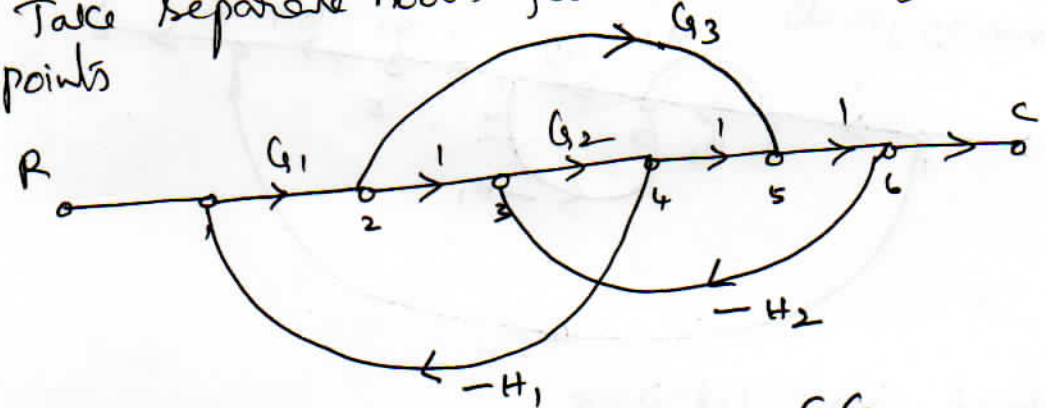
$$= \frac{\beta_3 s(s+\alpha_1) + \beta_2 s + \beta_1}{s(s+\alpha_1) + \alpha_2}$$

$$= \frac{(s^2+\alpha_1 s)\beta_3 + \beta_2 s + \beta_1}{s^2 + \alpha_1 s + \alpha_2}$$

① Draw the signal flow graph and determine the transfer function of the system.



(Sol) Take separate nodes for both summing and take-off points



(i) Number of forward paths $P_1 = G_1G_2$

$P_2 = G_1G_3$

(ii) Number of individual loops $\sum_m P_{m_1} = ?$

$P_{11} = G_2(-H_2) = -G_2H_2$; $P_{21} = G_1G_2(-H_1)$

$P_{31} = G_1G_3(-H_2)G_2(-H_1) = G_1G_2G_3H_1H_2$

Every forward path touching all the loops, hence

$\Delta_1 = \Delta_2 = 1$

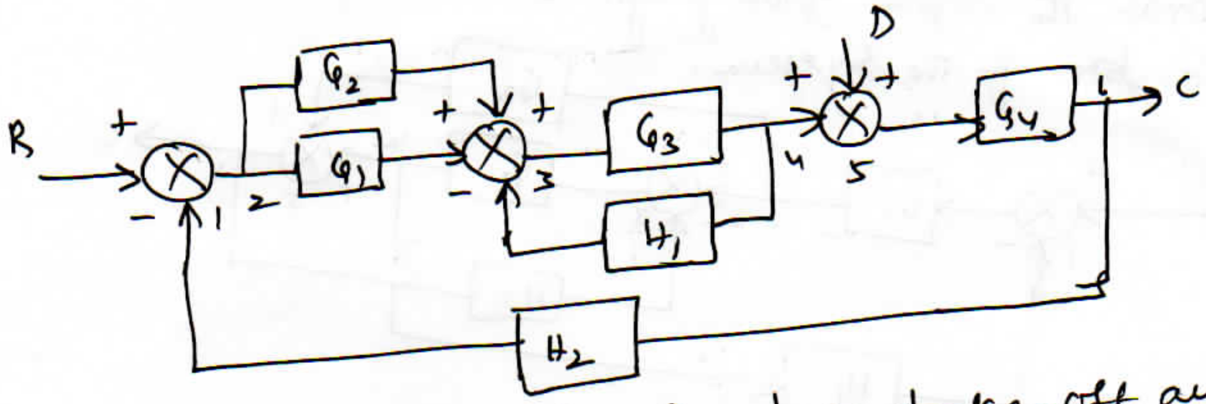
There are no combinations of non-touching loops

hence $\sum_m P_{m_2} = \sum_m P_{m_3} = 0$

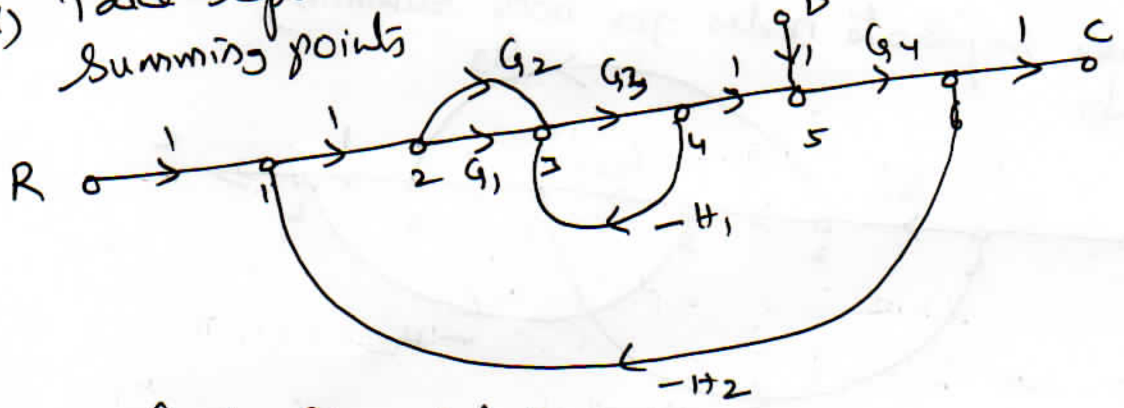
$$\therefore \frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_2 + G_1G_3}{1 - \{-G_2H_2 - G_1G_2H_1 + G_1G_2G_3H_1H_2\}}$$

$$= \frac{G_1G_2 + G_1G_3}{1 + G_2H_2 + G_1G_2H_1 - G_1G_2G_3H_1H_2}$$

② Using Mason's gain formula determine $\frac{C}{R}$



(Sol) Take separate nodes for both take-off and summing points



To find C/R; let $D = 0$

$\therefore\ P_1 = G_1 G_3 G_4 \qquad \Delta_1 = 1$

$P_2 = G_2 G_3 G_4 \qquad \Delta_2 = 1$

$P_{11} = -G_3 H_1 \qquad P_{21} = -G_1 G_3 G_4 H_2$

$P_{31} = -G_2 G_3 G_4 H_2$

and $\sum_m P_{m2} = \sum_m P_{m3} = 0$

$\therefore\ \Delta = 1 - \sum_m P_{m1}$

$$\therefore\ \frac{C}{R} = \frac{P_1 \Delta_1 + P_2 \Delta_2}{1 - \sum_m P_{m1}}$$

$$= \frac{G_1 G_3 G_4 (1) + G_2 G_3 G_4 (1)}{1 + G_3 H_1 + G_1 G_3 G_4 H_2 + G_2 G_3 G_4 H_2}$$

Transfer function of DC Servo Motor:

There are two types of DC Motors namely
(1) Field Controlled DC Motor (2) Armature controlled DC Motor

Field Controlled DC Servo Motor:

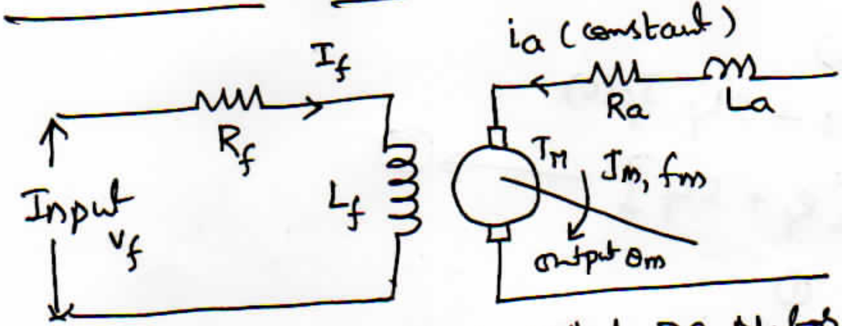


Figure: Field Controlled DC Motor

The input voltage V_f is applied to the field winding which has a resistance R_f and inductance L_f. The armature current i_a supplied to the armature is kept constant and thus the motor shaft is controlled by the input voltage V_f. The field current i_f produces a flux in the machine which in turn produces a torque at the motor shaft. The moment of inertia and the coefficient of viscous friction at the motor shaft are J_m and f_m respectively. The angular shift in the motor shaft is θ_m and the corresponding angular velocity being ω_m.

Since the armature current i_a kept constant, the relationship between the developed motor torque T_M and the field current i_f is given by

$$T_M \propto i_f \quad \text{or}$$

$$T_M = K_f \, i_f \longrightarrow (1)$$

where K_f is motor torque constant in Nm/A.

The relation V_f and i_f is given by

$$V_f = R_f i_f + L_f \frac{di_f}{dt} \longrightarrow ②$$

The relation between T_M, J_m and f_m is given by

$$T_M = J_m \frac{d^2\theta_m}{dt^2} + f_m \frac{d\theta_m}{dt} \longrightarrow ③$$

Taking LT of eq ②

$$V_f(s) = R_f I_f(s) + sL_f I_f(s)$$

$$= I_f(s)[R_f + sL_f] \longrightarrow \text{I}$$

Taking LT of eq ①

$$T_M(s) = K_f I_f(s) \longrightarrow \text{II}$$

Taking LT of eq ③

$$T_M(s) = [J_m s^2 + f_m s]\,\theta_m(s) \longrightarrow \text{III}$$

The relation between. V_f, I_f and T_M and θ_m is shown in figure

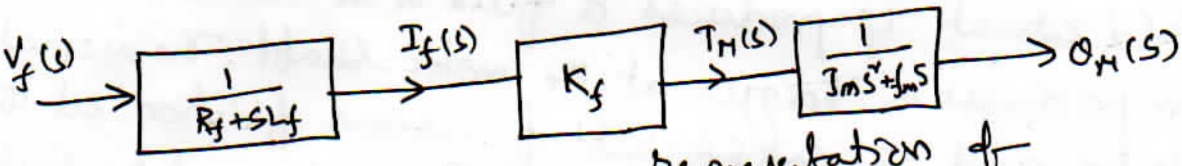


Figure : Block Diagram representation of field Controlled DC Motor.

The transfer function relating the input and output is given by

$$\frac{\theta_M(s)}{V_f(s)} = \frac{K_f}{(R_f + sL_f)(J_m s^2 + f_m s)}$$

The relation between angular velocity ω_m and angular displacement θ_m is given by

$$\omega_m = \frac{d}{dt}\theta_m \quad \text{Taking LT} \quad \omega_m(s) = s\,\theta_m(s)$$

$$\therefore \frac{\omega_m(s)/s}{V_f(s)} = \frac{K_f}{(R_f + sL_f)(J_m s^2 + f_m s)}$$

$$\text{or} \quad \frac{\omega_m(s)}{V_f(s)} = \frac{K_f}{(R_f + sL_f)(J_m s + f_m)}$$

(2) Armature Controlled DC Motor: The relation between applied armature voltage Va and motor shaft displacement θ_m can be derived as follows.

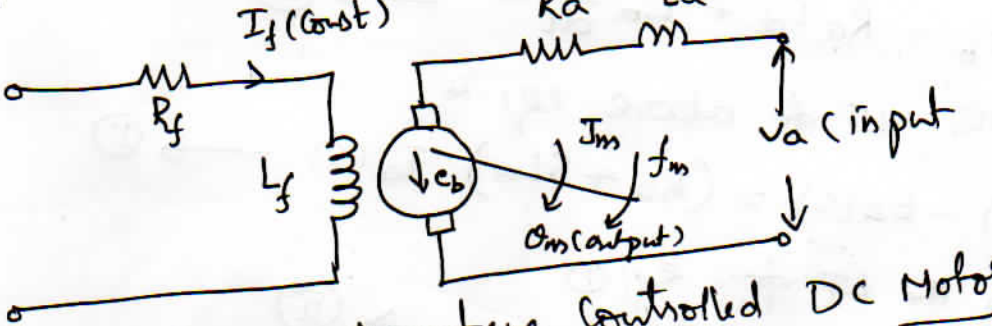


Figure: Armature Controlled DC Motor

The input voltage Va is applied to the armature which has a resistance of Ra and inductance of La. The field current supplied to the field winding is kept constant and thus the armature input voltage Va controls the motor shaft output θ_m. The moment of inertia and the coefficient of viscous friction at the motor shaft being J_m and f_m respectively. The angular shift in the motor shaft being θ_m and the motor shaft velocity is being ω_m.

As the field current I_f is kept constant, the relation between the torque developed T_M and i_a is

$$T_M \propto i_a$$

$$\text{or } T_M = K_T i_a \longrightarrow ①$$

where K_T is motor torque constant K_T in Nm/A

The applied input voltage v_a is being opposed by the back emf e_b developed in armature. The relation between e_b and the motor speed ω_m is given by

$$e_b \propto \omega_m, \text{ where } \omega_m = \frac{d}{dt}\theta_m$$

$$\therefore e_b = K_b \frac{d}{dt}\theta_m \longrightarrow ②$$

where K_b is the back emf constant expressed in V/(rad/sec).

The resultant KVL equation of armature circuit is

$$V_a - e_b = R_a i_a + L_a \frac{d}{dt} i_a \longrightarrow ③$$

Taking the LT of above eq is

$$V_a(s) - E_b(s) = (R_a + sL_a)\, I_a(s) \longrightarrow \text{I}$$

Taking the LT of eq ①

$$T_M(s) = K_T I_a(s) \longrightarrow \text{II}$$

Taking the LT of eq ②

$$T_M(s) = (J_m s^2 + f_m s)\,\theta_m(s) \longrightarrow \text{III}$$

The relation between all the above eqs is as follows

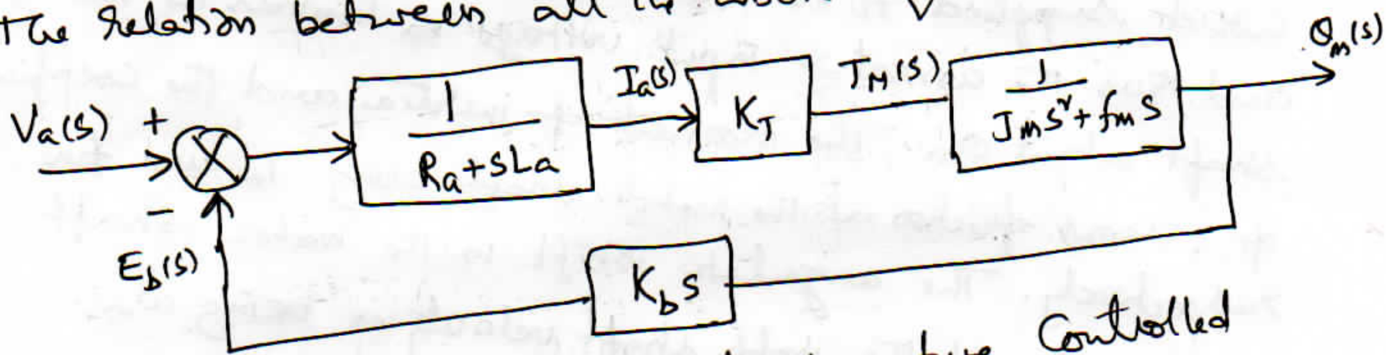
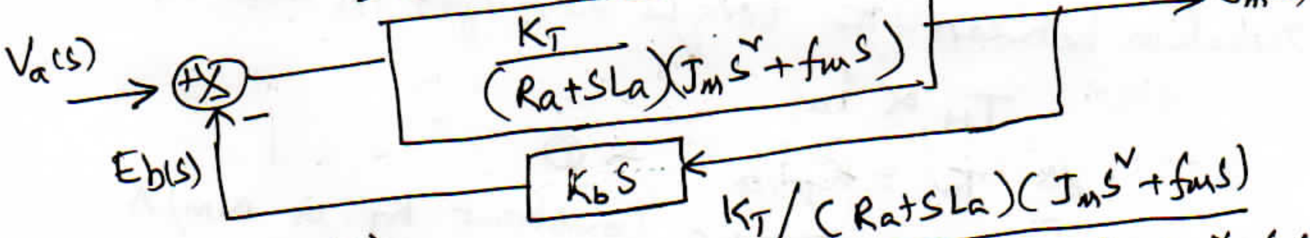


Figure: Block Diagram of Armature Controlled DC Motor.



$$\therefore \frac{\theta_m(s)}{V_a(s)} = \frac{K_T/(R_a+sL_a)(J_m s^2+f_m s)}{1+\left[K_T/(sL_a+R_a)(J_m s^2+f_m s)\right] sK_b}$$

$$= \frac{K_T}{s(R_a+sL_a)(J_m s+f_m) + s K_T K_b}$$

If the armature inductance L_a is neglected

$$\frac{\theta_m(s)}{V_a(s)} = \frac{K_T}{sR_a(J_m s+f_m) + sK_T K_b}$$

$$\frac{\theta_m(s)}{V_a(s)} = \frac{K_T}{s(sR_aJ_m + R_af_m + K_TK_b)}$$

$$= \frac{K_T/(R_af_m + K_TK_b)}{s\left[\frac{sR_aJ_m}{R_af_m + K_TK_b} + 1\right]}$$

$$\text{or } \frac{\theta_m(s)}{V_a(s)} = \frac{K_m}{s(1+sT_M)} \longrightarrow (i)$$

where $K_m = \frac{K_T}{R_af_m + K_TK_b}$ is motor gain Constant

$T_m = \frac{R_aJ_m}{(R_af_m + K_TK_b)}$ is motor time Constant

Also $\omega_m = \frac{d}{dt}\theta_m$ and $\omega_m(s) = s\theta_m(s)$

$$\therefore \frac{\omega_m(s)/s}{V_a(s)} = \frac{K_m}{s(1+sT_M)}$$

$$\text{or } \frac{\omega_m(s)}{V_a(s)} = \frac{K_m}{(1+sT_m)}$$

The relation between torque Constant K_T and back emf Constant K_b:- The mechanical power output of the motor is $T_M\omega_m$, which is equal to armature input e_bi_a

Therefore $T_M\omega_M = e_bi_a$

where $K_M = K_Ti_a$ and $e_b = K_b\omega_m$

$$\therefore K_Ti_a\omega_m = K_b\omega_mi_a$$

Hence $\boxed{K_T = K_b}$

Transfer function of AC Servo Motor :

The transfer function of AC Servo Motor relates the angular shift θ_m in the shaft to the input control $V_c(t)$.

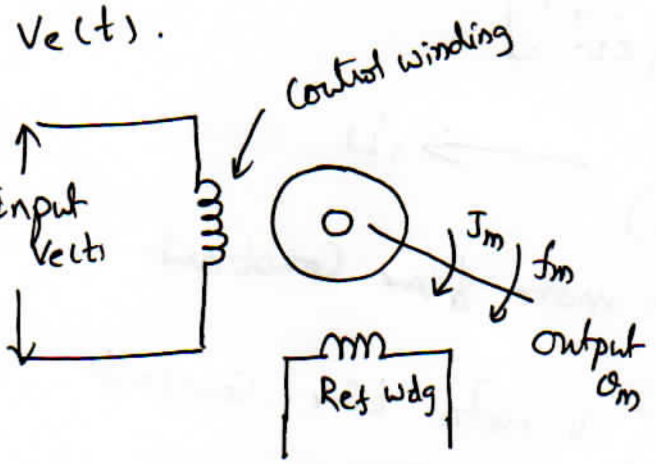


Figure (1) : Two-phase AC Servo Motor.

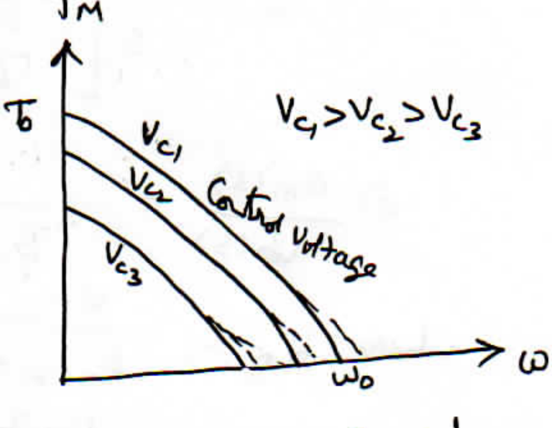


Figure (2) : Torque Speed characteristics of a two phase AC Servo Motor

Two-phase ac servo motor is a two phase induction motor having drag cup type rotor construction. The control voltage $V_c(t)$ is applied to the control winding and a fixed voltage having a phase difference of 90° w.r.t control voltage is applied to the reference winding. winding voltage is applied to the reference winding. The control voltage results in the development of the motor torque T_M. The torque-speed characteristics of motor are shown in figure (2).

The moment of inertia and the viscous friction coefficient of motor are given by J_m and f_m respectively. The angular shift of motor shaft and velocity are given by θ_m and ω_m respectively.

From the Torque-Speed characteristics, the dynamic relation between the motor torque and the speed is given by

$$T_M = m\omega_m + KV_c \longrightarrow ①$$

where m and K can be derived as follows

(i) when the speed $\omega_m = 0$, the torque is T_0 (stalling torque) and this stalling torque is proportional to the control voltage V_c.

$$\therefore T_0 = KV_c \quad \text{or} \quad K = \frac{T_0}{V_c} \quad \text{in Nm/V}$$

(ii) The slope of the torque-speed characteristics is

$$m = -\frac{T_0}{\omega_0} \quad \text{in Nm/rad/sec}$$

also $\omega_m = \frac{d\theta_m}{dt}$;

Now equation ① can be expressed as

$$T_M = m\frac{d\theta_m}{dt} + KV_c \longrightarrow ②$$

$$\text{Also } T_M = J_m\frac{d^2\theta_m}{dt^2} + f_m\frac{d\theta_m}{dt} \longrightarrow ③$$

Taking LT of eq ② $T_M(s) = ms\,\theta_m(s) + KV_c(s) \rightarrow (i)$

Taking LT of eq ③; $T_M(s) = (s^2 J_m + s f_m)\,\theta_m(s) \rightarrow (ii)$

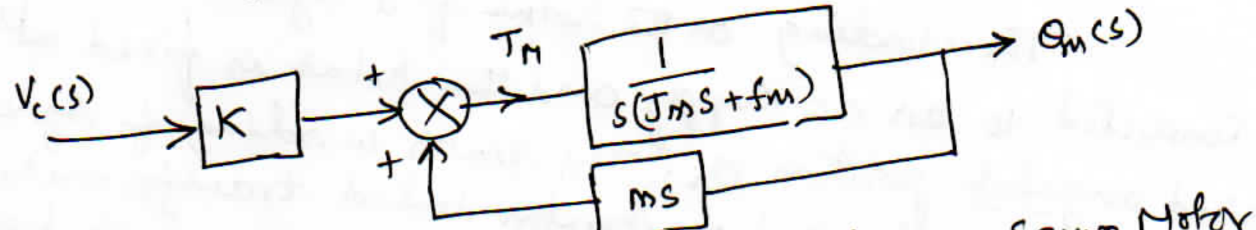
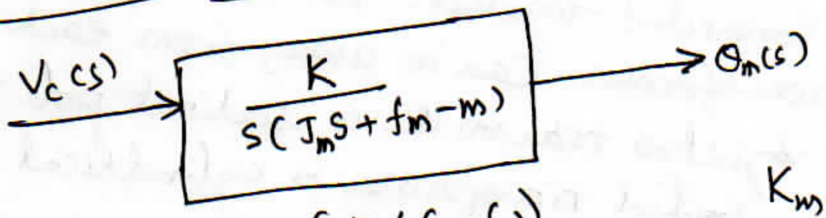


Figure: Block Diagram representation of A.C Servo Motor



$$\therefore TF \quad \frac{\theta_m(s)}{V_c(s)} = \frac{(K/(f_m - f))}{s\left(\frac{J_m s}{f_m - f} + 1\right)} = \frac{K_m}{s(1 + sT_M)}$$

where $K_m = \frac{K}{f_m - f}$ is motor gain constant

$T_M = \frac{J_m}{f_m - f}$ is motor time constant

$$\text{Also} \quad \frac{[\theta_m(s)/s]}{V_c(s)} = \frac{\omega_m(s)}{V_c(s)} = \frac{K_m}{(1 + sT_M)}$$

Synchro Error Detector (Selsyn):

The synchro transmitter and synchro control transformer converts an angular position difference into a proportional ac voltage.

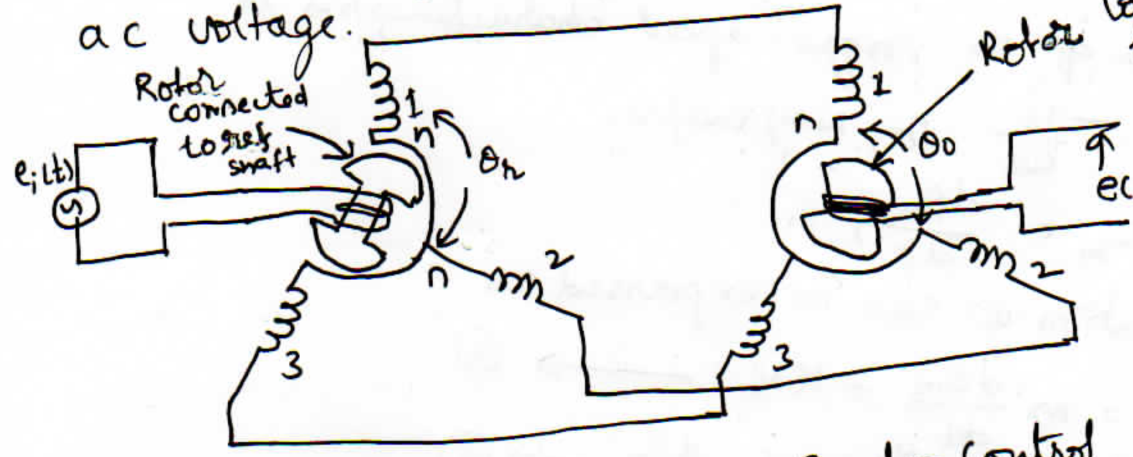


Figure : Synchro Error Detector

The winding on the rotor of a synchro transmitter is connected to an ac supply and this rotar is fixed at a desired angular position θ_r. The stator winding of synchro transmitter and also that of synchro control transformer are wound at 120° in space on the stator. The two stator windings are connected together. The locations of transmitter and control transformer can be away from each other. The rot of synchro transmitter is salient pole type and that of synchro control transformer is cylindrical type.

The rotor of the control transformer is coupled to the output shaft of the control system. If the position of the output shaft is indicated as θ_0, this results in an angular error $\theta_e = (\theta_r - \theta_0)$ between the positions of reference and output shafts.

The process of conversion of the angular difference into a proportional voltage is explained as follows.

If $e_i(t) = E_m \sin(2\pi f t)$ → ① is applied to the rotor winding of the synchro transmitter, then if $\theta_r = 0$, the corresponding voltage induced by transformer section across the stator winding 1n is given by

$$e_{1n} = K E_m \sin(2\pi f t) \longrightarrow ②$$

where K is constant of proportionality

As the stator windings 2n and 3n are 240° and 120° apart in anti-clockwise direction w.r.t the winding 1n, the voltages induced across them are

$$e_{2n} = K E_m \sin(2\pi f t) \cos 240^\circ$$
$$= -0.5 K E_m \sin(2\pi f t) \longrightarrow ③$$

$$e_{3n} = K E_m \sin(2\pi f t) \cos 120^\circ$$
$$= -0.5 K E_m \sin(2\pi f t) \longrightarrow ④$$

Now, if the rotor of the synchro transmitter shifts in anti-clock wise direction through an angle θ, the voltages induced in stator coil are

$$e_{1n} = K E_m \sin 2\pi f t \cos\theta \longrightarrow ⑤$$
$$e_{2n} = K E_m \sin 2\pi f t \cos(240-\theta) \longrightarrow ⑥$$
$$e_{3n} = K E_m \sin 2\pi f t \cos(120-\theta) \longrightarrow ⑦$$

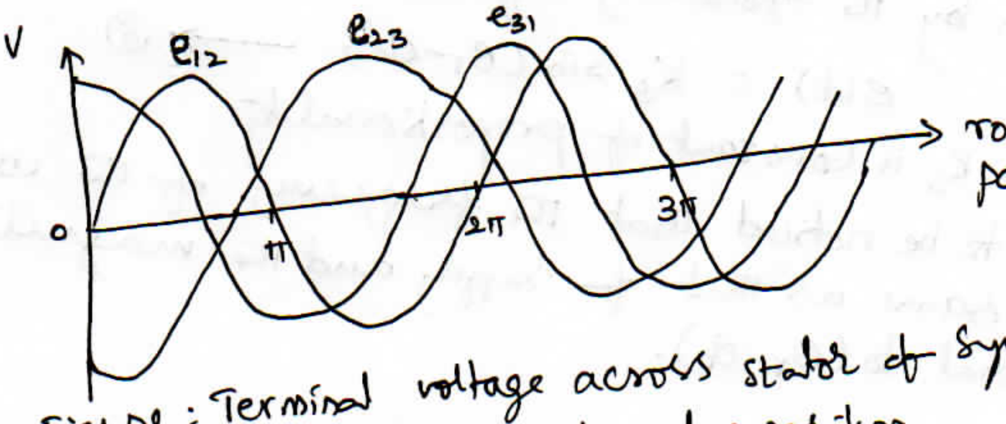


Figure : Terminal voltage across stator of synchro transmitter w.r.t rotor position

The three voltages e_{1n}, e_{2n} and e_{3n} are connected consecutively to three stator windings of the control transformer and produce a resultant flux in the air gap of the same stator windings, which in turn induces a voltage across the rotor winding of the control transformer. The magnitude of this induced voltage depends on the difference $(\theta_r - \theta_0)$. If the difference $(\theta_r - \theta_0)$ is zero, the induced voltage across the rotor winding terminals of the control transformer is zero, maximum for $\theta_r - \theta_0 = 90°$ and again zero when $\theta_r - \theta_0 = 180°$. After 180°, the phase of the induced voltage reverses. The magnitude is again maximum with a reversed phase for $\theta_r - \theta_0 = 270°$ and finally zero for $\theta_r - \theta_0 = 360°$. The variation of the amplitude of induced voltage e(t) across the rotor of the control transformer w.r.t $(\theta_r - \theta_0)$ is shown in figure.

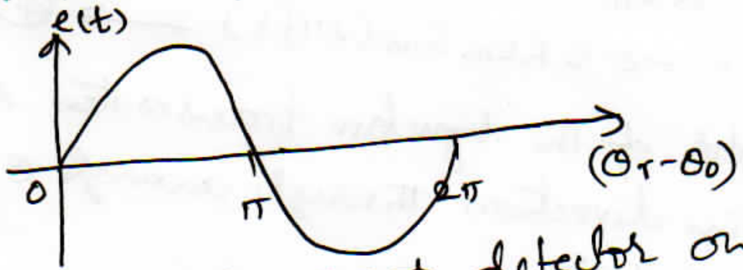


Figure : Synchro error detector output

Therefore, the magnitude of the output induced voltage e(t) developed across the rotor of synchro transformer is given by the following equation

$$e(t) = K_s \sin(\theta_r - \theta_0) \longrightarrow (8)$$

where K_s is constant of proportionality

It is to be noticed that the frequency of the voltage e(t) is same as that of supply and the magnitude is proportional to $(\theta_r - \theta_0)$.

In general, the angular error $(\theta_r - \theta_o)$ is usually small and $\theta_r - \theta_o$ is expressed in radians, therefore

$$\sin(\theta_r - \theta_o) \simeq \theta_r - \theta_o \longrightarrow (9)$$

$$\therefore e(t) = K_s(\theta_r - \theta_o) \longrightarrow (10)$$

Taking LT on both sides

$$E(s) = K_s[\theta_r(s) - \theta_o(s)] \longrightarrow (11)$$

$$= K_s\,\theta_e(s) \longrightarrow (12)$$

where $\theta_e(s) = \theta_r(s) - \theta_o(s) \longrightarrow (13)$

The block diagram represetation of synchro error is shown below

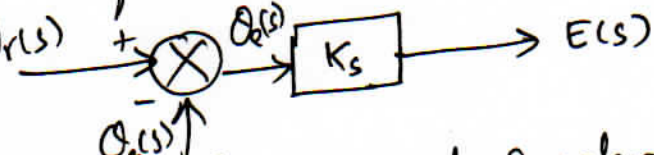


Figure : Block diagram of Synchro error detector

The transfer function of Synchro error detector is

$$\frac{E(s)}{\theta_e(s)} = K_s \quad \text{or} \quad \frac{E(s)}{[\theta_r(s) - \theta_e(s)]} = K_s$$

where K_s is known as the sensitivity or the gain of synchro error detector.

The variation of the magnitude of the output voltage 'e' of synchro error detector is a function of time and shown in figure.

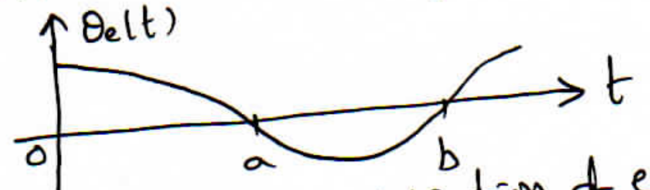


Figure : Variation of error w.r.t time

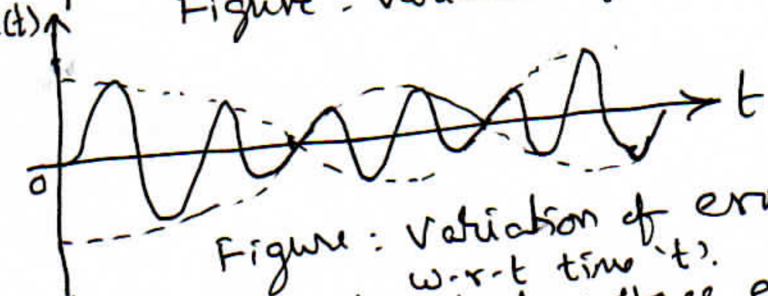


Figure : Variation of error e(t) w.r.t time 't'.

It is to be noted that the phase of output voltage e(t) reverses at points 'a' and 'b' as the error $\theta_e(t)$ changes its sign.