UNIT-V

TRANSMISSION LINES-I

- Types of transmission lines
- Transmission line Parameters- Primary & Secondary Constants
- Transmission Line Equations
- Expressions for Characteristics Impedance
- Propagation Constant
- Phase and Group Velocities
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TRANSMISSION LINE THEORY

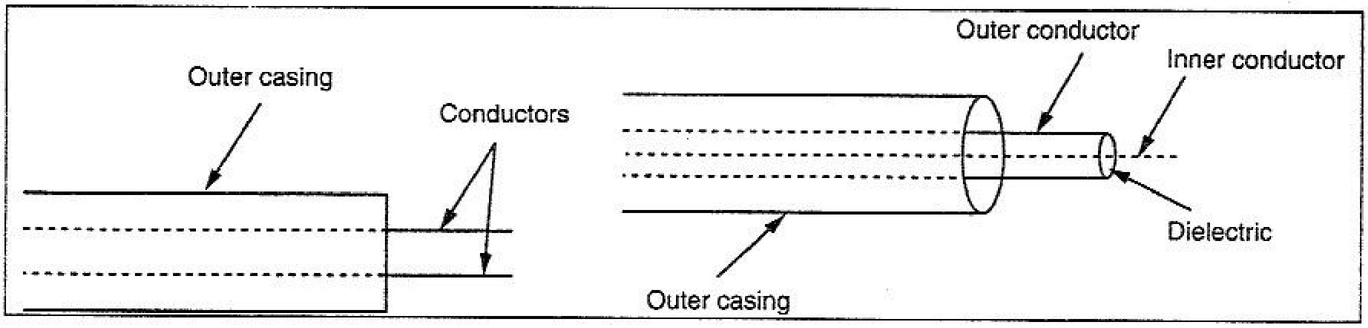
1.1. INTRODUCTION

The transfer of energy from one point to another takes place through either wave guides or transmission lines. Transmission lines always consist of atleast two separate conductors between which a voltage can exist, but the wave guides involve only one conductor; for example, a hollow rectangular or circular waveguide within which the wave propagates. Transmission lines are a means of conveying power from one point to another. There are two types of commonly used transmission lines.

1. Parallel wire (balanced) line
2. Coaxial (unbalanced) line

Parallel wire line : It is a common form of transmission line known as open wire line as shown in Fig. 1.1(*a*). It is employed where balanced properties are required. Telephone lines, line connecting between folded dipole antenna and TV receiver are good examples of parallel or balanced or open wire line. The parallel wire lines are not used for microwave transmission.

Coaxial line : Coaxial lines consist of inner and outer conductor spacers of dielectric as shown in Fig. 1.1(*b*). It is used when unbalanced properties are needed, as in the interconnection of a broadcast transmitter to its grounded antenna. It is employed at UHF and microwave frequencies.



(a) Parallel wire (balanced) line *(b) Coaxial (unbalanced) line*

Fig. 1.1. Transmission lines

1.2. TRANSMISSION LINE AS CASCADED T SECTIONS

To study the behaviour of transmission line, a transmission can be considered to be made up of a number of identical symmetrical T sections connected in series as in Fig.1.2. If the last section is terminated with its characteristic impedance, the input impedance at the first section is Z_0. Each section is terminated by the input impedance of the following section.

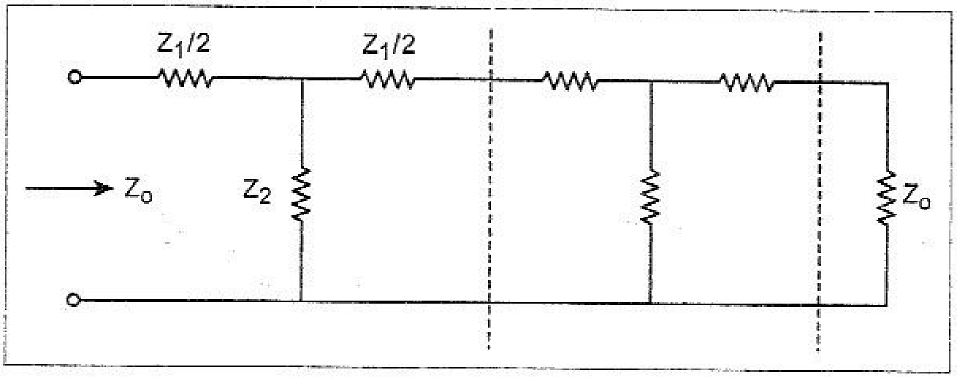


Fig. 1.2. A line of cascaded T sections

The characteristic impedance for a T section is

$$Z_{0T} = \sqrt{Z_1 Z_2 \left(1 + \frac{Z_1}{4 Z_2}\right)}$$

If 'n' number of T sections are cascaded and if the sending and receiving currents are I_S and I_R respectively, then

$$I_S = I_R \, e^{n\gamma}$$

where γ is the propagation constant for one T section.

$$\gamma = \alpha + j\beta$$

$$e^{\gamma} = e^{\alpha + j\beta} = 1 + \frac{Z_1}{2 Z_2} + \sqrt{\frac{Z_1}{Z_2}\left(1 + \frac{Z_1}{4 Z_2}\right)}$$

One T section representing an incremental length Δx of the line has a series impedance $Z_1 = Z\,\Delta x$ and shunt impedance $Z_2 = \frac{1}{Y\,\Delta x}$. The characteristic impedance of any small T section is that of the line as a whole.

$$Z_0 = \sqrt{Z_1 Z_2 \left(1 + \frac{Z_1}{4 Z_2}\right)}$$

Substituting the values of Z_1 and Z_2,

$$Z_0 = \sqrt{\frac{Z\,\Delta x}{Y\,\Delta x}\left(1 + \frac{Z\,\Delta x\, Y\,\Delta x}{4}\right)}$$

$$= \sqrt{\frac{Z}{Y}\left(1 + \frac{ZY\,(\Delta x)^2}{4}\right)}$$

If Δx tends to zero, then Z_0 becomes,

$$Z_0 = \sqrt{\frac{Z}{Y}}$$

$$\sqrt{\frac{Z_1}{Z_2}\left(1+\frac{Z_1}{4Z_2}\right)} = \sqrt{\frac{Z_1}{Z_2}}\left(1+\frac{Z_1}{4Z_2}\right)^{\frac{1}{2}}$$

By the binomial theorem,

$$\sqrt{\frac{Z_1}{Z_2}\left(1+\frac{Z_1}{4Z_2}\right)} = \sqrt{\frac{Z_1}{Z_2}}\left[1+\frac{1}{2}\left(\frac{Z_1}{4Z_2}\right)-\frac{1}{8}\left(\frac{Z_1}{4Z_2}\right)^2+\ldots\ldots\right]$$

Substituting this value in e^{γ} equation,

$$e^{\gamma} = 1+\frac{Z_1}{2Z_2}+\sqrt{\frac{Z_1}{Z_2}\left(1+\frac{Z_1}{4Z_2}\right)}$$

$$= 1+\frac{Z_1}{2Z_2}+\sqrt{\frac{Z_1}{Z_2}}+\frac{1}{8}\left(\frac{Z_1}{Z_2}\right)\sqrt{\frac{Z_1}{Z_2}}-\frac{1}{128}\left(\frac{Z_1}{Z_2}\right)^2\sqrt{\frac{Z_1}{Z_2}}+\ldots\ldots$$

$$= 1+\sqrt{\frac{Z_1}{Z_2}}+\frac{1}{2}\left(\sqrt{\frac{Z_1}{Z_2}}\right)^2+\frac{1}{8}\left(\sqrt{\frac{Z_1}{Z_2}}\right)^3-\frac{1}{128}\left(\sqrt{\frac{Z_1}{Z_2}}\right)^5+\ldots\ldots$$

When applied to the incremental length of line Δx, then $Z_1 = Z\,\Delta x$, $Z_2 = \frac{1}{Y\,\Delta x}$ and propagation constant becomes $\gamma\,\Delta x$,

$$e^{\gamma\Delta x} = 1+\sqrt{ZY}\,\Delta x+\frac{1}{2}(\sqrt{ZY})^2(\Delta x)^2+\frac{1}{8}(\sqrt{ZY})^3(\Delta x)^3-128(\sqrt{ZY})^5(\Delta x)^5$$

Series expansion for an exponential $e^{\gamma\Delta x}$ is

$$e^{\gamma\Delta x} = 1+\gamma\Delta x+\frac{\gamma^2(\Delta x)^2}{2!}+\frac{\gamma^3(\Delta x)^3}{3!}+\ldots\ldots\ldots$$

Equating the above two expressions,

$$\sqrt{ZY}\,\Delta x+\frac{(\sqrt{ZY})^2(\Delta x)^2}{2}+\frac{(\sqrt{ZY})^2(\Delta x)^3}{8}+\ldots = \gamma\Delta x+\frac{\gamma^2(\Delta x)^2}{2}+\frac{\gamma^3(\Delta x)^3}{6}+\ldots$$

$$\gamma+\frac{\gamma^2\,\Delta x}{2}+\frac{\gamma^3(\Delta x)^2}{6}+\ldots\ldots = \sqrt{ZY}+\frac{(\sqrt{ZY})^2}{2}\Delta x+\frac{(\sqrt{ZY})^3(\Delta x)^2}{8}+\ldots\ldots$$

If Δx tends to zero then,

$$\gamma = \sqrt{ZY}$$

This is the value of propagation constant in terms of Z and Y.

Since each conductor of transmission line has a certain length and diameter, it must have resistance and inductance; moreover the two conductors are separated by a dielectric medium (say, air), therefore there must be a capacitance between them. This dielectric between the conducting wires may not be perfect, and hence a leakage current will flow creating leakage (shunt) capacitance between the conductors. These four parameters resistance (R), inductance (L), capacitance (C) and conductance (G), all distributed along the lines are known as distributed parameters. The equivalent circuit diagram of transmission line is shown in Fig. 1.3.

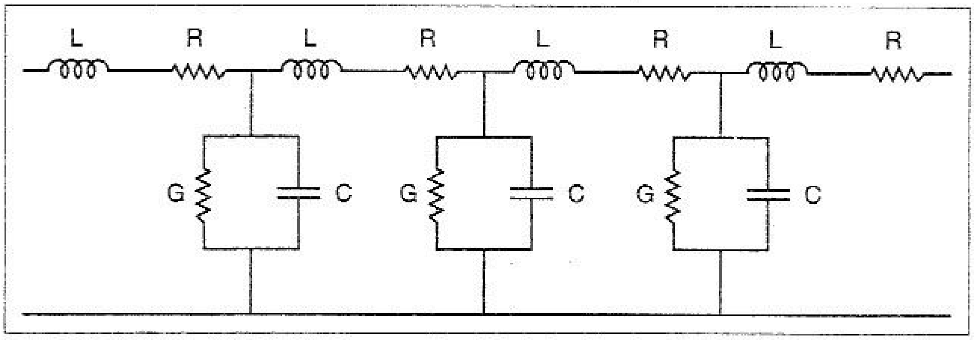


Fig. 1.3. Equivalent circuit diagram of transmission line

The four line parameters resistance (R), inductance (L), capacitance (C) and conductance (G) are also known as ***primary constants*** of the transmission line.

Resistance (R) is defined as the loop resistance per unit length of the transmission line. It is measured in ohms/km.

Inductance (L) is defined as the loop inductance per unit length of the transmission line. It is measured in Henries/km.

Capacitance (C) is defined as the shunt capacitance per unit length between the two transmission lines. It is measured in Farads/km.

Conductance (G) is defined as the shunt conductance per unit length between the two transmission lines. It is measured in mhos/km.

1.3. TRANSMISSION LINE EQUATION

Transmission line is a conductive method of guiding electrical energy from one place to another. A uniform transmission line can be considered to be made up of an infinite number of T sections, each of infinitesimal size dx. The equivalent circuit of T section of transmission line is shown in Fig. 1.4.

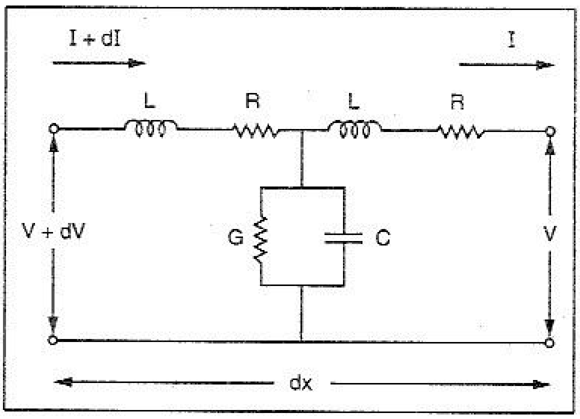


Fig. 1.4. Equivalent circuit of T section of Transmission line

The parameters R, L, G and C are distributed throughout the transmission line. The constants of an incremental length dx of a line are shown in Fig. 1.4. The series impedance per unit length and shunt admittance per unit length are given by

$$Z = R + j\omega L$$

$$Y = G + j\omega C$$

Consider a T section of transmission line of length dx. Let V + dV be the voltage and I + dI be the current at one end of T section. Let V be the voltage and I be the current at the other end of this section.

The series impedance of a small section dx is $(R + jL\omega)\,dx$. The shunt admittance of this section dx is $(G + jC\omega)\,dx$.

The voltage drop across the series impedance of T sections *i.e.*, the potential difference between the two ends of T section is

$$V + dV - V = I\,(R + j\omega L)\,dx$$

$$dV = I\,(R + j\omega L)\,dx$$

$$\frac{dV}{dx} = I\,(R + j\omega L) \qquad \ldots (1.1)$$

$$\frac{dV}{dx} = IZ$$

The current difference between the two ends of T section is due to the voltage drop across the shunt admittance.

$$I + dI - I = V\,(G + j\omega C)\,dx$$

$$dI = V\,(G + j\omega C)\,dx$$

$$\frac{dI}{dx} = V\,(G + j\omega C) \qquad \ldots (1.2)$$

$$\frac{dI}{dx} = VY$$

Differentiating equation (1.1) w.r.t. 'x',

$$\frac{d^2V}{dx^2} = (R + j\omega L)\,\frac{dI}{dx}$$

Substituting the value of $\frac{dI}{dx}$ in the above equation

$$\frac{d^2V}{dx^2} = (R + j\omega L)\,(G + j\omega C)\,V \qquad \ldots (1.3)$$

Differentiating equation (1.2) w.r.t. 'x'

$$\frac{d^2I}{dx^2} = (G + j\omega C)\,\frac{dV}{dx}$$

Substituting the value of $\frac{dV}{dx}$ in the above equation

$$\frac{d^2I}{dx^2} = (R + j\omega L)\,(G + j\omega C)\,I \qquad \ldots (1.4)$$

But propagation constant is given by

$$\gamma = \sqrt{(R + j\omega L)\,(G + j\omega C)} = \sqrt{ZY}$$

Substituting the value of γ in equation (1.3) and (1.4),

$$\text{then} \quad \frac{d^2V}{dx^2} = \gamma^2 V$$

$$\frac{d^2I}{dx^2} = \gamma^2 I$$

The solutions of the above linear differential equations are

$$V = A\,e^{\gamma x} + B\,e^{-\gamma x} \quad \ldots (1.5)$$

$$I = C\,e^{\gamma x} + D\,e^{-\gamma x} \quad \ldots (1.6)$$

where A, B, C and D are arbitrary constants.

Differentiating the equation (1.5), w.r.t. 'x'

$$\frac{dV}{dx} = A\gamma\,e^{\gamma x} - B\gamma\,e^{-\gamma x}$$

$$\text{But} \quad \frac{dV}{dx} = IZ$$

$$IZ = A\gamma\,e^{\gamma x} - B\gamma\,e^{-\gamma x}$$

$$= A\sqrt{ZY}\;e^{\sqrt{ZY}\,x} - B\sqrt{ZY}\;e^{-\sqrt{ZY}\,x} \qquad [\because \gamma = \sqrt{ZY}\,]$$

$$I = A\sqrt{\frac{Y}{Z}}\;e^{\sqrt{ZY}\,x} - B\sqrt{\frac{Y}{Z}}\;e^{-\sqrt{ZY}\,x} \quad \ldots (1.7)$$

Similarly, differentiating the equation (1.6) w.r.t. 'x'

$$\frac{dI}{dx} = C\gamma\,e^{\gamma x} - D\gamma\,e^{-\gamma x}$$

$$\text{But} \quad \frac{dI}{dx} = VY$$

$$VY = C\gamma\,e^{\gamma x} - D\gamma\,e^{-\gamma x}$$

$$= C\sqrt{ZY}\;e^{\sqrt{ZY}\,x} - D\sqrt{ZY}\;e^{-\sqrt{ZY}\,x}$$

$$V = C\sqrt{\frac{Z}{Y}}\;e^{\sqrt{ZY}\,x} - D\sqrt{\frac{Z}{Y}}\;e^{-\sqrt{ZY}\,x} \quad \ldots (1.8)$$

Since the distance x is measured from the receiving end of the transmission line,

$$x = 0, \qquad \therefore \quad I = I_R$$

$$V = V_R$$

$$V_R = I_R\,Z_R$$

where I_R is the current in the receiving end of line

V_R is the voltage across the receiving end of the lines

Z_R is the impedance of receiving end

Substituting this condition in equations (1.5), (1.6), (1.7) and (1.8).

$$V_R = A + B \qquad \dots (1.9)$$

$$I_R = C + D \qquad \dots (1.10)$$

$$I_R = A\sqrt{\frac{Y}{Z}} - B\sqrt{\frac{Y}{Z}} \qquad \dots (1.11)$$

$$V_R = C\sqrt{\frac{Z}{Y}} - D\sqrt{\frac{Z}{Y}} \qquad \dots (1.12)$$

To solve these equations,

$$\text{Let } x = \sqrt{\frac{Z}{Y}} \quad \text{and} \quad \frac{1}{x} = \sqrt{\frac{Y}{Z}}$$

$$\text{Then } I_R = \frac{A}{x} - \frac{B}{x}$$

$$= \frac{1}{x}(A - B)$$

$$\text{But } I_R = C + D$$

$$C + D = \frac{1}{x}(A - B)$$

$$Cx + Dx = A - B$$

$$A - B = Cx + Dx \qquad \dots (1.13)$$

Similarly, equation (1.12) becomes,

$$V_R = Cx - Dx$$

$$\text{But } V_R = A + B$$

$$A + B = Cx - Dx \qquad \dots (1.14)$$

$$A - B = Cx + Dx \qquad \dots (1.13)$$

Adding the equations (1.13) and (1.14),

$$2A = 2Cx$$

$$A = Cx$$

Similarly subtracting the equations (1.13) and (1.14),

$$2B = -2x\,Dx$$

$$B = -Dx$$

Substituting the values of A and B in the following equations.

$$V_R = A + B$$

$$= Cx - Dx$$

$$\text{But } I_R = C + D$$

$$I_R x = Cx + Dx \quad \ldots (1.15)$$

$$V_R = Cx - Dx \quad \ldots (1.16)$$

Adding the equations (1.15) and (1.16),

$$2Cx = I_R x + V_R$$

$$C = \frac{I_R}{2} + \frac{V_R}{2x}$$

$$\boxed{\therefore C = \frac{I_R}{2} + \frac{V_R}{2}\sqrt{\frac{Y}{Z}}} \quad \ldots (1.17) \qquad \left[\because x = \sqrt{\frac{Z}{Y}}\right]$$

Subtracting the equations (1.15) and (1.16),

$$2Dx = I_R x - V_R$$

$$D = \frac{I_R}{2} - \frac{V_R}{2x}$$

$$\boxed{\therefore D = \frac{I_R}{2} - \frac{V_R}{2}\sqrt{\frac{Y}{Z}}} \quad \ldots (1.18)$$

$$\text{But } A = Cx$$

$$A = \frac{I_R}{2}x + \frac{V_R}{2}$$

$$\boxed{\therefore A = \frac{V_R}{2} + \frac{I_R}{2}\sqrt{\frac{Z}{Y}}} \quad \ldots (1.19)$$

$$B = -Dx$$

$$B = -\frac{I_R}{2}x + \frac{V_R}{2}$$

$$\boxed{\therefore B = \frac{V_R}{2} - \frac{I_R}{2}\sqrt{\frac{Z}{Y}}} \quad \ldots (1.20)$$

The characteristic impedance is defined as

$$Z_o = \sqrt{\frac{Z}{Y}}$$

$$= \sqrt{\frac{R + j\omega L}{G + j\omega C}} \quad \ldots (1.21)$$

Substituting the value of Z_o in equations (1.19), (1.20), (1.17) and (1.18),

$$A = \frac{V_R}{2} + \frac{I_R}{2}\sqrt{\frac{Z}{Y}}$$

$$A = \frac{V_R}{2} + \frac{V_R}{2\,Z_R}\,Z_o$$

$$\boxed{A = \frac{V_R}{2}\left[1 + \frac{Z_o}{Z_R}\right]} \qquad \dots (1.22)$$

$$B = \frac{V_R}{2} - \frac{I_R}{2}\sqrt{\frac{Z}{Y}}$$

$$= \frac{V_R}{2} - \frac{V_R}{2\,Z_R}\,Z_o$$

$$\boxed{B = \frac{V_R}{2}\left[1 - \frac{Z_o}{Z_R}\right]} \qquad \dots (1.23)$$

$$C = \frac{I_R}{2} + \frac{V_R}{2}\sqrt{\frac{Y}{Z}}$$

$$= \frac{I_R}{2} + \frac{I_R\,Z_R}{2\,Z_o} \qquad [\because V_R = I_R\,Z_R]$$

$$\boxed{C = \frac{I_R}{2}\left[1 + \frac{Z_R}{Z_o}\right]} \qquad \dots (1.24)$$

$$D = \frac{I_R}{2} - \frac{V_R}{2}\sqrt{\frac{Y}{Z}}$$

$$= \frac{I_R}{2} - \frac{I_R\,Z_R}{2\,Z_o}$$

$$\boxed{D = \frac{I_R}{2}\left[1 + \frac{Z_R}{Z_o}\right]} \qquad \dots (1.25)$$

Substituting the values of A, B, C and D in equations (1.5) and (1.6), the solutions of the differential equations are

$$V = \frac{V_R}{2}\left(1 + \frac{Z_o}{Z_R}\right) e^{\sqrt{ZY}\,x} + \frac{V_R}{2}\left(1 - \frac{Z_o}{Z_R}\right) e^{-\sqrt{ZY}\,x} \qquad \dots (1.26)$$

$$I = \frac{I_R}{2}\left(1 + \frac{Z_R}{Z_o}\right) e^{\sqrt{ZY}\,x} + \frac{I_R}{2}\left(1 - \frac{Z_R}{Z_o}\right) e^{-\sqrt{ZY}\,x} \qquad \dots (1.27)$$

$$V = \frac{V_R}{2}\left[\left(1 + \frac{Z_o}{Z_R}\right) e^{\sqrt{ZY}\,x} + \left(1 - \frac{Z_o}{Z_R}\right) e^{-\sqrt{ZY}\,x}\right] \qquad \dots (1.28)$$

$$I = \frac{I_R}{2}\left[\left(1+\frac{Z_R}{Z_o}\right)e^{\sqrt{ZY}\,x}+\left(1-\frac{Z_R}{Z_o}\right)e^{-\sqrt{ZY}\,x}\right] \quad \text{... (1.29)}$$

After simplification,

$$V = \frac{V_R}{2}e^{\sqrt{ZY}\,x}+\frac{V_R}{2}\frac{Z_o}{Z_R}e^{\sqrt{ZY}\,x}+\frac{V_R}{2}e^{-\sqrt{ZY}\,x}-\frac{V_R}{2}\frac{Z_o}{Z_R}e^{-\sqrt{ZY}\,x}$$

$$I = \frac{I_R}{2}e^{\sqrt{ZY}\,x}+\frac{I_R}{2}\frac{Z_R}{Z_o}e^{\sqrt{ZY}\,x}+\frac{I_R}{2}e^{-\sqrt{ZY}\,x}-\frac{I_R}{2}\frac{Z_R}{Z_o}e^{-\sqrt{ZY}\,x}$$

$$V = V_R\left(\frac{e^{\sqrt{ZY}\,x}+e^{-\sqrt{ZY}\,x}}{2}\right)+I_R Z_o\left(\frac{e^{\sqrt{ZY}\,x}-e^{-\sqrt{ZY}\,x}}{2}\right) \quad [\because V_R = I_R Z_R]$$

$$I = I_R\left(\frac{e^{\sqrt{ZY}\,x}+e^{-\sqrt{ZY}\,x}}{2}\right)+\frac{V_R}{Z_o}\,(e^{\sqrt{ZY}\,x}-e^{-\sqrt{ZY}\,x}) \quad \left[\because I_R = \frac{V_R}{Z_R}\right]$$

Then equations can be written in terms of hyperbolic functions.

$$V = V_R\cosh\sqrt{ZY}\;x + I_R Z_o \sinh\sqrt{ZY}\;x \quad \text{... (1.30)}$$

$$I = I_R\cosh\sqrt{ZY}\;x + \frac{V_R}{Z_o}\sinh\sqrt{ZY}\;x \quad \text{... (1.31)}$$

These are the equations for voltage and current of a transmission line at any distance 'x' from the receiving end of transmission line.

The equations for voltage and current at the sending send of a transmission line of length 'l' are given by

$$V_S = V_R\cosh\sqrt{ZY}\,l + \frac{V_R}{Z_R}Z_o\sinh\sqrt{ZY}\,l \quad \left[\because I_R = \frac{V_R}{Z_R}\right]$$

$$I_S = I_R\cosh\sqrt{ZY}\,l + \frac{I_R Z_R}{Z_o}\sinh\sqrt{ZY}\;l \quad [\because V_R = I_R Z_R]$$

$$V_S = V_R\left[\cos\sqrt{ZY}\;l + \frac{Z_0}{Z_R}\sinh\sqrt{ZY}\;l\right] \quad \text{... (1.32)}$$

$$I_S = I_R\left[\cos\sqrt{ZY}\;l + \frac{Z_R}{Z_0}\sinh\sqrt{ZY}\;l\right] \quad \text{... (1.33)}$$

1.4. WAVELENGTH AND VELOCITY OF PROPAGATION

The propagation constant (γ) and characteristic impedance (Z_0) are called secondary constants of a transmission line.

Propagation constant is usually a complex quantity.

$$\gamma = \alpha + j\beta$$

where α is the attenuation constant.

β is the phase shift.

$$\gamma = \sqrt{ZY}$$

where $Z = R + j\omega L$

$$Y = G + j\omega C$$

The characteristic impedance of the transmission line is also a complex quantity.

$$Z_0 = \sqrt{\frac{Z}{Y}}$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \qquad \ldots (1.34)$$

Propagation constant is $\gamma = \alpha + i\beta$

$$= \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$\alpha + i\beta = \sqrt{RG - \omega^2 LC + j\omega(LG + RC)} \qquad \ldots (1.35)$$

Squaring on both sides,

$$(\alpha + j\beta)^2 = RG - \omega^2 LC + j\omega(LG + RC)$$

$$\alpha^2 - \beta^2 + 2j\,\alpha\beta = RG - \omega^2 LC + j\omega\,(LG + RC) \qquad \ldots (1.36)$$

Equating real parts,

$$\alpha^2 - \beta^2 = RG - \omega^2 LC$$

$$\alpha^2 = \beta^2 + RG - \omega^2 LC \qquad \ldots (1.37)$$

Equating imaginary parts,

$$2\,\alpha\beta = \omega\,(LG + RC)$$

Squaring on both sides,

$$4\,\alpha^2\,\beta^2 = \omega^2\,(LG + RC)^2$$

$$\alpha^2\,\beta^2 = \frac{\omega^2}{4}\,(LG + RC)^2$$

Substituting the value of α^2 [eqn. (1.37)] in the above equation,

$$(\beta^2 + RG - \omega^2 LC)\,\beta^2 = \frac{\omega^2}{4}\,(LG + RC)^2$$

$$\beta^4 + \beta^2\,(RG - \omega^2 LC) - \frac{\omega^2}{4}\,(LG + RC)^2 = 0$$

The solution of the quadratic equation is

$$\beta^2 = \frac{-(RG - \omega^2 LC) \pm \sqrt{(RG - \omega^2 LC)^2 + \omega^2\,(LG + RC)^2}}{2}$$

By neglecting the negative values,

$$\therefore \quad \beta = \sqrt{\frac{\omega^2 LC - RG + \sqrt{(RG - \omega^2 LC)^2 + \omega^2 (LG + RC)^2}}{2}} \qquad \ldots (1.38)$$

$$\alpha^2 = \beta^2 + RG - \omega^2 LC \qquad \ldots (1.37)$$

Substituting the value of β [eqn. (1.38)] in the above equation,

$$\alpha^2 = \frac{\omega^2 LC - RG + \sqrt{(RG - \omega^2 LC)^2 + \omega^2 (LG + RC)^2}}{2} + RG - \omega^2 LC$$

$$= \frac{RG - \omega^2 LC + \sqrt{(RG - \omega^2 LC)^2 + \omega^2 (LG + RC)^2}}{2}$$

$$\therefore \quad \alpha = \sqrt{\frac{RG - \omega^2 LC + \sqrt{(RG - \omega^2 LC)^2 + \omega^2 (LG + RC)^2}}{2}} \qquad \ldots (1.39)$$

For a perfect transmission line $R = 0$ and $G = 0$,

$$\beta^2 = \omega^2 LC$$

$$\therefore \quad \beta = \omega \sqrt{LC} \qquad \text{[only positive value]}$$

Velocity :

The velocity of propagation is given by,

$$v = \lambda f$$

$$= 2\pi f \frac{\lambda}{2\pi}$$

$$v = \frac{\omega}{\beta} \qquad \left[\because \beta = \frac{2\pi}{\lambda} \text{ and } \omega = 2\pi f\right]$$

Substituting the value of $\beta = \omega \sqrt{LC}$

$$\therefore \quad v = \frac{\omega}{\omega \sqrt{LC}}$$

$$v = \frac{1}{\sqrt{LC}}$$

This is the velocity of propagation for an ideal line.

Wavelength :

The distance travelled by the wave along the line while the phase angle is changing through 2π radians is called wavelength.

$$\beta\lambda = 2\pi$$

$$\lambda = \frac{2\pi}{\beta} \quad \text{or} \quad \lambda = \frac{v}{f}$$

1.5. INPUT IMPEDANCE AND TRANSFER IMPEDANCE OF TRANSMISSION LINE

Input impedance :

The equations for voltage and current at the sending end of a transmission line of length 'l' are given by

$$V_S = V_R\left(\cosh\sqrt{ZY}\,l + \frac{Z_0}{Z_R}\sinh\sqrt{ZY}\,l\right) \quad \dots (1.32)$$

$$I_S = I_R\left(\cosh\sqrt{ZY}\,l + \frac{Z_R}{Z_0}\sinh\sqrt{ZY}\,l\right) \quad \dots (1.33)$$

The input impedance of the transmission line is,

$$Z_S = \frac{V_S}{I_S}$$

$$= \frac{V_R\left(\cosh\sqrt{ZY}\,l + \frac{Z_0}{Z_R}\sinh\sqrt{ZY}\,l\right)}{I_R\left(\cosh\sqrt{ZY}\,l + \frac{Z_R}{Z_0}\sinh\sqrt{ZY}\,l\right)}$$

$$= \frac{I_R Z_R\left(\cosh\sqrt{ZY}\,l + \frac{Z_0}{Z_R}\sinh\sqrt{ZY}\,l\right)}{I_R\left(\cosh\sqrt{ZY}\,l + \frac{Z_R}{Z_0}\sinh\sqrt{ZY}\,l\right)}$$

$$Z_S = \frac{Z_0\,(Z_R\cosh\sqrt{ZY}\,l + Z_0\sinh\sqrt{ZY}\,l)}{(Z_0\cosh\sqrt{ZY}\,l + Z_R\sinh\sqrt{ZY}\,l)} \quad \dots (1.40)$$

Let $\sqrt{ZY} = \gamma$

The input impedance of the line is

$$Z_S = Z_0\left[\frac{Z_R\cosh\gamma l + Z_0\sinh\gamma l}{Z_0\cosh\gamma l + Z_R\sinh\gamma l}\right]$$

or $$Z_S = Z_0\left[\frac{Z_R + Z_0\tanh\gamma l}{Z_0 + Z_R\tanh\gamma l}\right]$$

In a different form, the equations for voltage and current at transmitting end of a line is given by equations (1.28) and (1.29),

$$V_S = \frac{V_R}{2}\left[\left(1 + \frac{Z_0}{Z_R}\right)e^{\sqrt{ZY}\,l} + \left(1 - \frac{Z_0}{Z_R}\right)e^{-\sqrt{ZY}\,l}\right] \quad \dots (1.28)$$

$$I_S = \frac{I_R}{2}\left[\left(1 + \frac{Z_R}{Z_0}\right)e^{\sqrt{ZY}\,l} + \left(1 - \frac{Z_R}{Z_0}\right)e^{-\sqrt{ZY}\,l}\right] \quad \dots (1.29)$$

or
$$V_S = \frac{V_R}{2}\left[\left(\frac{Z_R+Z_0}{Z_R}\right)e^{\sqrt{ZY}\,l} + \left(\frac{Z_R-Z_0}{Z_R}\right)e^{-\sqrt{ZY}\,l}\right]$$

$$I_S = \frac{I_R}{2}\left[\left(\frac{Z_R+Z_0}{Z_0}\right)e^{\sqrt{ZY}\,l} + \left(\frac{Z_0-Z_R}{Z_0}\right)e^{-\sqrt{ZY}\,l}\right]$$

or
$$V_S = \left(\frac{V_R}{2}\right)\left(\frac{Z_R+Z_0}{Z_R}\right)\left[e^{\sqrt{ZY}\,l} + \left(\frac{Z_R-Z_0}{Z_R+Z_0}\right)e^{-\sqrt{ZY}\,l}\right] \quad \ldots (1.41)$$

$$I_S = \frac{I_R}{2}\left(\frac{Z_R+Z_0}{Z_0}\right)\left[e^{\sqrt{ZY}\,l} - \left(\frac{Z_R-Z_0}{Z_R+Z_0}\right)e^{-\sqrt{ZY}\,l}\right] \quad \ldots (1.42)$$

The input impedance of the transmission line is given by,

$$Z_S = \frac{V_S}{I_S} = Z_0\left[\frac{e^{\sqrt{ZY}\,l} + \left(\frac{Z_R-Z_0}{Z_R+Z_0}\right)e^{-\sqrt{ZY}\,l}}{e^{\sqrt{ZY}\,l} - \left(\frac{Z_R-Z_0}{Z_R+Z_0}\right)e^{-\sqrt{ZY}\,l}}\right] \quad [\because V_R = I_R Z_R] \quad \ldots (1.43)$$

Let $\sqrt{ZY} = \gamma$

The input impedance of the transmission line is,

$$Z_S = Z_0\left[\frac{e^{\gamma l} + \left(\frac{Z_R-Z_0}{Z_R+Z_0}\right)e^{-\gamma l}}{e^{\gamma l} - \left(\frac{Z_R-Z_0}{Z_R+Z_0}\right)e^{-\gamma l}}\right] \quad \ldots (1.44)$$

If the line is terminated with its characteristic impedance *i.e.*, $Z_R = Z_0$, then the input impedance becomes equal to its characteristic impedance.

$$Z_S = Z_0$$

The input impedance of an infinite line is determined by letting $l \to \infty$.

$$\therefore \; Z_S = Z_0$$

It is found that a line of finite length, terminated with its characteristic impedance, appears to the transmitting end generator as an infinite line. A finite line terminated with Z_0 and an infinite line are same by measurements at the source.

If $K = \frac{Z_R-Z_0}{Z_R+Z_0}$, then

$$Z_S = Z_0\left[\frac{e^{\gamma l} + K e^{-\gamma l}}{e^{\gamma l} - K e^{-\gamma l}}\right] \quad \ldots (1.45)$$

Transfer impedance :

Transfer impedance is used to determine the current at the receiving end if voltage at transmitting end is known. Transfer impedance of a transmission line is defined as the ratio of voltage at the sending end (transmitted voltage) to the current at the receiving end (received current).

$$Z_T = \frac{V_S}{I_R}$$

Equation (1.41) becomes

$$V_S = \frac{V_R (Z_R + Z_0)}{2 Z_R} (e^{\gamma l} + K e^{-\gamma l})$$

$$V_S = \frac{I_R (Z_R + Z_0)}{2} (e^{\gamma l} + K e^{-\gamma l}) \qquad [\because V_R = I_R Z_R]$$

$$Z_T = \frac{V_S}{I_R} = \frac{Z_R + Z_0}{2} (e^{\gamma l} + K e^{-\gamma l})$$

$$= \frac{Z_R + Z_0}{2} \left(e^{\gamma l} + \frac{Z_R - Z_0}{Z_R + Z_0} e^{-\gamma l} \right)$$

$$= \left(\frac{Z_R + Z_0}{2} \right) e^{\gamma l} + \left(\frac{Z_R - Z_0}{2} \right) e^{-\gamma l}$$

$$= Z_R \left(\frac{e^{\gamma l} + e^{-\gamma l}}{2} \right) + Z_0 \left(\frac{e^{\gamma l} - e^{-\gamma l}}{2} \right)$$

$$= Z_R \cosh \gamma l + Z_0 \sinh \gamma l$$

$$\left[\because \frac{e^{\gamma l} + e^{-\gamma l}}{2} = \cosh \gamma l \quad \text{and} \quad \frac{e^{\gamma l} - e^{-\gamma l}}{2} = \sinh \gamma l \right]$$

$$Z_T = Z_R \cosh \gamma l + Z_0 \sinh \gamma l$$

1.6. LINE DISTORTION

Signal (*e.g.*, voice) transmitted over a transmission line is normally complex and consists of many frequency components. Such voice voltage will not have all frequencies transmitted with equal attenuation and equal time delay, the received waveform will not be identical with the input waveform at the sending end. This variation is known as ***distortion***. There are two types of line distortions. They are frequency distortion and delay distortion.

Frequency Distortion : A complex (voice) voltage transmitted on a transmission line will not be attenuated equally and the received waveform will not be identical with the input waveform at the transmitting end. This variation is known as frequency distortion.

The attenuation constant is given by

$$\alpha = \sqrt{\frac{RG - \omega^2 LC + \sqrt{(RG - \omega^2 LC)^2 + \omega^2 (LG + CR)^2}}{2}}$$

α is a function of frequency and therefore the line will introduce frequency distortion.

Delay or Phase Distortion : For an applied voice-voltage wave the received waveform may not be identical with the input waveform at the sending end, since some frequency components will be delayed more than those of other frequencies. This phenomenon is known as delay or phase distortion.

The phase constant is

$$\beta = \sqrt{\frac{RG - \omega^2 LC + \sqrt{(RG - \omega^2 LC)^2 + \omega^2 (LG + CR)^2}}{2}}$$

β is not a constant multiplied by ω and therefore the line will introduce delay distortion.

Frequency distortion is reduced in the transmission of high quality over wire lines by the use of equalizers at the line terminals.

Delay distortion is of relatively less importance to voice and music transmission. But it can be very serious for video transmission. This can be avoided by the use of co-axial cables.

1.7. THE DISTORTIONLESS LINE

If a line is to have neither frequency nor delay distortion, then attenuation factor α and the velocity of propagation v cannot be functions of frequency.

If $\quad v = \dfrac{\omega}{\beta}$

β must be a direct function of frequency.

$$\beta = \sqrt{\frac{\omega^2 LC - RG + \sqrt{(RG - \omega^2 LC)^2 + \omega^2 (LG + CR)^2}}{2}}$$

For β to be a direct function of frequency, the term

$(RG - \omega^2 LC)^2 + \omega^2 (LG + CR)^2$ must be equal to $(RG + \omega^2 LC)^2$

$R^2G^2 + \omega^4 L^2C^2 - 2\omega^2 LCRG + \omega^2 L^2 G + \omega^2 C^2 R^2 + 2\omega^2 LCRG$

$$= R^2G^2 + \omega^4 L^2 C^2 + 2\omega^2 LCRG$$

$$\omega^2 L^2 G^2 + \omega^2 C^2 R^2 = 2\,\omega^2 LCRG$$

$$\omega^2 L^2 G^2 + \omega^2 C^2 R^2 - 2\omega^2 LCRG = 0$$

$$(LG - CR)^2 = 0$$

$$LG = CR$$

$$\frac{R}{L} = \frac{G}{C}$$

This is the condition for distortionless line.

$$\text{Propagation constant } \gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$= \sqrt{L\left(\frac{R}{L} + j\omega\right) C\left(\frac{G}{C} + j\omega\right)}$$

$$= \sqrt{LC}\sqrt{\left(\frac{R}{L} + j\omega\right)\left(\frac{G}{C} + j\omega\right)}$$

$$\text{But } \frac{R}{L} = \frac{G}{C}$$

$$\gamma = \sqrt{LC}\left(\frac{R}{L} + j\omega\right)$$

$$\text{Then} \quad \beta = \sqrt{\frac{\omega^2 LC - RG + RG + \omega^2 LC}{2}}$$

$$= \sqrt{\frac{2\omega^2 LC}{2}}$$

$$\beta = \omega\sqrt{LC}$$

Velocity of propagation is

$$v = \frac{\omega}{\beta}$$

$$v = \frac{1}{\sqrt{LC}}$$

This is the same velocity for all frequencies, thus eliminating delay distortion.

Attenuation factor

$$\alpha = \sqrt{\frac{RG - \omega^2 LC + \sqrt{(RG - \omega^2 LC)^2 + \omega^2 (LG + CR)^2}}{2}}$$

To make α is independent of frequency, the term $(RG - \omega^2 LC)^2 + \omega^2 (LG + CR)^2$ is forced to be equal to $(RG + \omega^2 LC)^2$.

$$(LG - CR)^2 = 0$$

$$LG = CR$$

$$\boxed{\frac{L}{C} = \frac{R}{G}}$$

This will make α and the velocity independent of frequency simultaneously. To achieve this condition, it requires a very large value of L, since G is small.

The attenuation factor

$$\alpha = \sqrt{\frac{RG - \omega^2 LC + \sqrt{(RG + \omega^2 LC)^2}}{2}}$$

$$= \sqrt{\frac{RG - \omega^2 LC + RG + \omega^2 LC}{2}}$$

$$= \sqrt{\frac{2RG}{2}}$$

$$\alpha = \sqrt{RG}$$

It is independent of frequency, thus eliminating frequency distortion on the line.

The characteristic impedance Z_0 is given by

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

$$= \sqrt{\frac{L\left(\frac{R}{L} + j\omega\right)}{C\left(\frac{G}{C} + j\omega\right)}}$$

But $\frac{R}{L} = \frac{G}{C}$ for distortionless line.

$$\therefore \; Z_0 = \sqrt{\frac{L}{C}}$$

It is purely real and is independent of frequency.

1.8. TELEPHONE CABLE

In the telephone cable the wires are insulated with paper and twisted in pairs. This construction results in negligible values of inductance and conductance. Therefore $L\omega << R$ and $G << C\omega$.

$$Z = R + j\omega L \approx R$$

$$Y = G + j\omega C \approx j\omega C$$

Propagation constant

$$\gamma = \sqrt{ZY}$$

$$= \sqrt{j\omega RC}$$

$$= \sqrt{\frac{j2\omega RC}{2}}$$

But $\gamma = \alpha + j\beta$

$$\alpha + j\beta = (1 + j)\sqrt{\frac{\omega RC}{2}}$$

Equating real and imaginary parts

$$\alpha = \sqrt{\frac{\omega RC}{2}}$$

$$\beta = \sqrt{\frac{\omega RC}{2}}$$

Velocity of propagation $v = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\frac{\omega RC}{2}}} = \sqrt{\frac{2\omega}{RC}}$

The characteristic impedance $Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R}{j\omega C}} = \sqrt{\frac{R}{\omega C}} \angle -45°$

It is found that the propagation constant α and velocity of propagation v are functions of frequency. Thus, the higher frequencies are attenuated more and travel faster than the lower frequencies resulting in considerable frequency and delay distortion.

1.9. LOADING OF LINES

It is necessary to increase L/C ratio to achieve distortionless condition in a transmission line. This can be done by increasing the inductance of a transmission line. Increasing inductance by inserting inductances in series with line is termed as ***loading*** and such lines are called ***loaded lines.*** The lumped inductors, known as ***loading coils*** are placed at suitable intervals along the transmission line to increase the effective distributed inductance.

The effect of loading can be realised by comparing the unloading of a transmission line in the attenuation Vs frequency graph. Fig.1.5 shows that the loaded line offers a low attenuation when compared to the unloaded line only for limited range of frequencies.

The important aspect of loading coil design is that saturation and stray fields should be avoided. It should have a low resistance and should be in small size. In general toroidal cores are used for loading coils.

Types of Loading

The open wire lines have more inductance of their own and so have much less distortion than cable. Therefore, the loading practice is not applicable to open wires but it is restricted to cables only. There are three types of loading in practice. They are

(a) Lumped loading

(b) Continuous loading

(c) Patch loading

(a) Lumped loading : The inductance of a transmission line can be increased by the introduction of loading coil at uniform intervals. This is called lumped loading. It acts as a low pass filter. So, it is applicable only for a limited range of frequency. The loading coils have an internal resistance R thus, increasing the total effective inductance increases R. Further hysteresis and eddy current losses which occur in the loading coils resulting in further apparent increase in R. Therefore, there is a practical limitation on the value of inductance that can be increased for the reduction of attenuation. Thus the loading coil should be carefully designed so that it will not introduce any distortion.

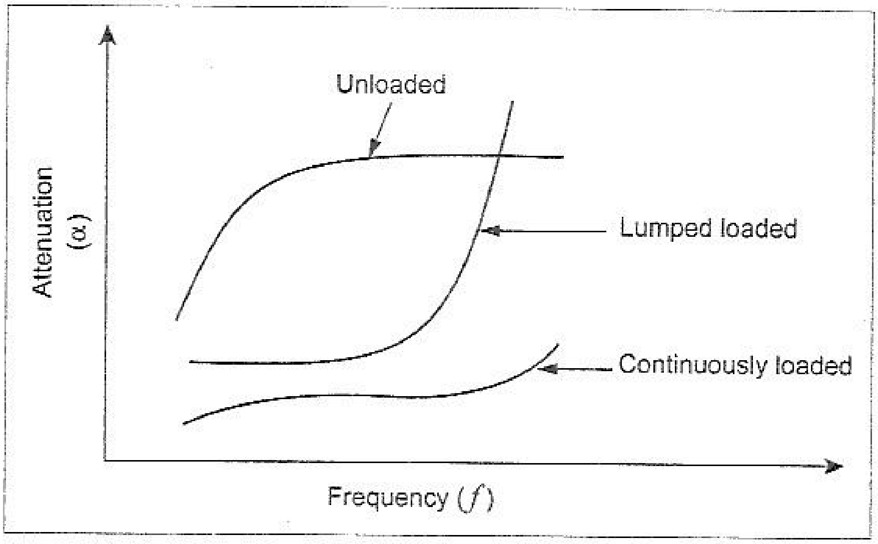


Fig. 1.5. Comparison of loaded and unloaded cable characteristics

(b) Continuous loading : A type of iron or some other magnetic material is wound on the transmission line (cable) to increase the permeability of the surrounding medium and thereby increase the inductance. It is a quite expensive method. Further eddy current and hysteresis losses in the magnetic material increases the primary constant R. Therefore, continuous loading is used only on ocean cables where lumped loading is difficult. The advantage of continuous loading over lumped loading is that attenuation factor α increases uniformly with increase in frequency.

(c) Patch loading : It employs sections of continuously loaded cable separated by sections of unloaded cable. The typical length for the section is normally a quarter kilometer. In this method the advantage of continuous loading is obtained and the cost is reduced considerably.

1.9.1. Inductance Loading of Telephone Cables

Distortionless line with distributed parameters is used to avoid the frequency and delay distortion experienced on telephone cables. It is necessary to increase the L/C to achieve distortionless condition $\frac{L}{C} = \frac{R}{G}$. Heaviside suggested that the inductance be increased and Pupin suggested that this increase in the inductance by lumped inductors spaced at intervals along the line. This use of inductance is called loading the line. The distributed loading is obtained by winding the cable with a high permeability steel tape such as permalloy in some submarine cables.

Consider an uniformly loaded cable with G = 0. Then,

$$Z = R + j\omega L$$

$$Y = j\omega C \qquad [\because G = 0]$$

$$Z = \sqrt{R^2 + (L\omega)^2} \; \Big|\; \tan^{-1}\left(\frac{L\omega}{R}\right)$$

$$= \sqrt{R^2 + (L\omega)^2} \;\; \angle \frac{\pi}{2} - \tan^{-1} \frac{R}{L\omega}$$

Propagation constant $\gamma = \sqrt{ZY}$

$$= \sqrt{\sqrt{R^2 + (L\omega)^2} \;\; \angle \frac{\pi}{2} - \tan^{-1}\frac{R}{L\omega} \left(\omega C \angle \frac{\pi}{2} \right)}$$

$$= \sqrt{\omega C \sqrt{R^2 + (L\omega)^2} \;\; \angle \pi - \tan^{-1}\frac{R}{L\omega}}$$

$$= \sqrt{(\omega C)(L\omega)\sqrt{1 + \frac{R^2}{(L\omega)^2}}} \;\; \angle \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{R}{L\omega}$$

$$= \omega\sqrt{LC} \sqrt[4]{1 + \left(\frac{R}{L\omega}\right)^2} \;\; \angle \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{R}{L\omega}$$

Since R is small with respect to $L\omega$, the term $\left(\frac{R}{L\omega}\right)$ is neglected.

$$\therefore \gamma = \omega\sqrt{LC} \;\; \angle \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{R}{L\omega}$$

$$\text{If } \theta = \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{R}{L\omega}$$

$$\cos\theta = \cos\left(\frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{R}{L\omega}\right)$$

$$= \sin\left(\frac{1}{2}\tan^{-1}\frac{R}{L\omega}\right)$$

For small angle,

$$\sin\theta \approx \tan\theta \approx \theta$$

so that $\cos\theta = \frac{R}{2L\omega}$

Similarly, $\sin\theta = \sin\left(\frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{R}{L\omega}\right) = 1$

Propagation constant $\gamma = \omega\sqrt{LC}\,(\cos\theta + j\sin\theta)$

$$= \omega\sqrt{LC}\left(\frac{R}{2L\omega} + j\right)$$

$$\gamma = \frac{R\sqrt{LC}}{2L} + j\omega\sqrt{LC}$$

$$= \frac{R}{2}\sqrt{\frac{C}{L}} + j\omega\sqrt{LC}$$

$$\therefore \text{ Attenuation constant } \alpha = \frac{R}{2}\sqrt{\frac{C}{L}}$$

$$\text{Phase-shift } \beta = \omega\sqrt{LC}$$

$$\text{Velocity of propagation } v = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$

It is noted that if G = 0 and Lω >> R, the attenuation and velocity are both independent of frequency and the loaded cable will be distortionless. Attenuation may be reduced by increasing L. Continuous (uniform) loading is expensive and achieves only a small increase in L per unit length. Lumped loading is preferred for cables.

Campbell's Equation

An analysis for the performance of a line loaded at uniform intervals can be obtained by considering a symmetrical section of line from the centre of one loading coil to the centre of the next coil. The section of line may be replaced with an equivalent T section having symmetrical series arms as shown in Fig.1.6. The series arm of T section including loading coil is given by

$$\frac{Z_1'}{2} = \frac{Z_c}{2} + \frac{Z_1}{2} \qquad \text{[From the fig.]}$$

where $\frac{Z_1}{2}$ is the series arm of T section.

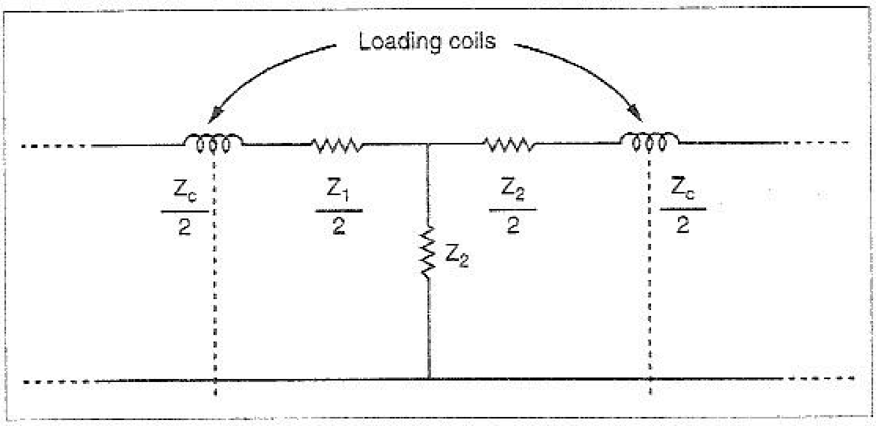


Fig. 1.6. Equivalent T section for part of a line between two lumped loading coils

$$\frac{Z_1}{2} = Z_0 \tanh\frac{\gamma l}{2}$$

$$\therefore \frac{Z_1'}{2} = \frac{Z_c}{2} + Z_0 \tanh\frac{\gamma l}{2}$$

Where l is the distance between two loading coils.

The shunt Z_2 arm of the equivalent T section is

$$Z_2 = \frac{Z_0}{\sinh \gamma l}$$

For loaded T section

$$\cosh \gamma' l = 1 + \frac{Z_1'}{2 Z_2}$$

$$= 1 + \frac{\frac{Z_c}{2} + Z_0 \tanh \frac{\gamma l}{2}}{\frac{Z_0}{\sinh \gamma l}}$$

$$\text{But } \tanh \frac{\gamma l}{2} = \frac{\cosh \gamma l - 1}{\sinh \gamma l}$$

Substituting this value in above equation

$$\therefore \cosh \gamma' l = 1 + \frac{\frac{Z_c}{2} + Z_0 \frac{\cosh \gamma l - 1}{\sinh \gamma l}}{\frac{Z_0}{\sinh \gamma l}}$$

$$= 1 + \frac{\frac{Z_c}{2} \sinh \gamma l + Z_0 (\cosh \gamma l - 1)}{Z_0}$$

$$= 1 + \frac{Z_c}{2 Z_0} \sinh \gamma l + \cosh \gamma l - 1$$

$$\cosh \gamma' l = \frac{Z_c}{2 Z_0} \sinh \gamma l + \cosh \gamma l$$

This equation is called as Campbell's equation and it is used to determine the value of γ' of a line section consisting of partially lumped and partially distributed elements. For a cable Z_2 is capacitance and the cable capacitance and lumped inductances appear similar to the circuit of the low pass filter. It is found that for frequencies below cutoff, the attenuation is reduced, but the cut-off attenuation is increased (as a result of filter action). In practice, pure distortionless line is not obtained by loading, because R and L are to some extent functions of frequency. Eddy current losses are more in these coils. However, there is a major improvement in the loaded cable over the unloaded cable for a reasonable frequency range.

1.10. OPEN CIRCUITED AND SHORT CIRCUITED LINES

The expressions for voltage and current at the sending end of a transmission line of length 'l' are given by

$$V_S = V_R\left[\cosh\sqrt{ZY}\,l + \frac{Z_o}{Z_R}\sinh\sqrt{ZY}\,l\right]$$

$$I_S = I_R\left[\cosh\sqrt{ZY}\,l + \frac{Z_R}{Z_o}\sinh\sqrt{ZY}\,l\right]$$

The input impedance of a transmission line is given by

$$Z_S = \frac{V_S}{I_S}$$

$$= \frac{V_R\left[\cosh\sqrt{ZY}\,l + \frac{Z_o}{Z_R}\sinh\sqrt{ZY}\,l\right]}{I_R\left[\cosh\sqrt{ZY}\,l + \frac{Z_R}{Z_o}\sinh\sqrt{ZY}\,l\right]}$$

$$= \frac{V_R}{I_R}\,\frac{Z_o}{Z_R}\,\frac{(Z_R\cosh\gamma l + Z_o\sinh\gamma l)}{(Z_o\cosh\gamma l + Z_R\sinh\gamma l)}$$

$$= Z_o\left(\frac{Z_R\cosh\gamma l + Z_o\sinh\gamma l}{Z_o\cosh\gamma l + Z_R\sinh\gamma l}\right) \qquad \left[\because Z_R = \frac{V_R}{I_R}\right]$$

$$Z_S = Z_o\left(\frac{Z_R\cosh\gamma l + Z_o\sinh\gamma l}{Z_o\cosh\gamma l + Z_R\sinh\gamma l}\right)$$

If short circuited, the receiving end impedance is zero.

$$i.e.,\quad Z_R = 0$$

$$\therefore Z_{sc} = Z_o\left(\frac{Z_o\sinh\gamma l}{Z_o\cosh\gamma l}\right)$$

Short circuited impedance

$$Z_{sc} = Z_o\tanh\gamma l$$

If open circuited, the receiving end impedance is infinite.

i.e., $$Z_R = \infty$$

Input impedance of transmission line can be written as

$$Z_S = Z_o\left[\frac{\cosh\gamma l + \frac{Z_o}{Z_R}\sinh\gamma l}{\frac{Z_o}{Z_R}\cosh\gamma l + \sinh\gamma l}\right]$$

Applying $Z_R = \infty$

$$\text{Then} \quad Z_{oc} = Z_0 \left[\frac{\cosh \gamma l}{\sinh \gamma l} \right]$$

The open circuited impedance

$$Z_{oc} = Z_0 \coth \gamma l$$

By multiplying open circuited impedance and short circuited impedances

$$Z_{oc} Z_{sc} = Z_0^2 \tanh \gamma l \coth \gamma l$$

$$= Z_0^2$$

The characteristic impedance is given by

$$Z_0 = \sqrt{Z_{oc} Z_{sc}}$$

By dividing short circuited impedance by open circuited impedance.

$$\frac{Z_{sc}}{Z_{oc}} = \frac{Z_0 \tanh \gamma l}{Z_0 \coth \gamma l}$$

$$= \tan^2 h \, \gamma l$$

$$\tanh \gamma l = \sqrt{\frac{Z_{sc}}{Z_{oc}}}$$

$$\gamma l = \tanh^{-1} \sqrt{\frac{Z_{sc}}{Z_{oc}}}$$

1.11. REFLECTION

When the load impedance is not equal to the characteristic impedance of transmission line, reflection takes place.

The expressions for voltage and current on the transmission line are

$$V = \frac{V_R}{2} \left[\left(1 + \frac{Z_0}{Z_R} \right) e^{\sqrt{ZY}\,x} + \left(1 - \frac{Z_0}{Z_R} \right) e^{-\sqrt{ZY}\,x} \right]$$

$$I = \frac{I_R}{2} \left[\left(1 + \frac{Z_R}{Z_0} \right) e^{\sqrt{ZY}\,x} + \left(1 - \frac{Z_R}{Z_0} \right) e^{-\sqrt{ZY}\,x} \right]$$

or

$$V = \frac{V_R}{2} \left[\frac{Z_R + Z_0}{Z_R} e^{\sqrt{ZY}\,x} + \frac{Z_R - Z_0}{Z_R} e^{-\sqrt{ZY}\,x} \right]$$

$$I = \frac{I_R}{2} \left[\frac{Z_R + Z_0}{Z_0} e^{\sqrt{ZY}\,x} - \frac{Z_R - Z_0}{Z_0} e^{-\sqrt{ZY}\,x} \right]$$

or

$$V = \frac{V_R (Z_R + Z_o)}{2 Z_R} \left[e^{\gamma x} + \left(\frac{Z_R - Z_o}{Z_R + Z_o} \right) e^{-\gamma x} \right]$$

$$I = \frac{I_R (Z_R + Z_o)}{2 Z_R} \left[e^{\gamma x} - \left(\frac{Z_R - Z_o}{Z_R + Z_o} \right) e^{-\gamma x} \right]$$

$$[\because \gamma = \sqrt{ZY}\,]$$

If the transmission line is not terminated with the characteristic impedance *i.e.*, $Z_R \neq Z_o$ (mismatch) the above expressions for voltage and current exist. It consists of two waves, one is moving in the forward (positive x) direction which is called incident wave and the other is moving in the opposite (negative x) direction which is called reflected ray. The term varying with $e^{\gamma x}$ represents a wave progressing from the sending end towards the receiving end and the amplitude decreasing with increased distance. The term varying with $e^{-\gamma x}$ represents a wave progressing from the receiving end towards the sending end, decreasing in amplitude with increased distance.

If the transmission line is terminated with characteristic impedance *i.e.*, $Z_R = Z_o$ (properly matched) then the voltage and current expressions are

$$V = V_R\, e^{\gamma x}$$

$$I = I_R\, e^{\gamma x}$$

The incident wave moves only in forward (positive x) direction. There is no reflected wave in the opposite direction.

1.11.1. Reflection Coefficient

Reflection coefficient is defined as the ratio of the reflected voltage to the incident voltage at the receiving end of the line.

$$K = \frac{\text{Reflected voltage at load}}{\text{Incident voltage at load}} = \frac{V_R}{V_S}$$

The equation for the voltage of a transmission line is

$$V = \frac{V_R (Z_R + Z_o)}{2 Z_R} \left[e^{\gamma x} + \left(\frac{Z_R - Z_o}{Z_R + Z_o} \right) e^{-\gamma x} \right]$$

$$V = \frac{V_R (Z_R + Z_o)}{2 Z_R}\, e^{\gamma x} + \frac{V_R (Z_R - Z_o)}{2 Z_R}\, e^{-\gamma x}$$

The first term ($e^{\gamma x}$) represents incident wave, whereas the second term ($e^{-\gamma x}$) represents the reflected wave. The ratio of amplitude of the reflected wave voltage to the amplitude of the incident wave voltage is nothing but reflection coefficient.

$$K = \frac{\frac{V_R (Z_R - Z_o)}{2 Z_R}}{\frac{V_R (Z_R + Z_o)}{2 Z_R}} = \frac{Z_R - Z_o}{Z_R + Z_o}$$

$$\boxed{K = \frac{Z_R - Z_o}{Z_R + Z_o}}$$

It is also defined as in terms of the ratio of the reflected current to the incident current. But it is negative.

$$-K = \frac{\text{Reflected current at load}}{\text{Incident current at load}} = \frac{I_R}{I_S}$$

If the transmission line is terminated by its characteristic impedance ($Z_R = Z_o$), the reflection coefficient becomes zero.

1.11.2. Reflection Factor and Reflection Loss

Consider a transmission line with a voltage source V_S and its impedance Z_1 and load impedance Z_2 as shown in Fig.1.7. If Z_2 is not equal to Z_1, reflection takes place. The power delivered to the load is less than that with impedance matching. Reflection results in power loss. This loss is known as reflection loss.

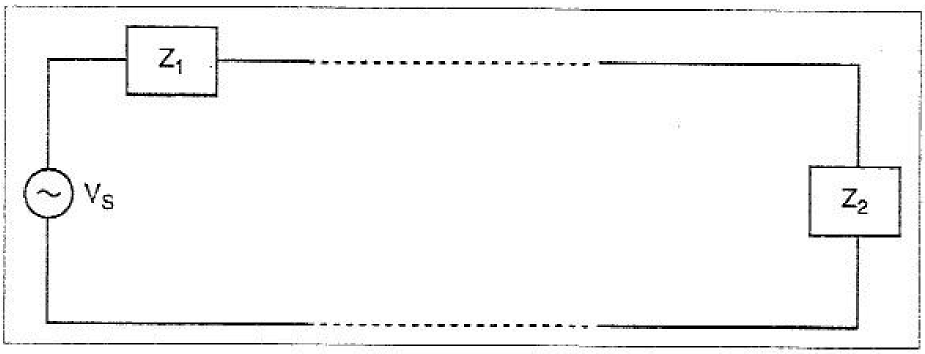


Fig. 1.7. Transmission line with voltage source V_S and impedance Z_1

Image matching between the impedances Z_1 and Z_2 can be obtained by inserting an ideal transformer and a phase shifting network between Z_1 and Z_2. If I_1 and I_2 be the currents in the primary and secondary of the transformer respectively, the current ratio of the transformer is given by

$$\frac{I_2}{I_1} = \sqrt{\frac{Z_1}{Z_2}}$$

Z_2 may be adjusted to that of Z_1 by choosing the proper transformation ratio and phase angle. Z_2 is the image impedance of Z_1. The current through the source is

$$I_1 = \frac{V_S}{2\,Z_1}$$

The current flow in the secondary of the transformer under image impedance matching is

$$I_2' = I_1\sqrt{\frac{Z_1}{Z_2}} = \frac{V}{2\,Z_1}\sqrt{\frac{Z_1}{Z_2}} = \frac{V_S}{2\sqrt{Z_1 Z_2}}$$

The current in the load impedance Z_2 without image matching.

$$|I_2| = \frac{|V_S|}{|Z_1 + Z_2|}$$

The ratio of the current actually flowing in the load to that which might flow under matched condition is known as ***reflection factor.***

$$\left|\frac{I_2}{I_2'}\right| = \frac{\dfrac{|V_S|}{|Z_1+Z_2|}}{\dfrac{|V_S|}{|2\sqrt{Z_1 Z_2}|}}$$

$$k = \left|\frac{2\sqrt{Z_1 Z_2}}{Z_1 + Z_2}\right|$$

The reflection factor indicates the change in current in the load due to reflection at the mismatched junction.

The ***reflection loss*** is the reciprocal of the reflection factor in nepers or dB.

$$\text{Reflection loss} = \ln\frac{1}{k}$$

$$= \ln\left|\frac{Z_1 + Z_2}{2\sqrt{Z_1 Z_2}}\right| \text{ nepers}$$

$$= 20\log\left|\frac{Z_1 + Z_2}{2\sqrt{Z_1 Z_2}}\right| \text{ dB}$$

1.12. T AND π SECTIONS EQUIVALENT TO LINES

A T section is shown in Fig.1.8 with two ports 1, 1 and 2, 2.

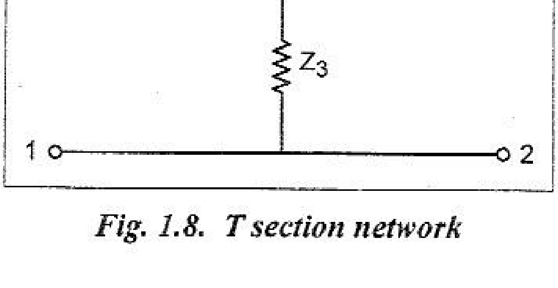


Fig. 1.8. T section network

Impedance measurements may be made at any port with the other port opened or shorted.

Let Z_{1OC} be the impedance at port 1 when port 2 is open circuited.

Z_{1SC} be the impedance at port 1 when port 2 is short circuited.

Z_{2OC} be the impedance at port 2 when port 1 is open circuited.

Z_{2SC} be the impedance at port 2 when port 1 is short circuited.

$$Z_{1OC} = Z_1 + Z_3$$

$$Z_{1SC} = Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3}$$

$$Z_{2OC} = Z_2 + Z_3$$

$$Z_{2SC} = Z_2 + \frac{Z_1 Z_3}{Z_1 + Z_3}$$

By solving these equations, the values of Z_1, Z_2 and Z_3 are determined.

$$Z_{1OC} - Z_{1SC} = Z_3 - \frac{Z_2 Z_3}{Z_2 + Z_3}$$

$$= \frac{Z_3 Z_2 + Z_3^2 - Z_2 Z_3}{Z_2 + Z_3}$$

$$= \frac{Z_3^2}{Z_2 + Z_3}$$

$$= \frac{Z_3^2}{Z_{2OC}} \qquad [\because Z_2 + Z_3 = Z_{2OC}]$$

$$Z_3^2 = Z_{2OC}(Z_{1OC} - Z_{1SC})$$

$$Z_3 = \pm\sqrt{Z_{2OC}(Z_{1OC} - Z_{1SC})}$$

Taking the positive value,

$$Z_3 = \sqrt{Z_{2OC}(Z_{1OC} - Z_{1SC})}$$

$$Z_1 = Z_{1OC} - Z_3 \qquad [\because Z_{1OC} = Z_1 + Z_3]$$

$$= Z_{1OC} - \sqrt{Z_{2OC}(Z_{1OC} - Z_{1SC})}$$

$$Z_2 = Z_{2OC} - Z_3 \qquad [\because Z_{2OC} = Z_2 + Z_3]$$

$$= Z_{2OC} - \sqrt{Z_{2OC}(Z_{1OC} - Z_{1SC})}$$

$$Z_1 = Z_{1OC} - \sqrt{Z_{2OC}(Z_{1OC} - Z_{1SC})}$$

$$Z_2 = Z_{2OC} - \sqrt{Z_{2OC}(Z_{1OC} - Z_{1SC})}$$

UNIT – V
Transmission Lines – II

- SC and OC Lines
- Input Impedance Relations
- Reflection Coefficient
- VSWR
- λ/4, λ 2, λ /8 Lines - Impedance Transformations
- Smith Chart - Configuration and Applications,
- Single Stub Matching
- Illustrative Problems.

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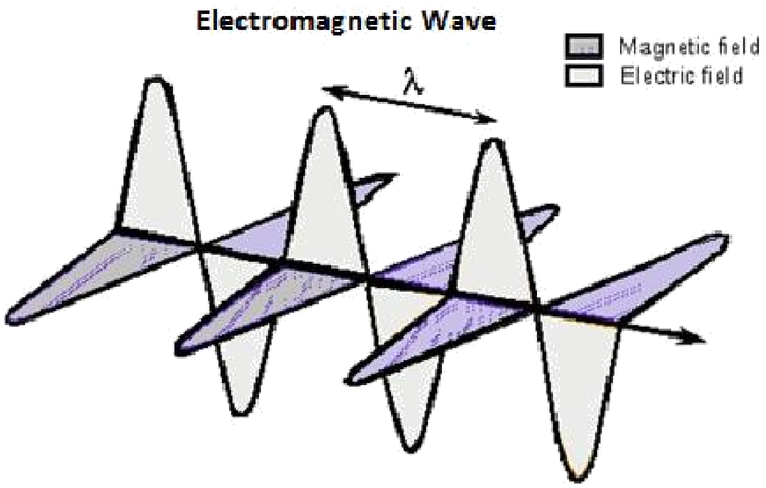
This means, more the current flows towards the surface of the conductor, it flows less towards the center, which is known as the **Skin Effect**.

Inductance

In an AC transmission line, the current flows sinusoidally. This current induces a magnetic field perpendicular to the electric field, which also varies sinusoidally. This is well known as Faraday's law. The fields are depicted in the following figure.



This varying magnetic field induces some EMF into the conductor. Now this induced voltage or EMF flows in the opposite direction to the current flowing initially. This EMF flowing in the opposite direction is equivalently shown by a parameter known as **Inductance**, which is the property to oppose the shift in the current.

It is denoted by "**L**". The unit of measurement is "**Henry** *H*".

Conductance

There will be a leakage current between the transmission line and the ground, and also between the phase conductors. This small amount of leakage current generally flows through the surface of the insulator. Inverse of this leakage current is termed as **Conductance**. It is denoted by "**G**".

The flow of line current is associated with inductance and the voltage difference between the two points is associated with capacitance. Inductance is associated with the magnetic field, while capacitance is associated with the electric field.

Capacitance

The voltage difference between the **Phase conductors** gives rise to an electric field between the conductors. The two conductors are just like parallel plates and the air in between them becomes dielectric. This pattern gives rise to the capacitance effect between the conductors.

Characteristic Impedance

If a uniform lossless transmission line is considered, for a wave travelling in one direction, the ratio of the amplitudes of voltage and current along that line, which has no reflections, is called as **Characteristic impedance**.

It is denoted by Z_0

$$Z_0 = \sqrt{\frac{voltage\ \ wave\ \ value}{current\ \ wave\ \ value}}$$

$$Z_0 = \sqrt{\frac{R + jwL}{G + jwC}}$$

For a lossless line, $R_0 = \sqrt{\frac{L}{C}}$

Where L & C are the inductance and capacitance per unit lengths.

Impedance Matching

To achieve maximum power transfer to the load, impedance matching has to be done. To achieve this impedance matching, the following conditions are to be met.

The resistance of the load should be equal to that of the source.

$$R_L = R_S$$

The reactance of the load should be equal to that of the source but opposite in sign.

$$X_L = -X_S$$

Which means, if the source is inductive, the load should be capacitive and vice versa.

Reflection Co-efficient

The parameter that expresses the amount of reflected energy due to impedance mismatch in a transmission line is called as **Reflection coefficient**. It is indicated by ρ *rho*.

It can be defined as "the ratio of reflected voltage to the incident voltage at the load terminals".

$$\rho = \frac{reflected\ voltage}{incident\ voltage} = \frac{V_r}{V_i}\ at\ load\ terminals$$

If the impedance between the device and the transmission line don't match with each other, then the energy gets reflected. The higher the energy gets reflected, the greater will be the value of ρ reflection coefficient.

Voltage Standing Wave Ratio $VSWR$

The standing wave is formed when the incident wave gets reflected. The standing wave which is formed, contains some voltage. The magnitude of standing waves can be measured in terms of standing wave ratios.

The ratio of maximum voltage to the minimum voltage in a standing wave can be defined as Voltage Standing Wave Ratio $VSWR$. It is denoted by "S".

$$S = \frac{|V_{max}|}{|V_{min}|} \quad 1 \leq S \leq \infty$$

VSWR describes the voltage standing wave pattern that is present in the transmission line due to phase addition and subtraction of the incident and reflected waves.

Hence, it can also be written as

$$S = \frac{1+\rho}{1-\rho}$$

The larger the impedance mismatch, the higher will be the amplitude of the standing wave. Therefore, if the impedance is matched perfectly,

$$V_{max} : V_{min} = 1 : 1$$

Hence, the value for VSWR is unity, which means the transmission is perfect.

Efficiency of Transmission Lines

The efficiency of transmission lines is defined as the ratio of the output power to the input power.

$$\%\ efficiency\ of\ transmission\ line\ \eta = \frac{Power\ delivered\ at\ reception}{Power\ sent\ from\ the\ transmission\ end} \times 100$$

Voltage Regulation

Voltage regulation is defined as the change in the magnitude of the voltage between the sending and receiving ends of the transmission line.

$$\%\ voltage\ regulation = \frac{sending\ end\ voltage - receiving\ end\ voltage}{sending\ end\ voltage} \times 100$$

Losses due to Impedance Mismatch

The transmission line, if not terminated with a matched load, occurs in losses. These losses are many types such as attenuation loss, reflection loss, transmission loss, return loss, insertion loss, etc.

Attenuation Loss

The loss that occurs due to the absorption of the signal in the transmission line is termed as Attenuation loss, which is represented as

$$Attenuation\ loss(dB) = 10\ log_{10}\left[\frac{E_i - E_r}{E_t}\right]$$

Where

- E_i = the input energy
- E_r = the reflected energy from the load to the input
- E_t = the transmitted energy to the load

Reflection Loss

The loss that occurs due to the reflection of the signal due to impedance mismatch of the transmission line is termed as Reflection loss, which is represented as

$$Reflection\ loss(dB) = 10\ log_{10}\left[\frac{E_i}{E_i - E_r}\right]$$

Where

- E_i = the input energy
- E_r = the reflected energy from the load

Transmission Loss

The loss that occurs while transmission through the transmission line is termed as Transmission loss, which is represented as

$$Transmission\ loss(dB) = 10\ log_{10}\frac{E_i}{E_t}$$

Where

- E_i = the input energy
- E_t = the transmitted energy

Return Loss

The measure of the power reflected by the transmission line is termed as Return loss, which is represented as

$$Return\ loss(dB) = 10\ log_{10}\frac{E_i}{E_r}$$

Where

- E_i = the input energy
- E_r = the reflected energy

Insertion Loss

The loss that occurs due to the energy transfer using a transmission line compared to energy transfer without a transmission line is termed as Insertion loss, which is represented as

$$Insertion\ loss(dB) = 10\ log_{10}\ \frac{E_1}{E_2}$$

Where

- E_1 = the energy received by the load when directly connected to the source, without a transmission line.
- E_2 = the energy received by the load when the transmission line is connected between the load and the source.

Stub Matching

If the load impedance mismatches the source impedance, a method called "Stub Matching" is sometimes used to achieve matching.

The process of connecting the sections of open or short circuit lines called **stubs** in the shunt with the main line at some point or points, can be termed as **Stub Matching**.

At higher microwave frequencies, basically two stub matching techniques are employed.

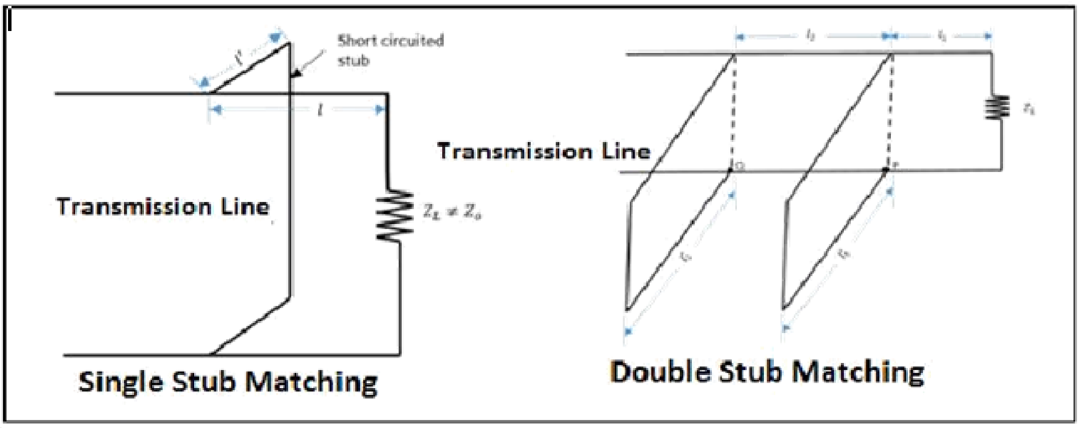
Single Stub Matching

In Single stub matching, a stub of certain fixed length is placed at some distance from the load. It is used only for a fixed frequency, because for any change in frequency, the location of the stub has to be changed, which is not done. This method is not suitable for coaxial lines.

Double Stub Matching

In double stud matching, two stubs of variable length are fixed at certain positions. As the load changes, only the lengths of the stubs are adjusted to achieve matching. This is widely used in laboratory practice as a single frequency matching device.

The following figures show how the stub matchings look.


Short circuited stub
Transmission Line
$Z_L \neq Z_o$
Single Stub Matching
Transmission Line
Double Stub Matching

Transmission Lines – Smith Chart & Impedance Matching (Intensive Reading)

1 The Smith Chart

Transmission line calculations − such as the determination of input impedance using equation (4.30) and the reflection coefficient or load impedance from equation (4.32) − often involves tedious manipulation of complex numbers. This tedium can be alleviated using a graphical method of solution. The best known and most widely used graphical chart is the Smith chart. The Smith chart is a circular plot with a lot of interlaced circles on it. When used correctly, impedance matching can be performed without any computation. The only effort required is the reading and following of values along the circles.

The Smith chart is a polar plot of the complex reflection coefficient, or equivalently, a graphical plot of normalized resistance and reactance functions in the reflection-coefficient plane. To understand how the Smith chart for a lossless transmission line is constructed, examine the voltage reflection coefficient of the load impedance defined by

$$\Gamma_L = \frac{V_{\text{refl}}}{V_{\text{inc}}} = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_{\text{re}} + j\Gamma_{\text{im}}, \tag{1}$$

where Γ_{re} and Γ_{im} are the real and imaginary parts of the complex reflection coefficient Γ_L. The characteristic impedance Z_0 is often a constant and a real industry normalized value, such as 50 Ω, 75 Ω, 100 Ω, and 600 Ω. We can then define the normalised load impedance by

$$z_L = Z_L / Z_0 = (R + jX)/Z_0 = r + jx. \tag{2}$$

With this simplification, we can rewrite the reflection coefficient formula in (1) as

$$\Gamma_L = \Gamma_{\text{re}} + j\Gamma_{\text{im}} = \frac{(Z_L - Z_0)/Z_0}{(Z_L + Z_0)/Z_0} = \frac{z_L - 1}{z_L + 1}. \tag{3}$$

The inverse relation of (3) is

$$z_L = \frac{1+\Gamma_L}{1-\Gamma_L} = \frac{1+|\Gamma_L|e^{j\theta}}{1-|\Gamma_L|e^{j\theta}} \tag{4}$$

or

$$r + jx = \frac{(1+\Gamma_{\text{re}}) + j\Gamma_{\text{im}}}{(1-\Gamma_{\text{re}}) - j\Gamma_{\text{im}}}. \tag{5}$$

Multiplying both the numerator and the denominator of (5) by the complex conjugate of the denominator and separating the real and imaginary parts, we obtain

$$r = \frac{1 - \Gamma_{\text{re}}^2 - \Gamma_{\text{im}}^2}{(1-\Gamma_{\text{re}})^2 + \Gamma_{\text{im}}^2} \tag{6}$$

and

$$x = \frac{2\Gamma_{\text{im}}^2}{(1-\Gamma_{\text{re}})^2 + \Gamma_{\text{im}}^2}. \tag{7}$$

Equation (6) can be rearranged as

$$\left(\Gamma_{\text{re}} - \frac{r}{1+r}\right)^2 + \Gamma_{\text{im}}^2 = \left(\frac{1}{1+r}\right)^2. \tag{8}$$

This equation is a relationship in the form of a parametric equation $(x-a)^2+(y-b)^2=R^2$ in the complex plane $(\Gamma_{re},\Gamma_{im})$ of a circle centred at the coordinates $\left(\frac{r}{r+1},0\right)$ and having a radius of $\frac{1}{r+1}$. Different values of r yield circles of different radii with centres at different positions on the Γ_{re}-axis. The following properties of the r-circles are noted:

- The centres of all r-circles lie on the Γ_{re}-axis.
- The circle where there is no resistance ($r = 0$) is the largest. It is centred at the origin and has a radius of 1.
- The r-circles become progressively smaller as r increases from 0 to ∞, ending at the $(\Gamma_{re}=1,\Gamma_{im}=0)$ point for an open circuit.
- All the r-circles pass through the point $(\Gamma_{re}=1,\Gamma_{im}=0)$.

See Figure 1 for further details.

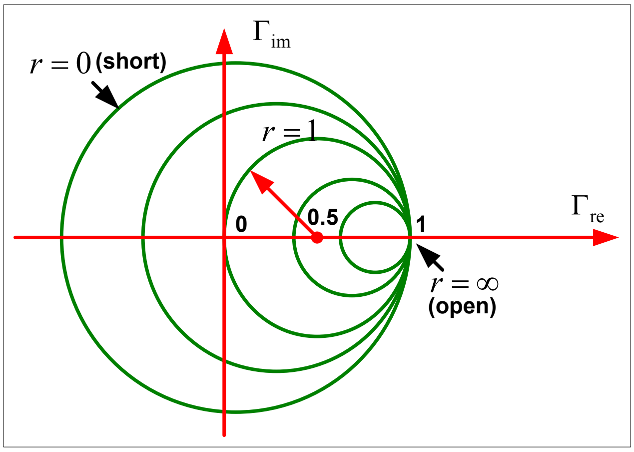


Figure 1: The r-circles in the complex plane $(\Gamma_{re},\Gamma_{im})$.

Similarly, (7) can be rearranged as

$$(\Gamma_{re}-1)^2+\left(\Gamma_{im}-\frac{1}{x}\right)^2=\left(\frac{1}{x}\right)^2. \tag{9}$$

Again, (9) is a parametric equation of the type $(x-a)^2+(y-b)^2=R^2$ in the complex plane (Γ_r,Γ_i) of a circle centred at the coordinates $\left(1,\frac{1}{x}\right)$ and having a radius of $\frac{1}{|x|}$. Different values of x yield circles of different radii with centres at different positions on the $\Gamma_{re}=1$ line. The following properties of the x-circles are noted:

- The centres of all x-circles lie on the $\Gamma_{re}=1$ line; those for $x>0$ (inductive reactance) lie above the Γ_{re}-axis, and those for $x<0$ lie below the Γ_{re}-axis.
- The $x = 0$ circle becomes the Γ_{re}-axis.
- The x-circles become progressively smaller as $|x|$ increases from 0 to ∞, ending at the $(\Gamma_{re}=1,\Gamma_{im}=0)$ point for an open circuit.
- All the x-circles pass through the point $(\Gamma_{re}=1,\Gamma_{im}=0)$.

See Figure 2 for further details.

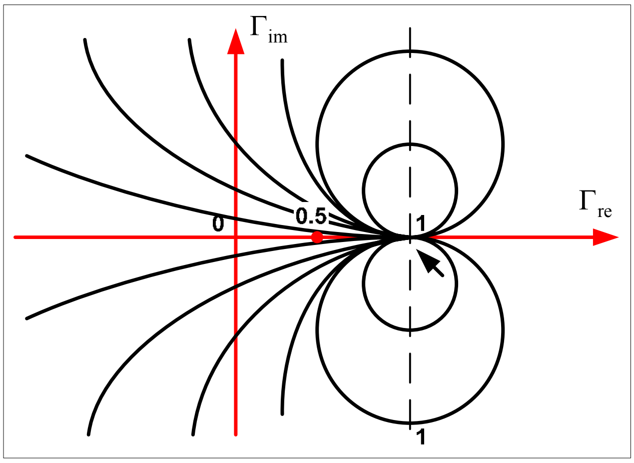


Figure 2: The x-circles in the complex plane $(\Gamma_{re}, \Gamma_{im})$.

To complete the Smith chart, the two circles' families are superimposed. The Smith chart therefore becomes a chart of r- and x-circles in the $(\Gamma_{re}, \Gamma_{im})$ -plane for $|\Gamma| \leq 1$. The intersection of an r-circle and an x-circle defines a point which represents a normalized load impedance $z_L = r + jx$. The actual load impedance is $Z_L = Z_0 z_L = Z_0 (r + jx)$. As an illustration, the impedance $Z_L = 85 + j30$ in a $Z_0 = 50\,\Omega$ -system is represented by the point P in Figure 3. Here $z_L = 1.7 + j0.6$ at the intersection of the $r = 1.7$ and the $x = 0.6$ circles. Values for Γ_{re} and Γ_{im} may then be obtained from the projections onto the horizontal and vertical axes (see Figure 4). These are approximately given by $\Gamma_{re} \approx 0.3$ and $\Gamma_{im} \approx 0.16$. Point P_{sc} at $(\Gamma_{re} = -1, \Gamma_{im} = 0)$ corresponds to $r = 0$ and $x = 0$ and therefore represents a short-circuit. P_{oc} at $(\Gamma_{re} = 1, \Gamma_{im} = 0)$ corresponds to an infinite impedance therefore represents an open circuit.

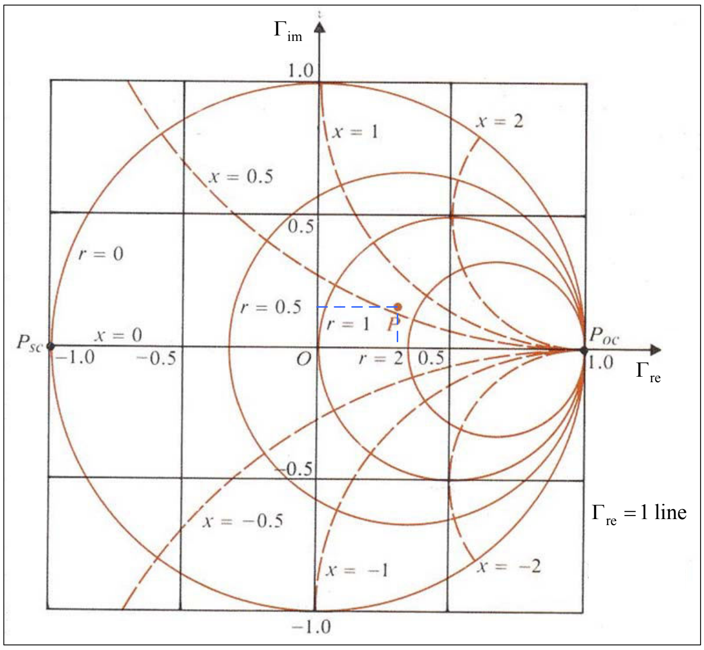


Figure 3: Smith chart with rectangular coordinates.

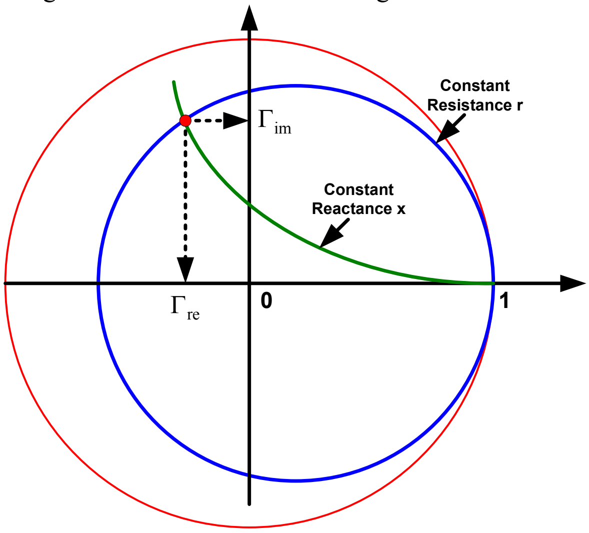


Figure 4: Direct extraction of the reflection coefficient $\Gamma = \Gamma_{re} + j\Gamma_{im}$ along the horizontal and vertical axes.

Instead of having a Smith chart marked with Γ_{re} and Γ_{im} marked in rectangular coordinates, the same chart can be marked in polar coordinates, so that every point in the Γ-plane is specified by a magnitude $|\Gamma|$ and a phase angle θ. This is illustrated in Figure 5, where several $|\Gamma|$-circles are shown in dashed lines and some θ-angles are marked around the $|\Gamma| = 1$ circle. The $|\Gamma|$-circles are normally not shown on commercially available Smith charts, but once the point representing a certain $z_L = r + jx$ is located, it is simply a matter of drawing a circle centred at the origin through the point. The ratio of the distance to the point and the radius to the edge of the chart is equal to the magnitude of $|\Gamma|$ of the load reflection coefficient, and the angle that a line to that point makes with the real axis represents θ. If, for example the point $z_L = 1.7 + j0.6$ is marked on the Smith chart at point *P*, we find that $|\Gamma_L| = 1/3$ and $\theta = 28°$.

Each $|\Gamma|$-circle intersects the real axis at two points. In Figure 5 we designate the point on the positive real axis as P_M and on the negative real axis as P_m. Since $x = 0$ along the real axis, both these points represent situations of a purely resistive load, $Z_L = R_L$. Obviously, $R_L > Z_0$ at P_M where $r > 1$, and $R_L < Z_0$ at P_m where $r < 1$. Since $S = R_L / Z_0$ for $R_L > Z_0$, the value of the *r*-circle passing through the point P_M is numerically equal to the standing wave ratio. For the example where $z_L = 1.7 + j0.6$, we find that $r = 2$ at P_M, so that $S = r = 2$.

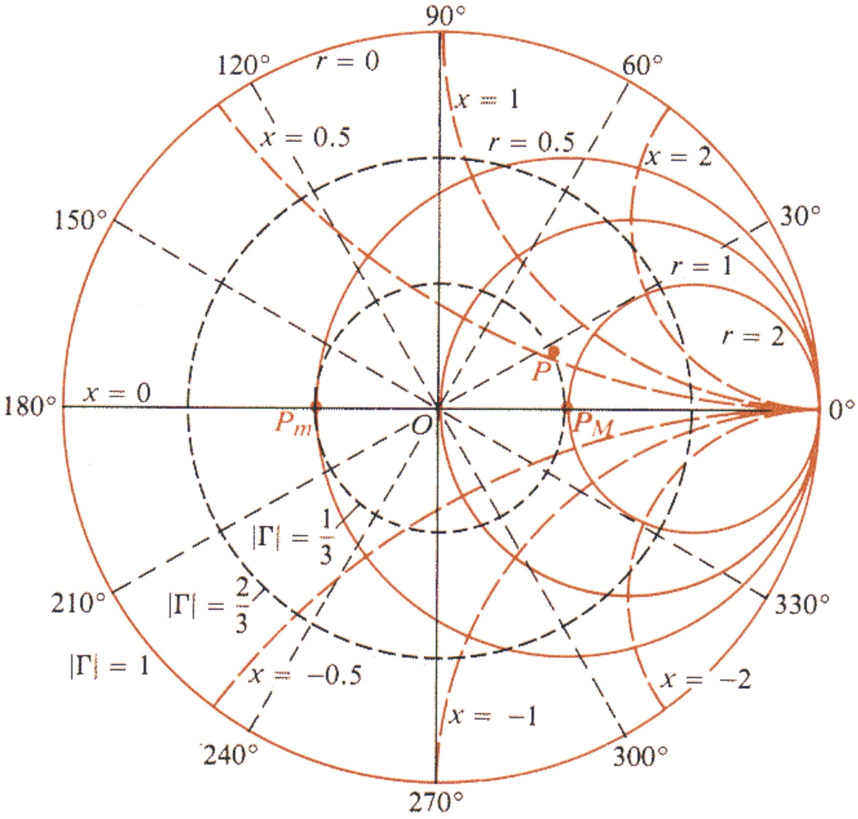


Figure 5: Smith chart in polar coordinates.

Example 1:

Consider a characteristic impedance of 50 Ω with the following impedances:

$Z_1 = 100 + j50\ \Omega$	$Z_2 = 75 - j100\ \Omega$	$Z_3 = j200\ \Omega$	$Z_4 = 150\ \Omega$
$Z_5 = \infty$ (an open circuit)	$Z_6 = 0$ (a short circuit)	$Z_7 = 50\ \Omega$	$Z_8 = 184 - j900\ \Omega$

The normalized impedances shown below are plotted in Figure 6.

$z_1 = 2 + j$	$z_2 = 1.5 - j2$	$z_3 = j4$	$z_4 = 3$
$z_5 = \infty$	$z_6 = 0$	$z_7 = 1$	$z_8 = 3.68 - j18$

It is also possible to directly extract the reflection coefficient Γ on the Smith chart of Figure 6. Once the impedance point is plotted (the intersection point of a constant resistance circle and of a constant reactance circle), simply read the rectangular coordinates projection on the horizontal and vertical axis. This will give Γ_{re}, the real part of the reflection coefficient, and Γ_{im}, the imaginary part of the reflection coefficient. Alternatively, the reflection coefficient may be obtained in polar form by using the scales provided on the commercial Smith chart.

$\Gamma_1 = 0.4 + 0.2j$ $= 0.45 \angle 27°$	$\Gamma_2 = 0.51 - 0.4j$ $= 0.65 \angle -38°$	$\Gamma_3 = 0.875 + 0.48j$ $= 0.998 \angle 29°$	$\Gamma_4 = 0.5$ $= 0.5 \angle 0°$
$\Gamma_5 = 1$ $= 1 \angle 0°$	$\Gamma_6 = -1$ $= 1 \angle 180°$	$\Gamma_7 = 0$ $= 0$	$\Gamma_8 = 0.96 - 0.1j$ $= 0.97 \angle -6°$

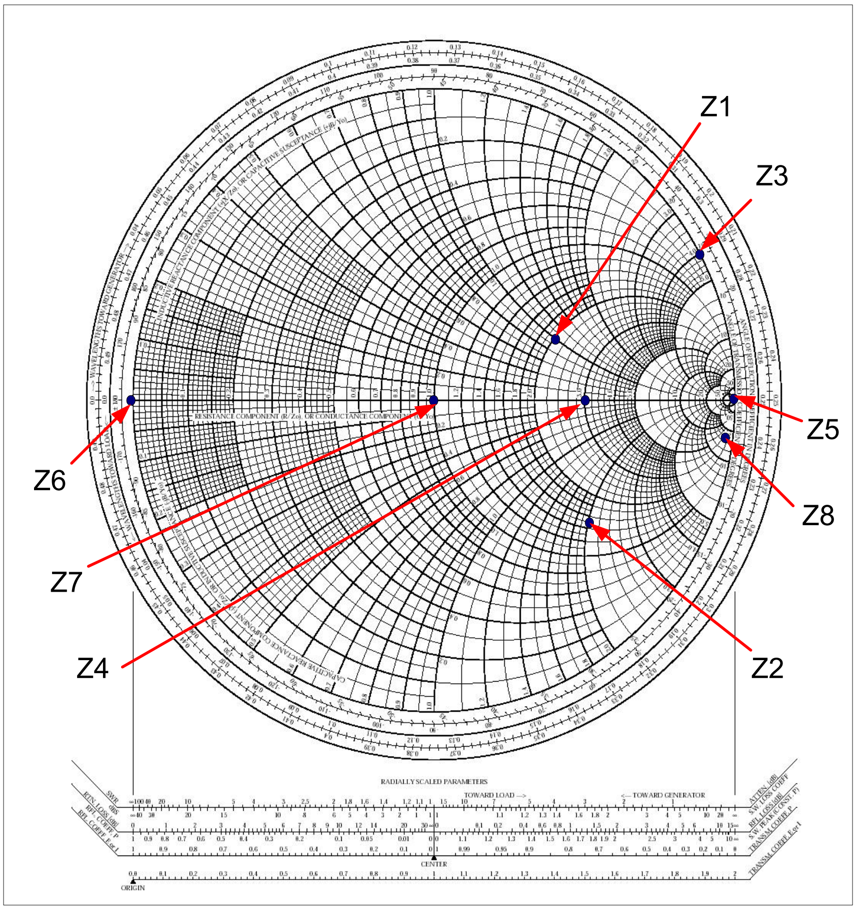


Figure 6: Points plotted on the Smith chart for Example 1.

The Smith chart is constructed by considering impedance (resistance and reactance). It can be used to analyse these parameters in both the series and parallel worlds. Adding elements in a series is straightforward. New elements can be added and their effects determined by simply moving along the circle to their respective values. However, summing elements in parallel is another matter, where admittances should be added.

We know that, by definition, $Y = 1/Z$ and $Z = 1/Y$. The admittance is expressed in mhos or Ω^{-1} or alternatively in Siemens or S. Also, as Z is complex, Y must also be complex. Therefore

$$Y = G + jB\,, \tag{10}$$

where G is called the conductance and B the susceptance of the element. When working with admittance, the first thing that we must do is normalize $y = Y/Y_0$. This results in $y = g + jb = 1/z$. So, what happens to the reflection coefficient? We note that

$$\Gamma = \frac{z-1}{z+1} = \frac{(z-1)/z}{(z+1)/z} = \frac{1-y}{1+y} = -\left(\frac{y-1}{1+y}\right). \qquad (11)$$

Thus, for a specific normalized impedance, say $z_1 = 1.7 + j0.6$, we can find the corresponding reflection coefficient as $\Gamma_1 = 0.33\angle 28°$. From (11), it then follows that the reflection coefficient for a normalized admittance of $y_2 = 1.7 + j0.6$ will be $\Gamma_2 = -\Gamma_1 = 0.33\angle(28° + 180°)$.

This also implies that for a specific normalized impedance *z*, we can find $y = 1/z$ by rotating through an angle of 180° around the centre of the Smith chart on a constant radius (see Figure 7).

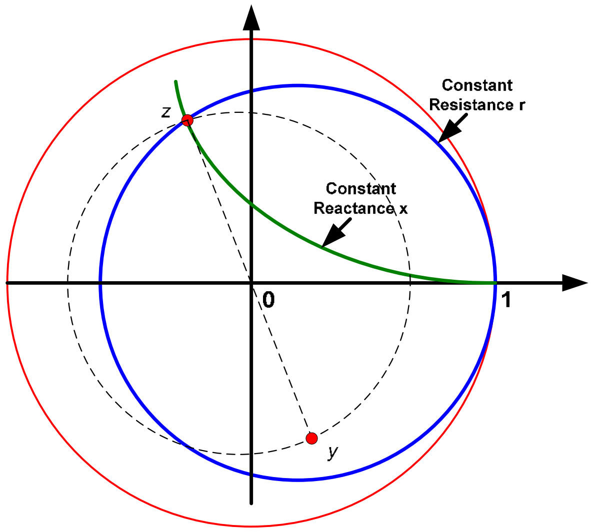


Figure 7: Results of the 180° rotation

Note that while *z* and *y* = 1/*z* represent the same component, the new point has a different position on the Smith chart and a different reflection value. This is due to the fact that the plot for *z* is an impedance plot, but for *y* it is an admittance plot. When solving problems where elements in series and in parallel are mixed together, we can use the same Smith chart by simply performing rotations where conversions from *z* to *y* or *y* to *z* are required.

2 Smith Charts and transmission line circuits

So far we have based the construction of the Smith chart on the definition of the voltage reflection coefficient at the load. The question is: what happens when we connect the load to a length of transmission line as in Figure 8.

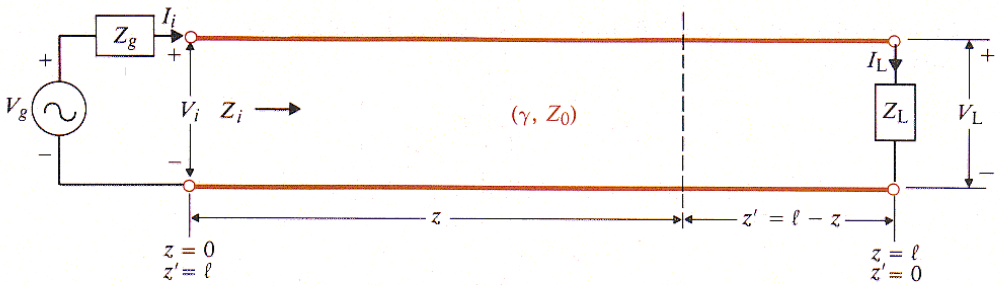


Figure 8: Finite transmission line terminated with load impedance Z_L.

On a lossless transmission line with $k = \beta$, the input impedance at a distance z' from the load is given by

$$Z_i = \frac{V(z')}{I(z')} = Z_0 \frac{1+\Gamma_L\, e^{-j2\beta z'}}{1-\Gamma_L\, e^{-j2\beta z'}} . \tag{12}$$

The normalised impedance is then

$$z_i = \frac{Z_i(z')}{Z_0} = \frac{1+\Gamma_L\, e^{-j2\beta z'}}{1-\Gamma_L\, e^{-j2\beta z'}} = \frac{1+\Gamma_i}{1-\Gamma_i} . \tag{13}$$

Consequently, the reflection coefficient seen looking into the lossless transmission line of length z' is given by

$$\Gamma_i = \Gamma_L\, e^{-j2\beta z'} = \left|\Gamma_L\right| e^{j\theta}\, e^{-j2\beta z'} \tag{14}$$

This implies that as we move along the transmission line towards the generator, the magnitude of the reflection coefficient does not change; the angle only changes from a value of θ at the load to a value of $(\theta - 2\beta z')$ at a distance z' from the load. On the Smith chart, we are therefore rotating on a constant $|\Gamma|$ circle. One full rotation around the Smith chart requires that $2\beta z' = 2\pi$, so that $z' = \pi/\beta = \lambda/2$ where λ is the wavelength on the transmission line.

Two additional scales in $\Delta z'/\lambda$ are usually provided along the perimeter of the $|\Gamma| = 1$ circle for easy reading of the phase change $2\beta\Delta z'$ due to a change in line length $\Delta z'$. The outer scale is marked in "wavelengths towards generator" in the clockwise direction (increasing z') and "wavelengths towards load" in the counter-clockwise direction (decreasing z'). Figure 9 shows a typical commercially available Smith chart.

Each $|\Gamma|$-circle intersects the real axis at two points. Refer to Figure 5. We designate the point on the positive real axis as P_M and on the negative real axis as P_m. Since $x = 0$ along the real axis, both these points represent situations of a purely resistive input impedance, $Z_i = R_i + j0$. Obviously, $R_i > Z_0$ at P_M where $r > 1$, and $R_i < Z_0$ at P_m where $r < 1$. At the point P_M we find that $Z_i = R_i = SZ_0$, while $Z_i = R_i = Z_0 / S$ at P_m. The point P_M on an impedance chart corresponds to the positions of a voltage maximum (and current minimum) on the transmission line, while P_m represents a voltage minimum (and current maximum). Given an arbitrary normalised impedance z, the value of the r-circle passing through the point P_M is numerically equal to the standing wave ratio. For the example, if $z = 1.7 + j0.6$, we find that $r = 2$ at P_M, so that $S = r = 2$.

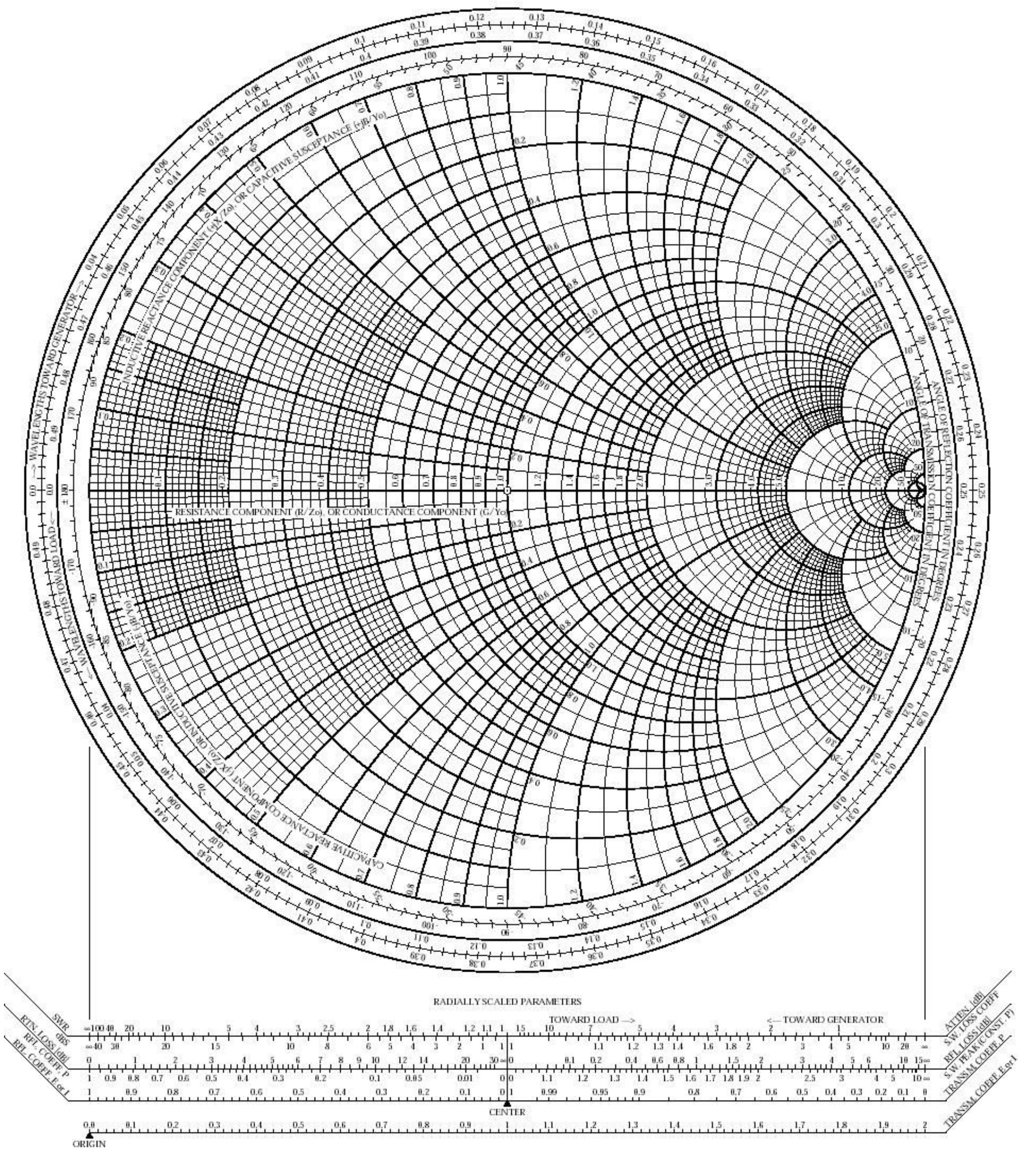


Figure 9: The Smith chart.

Example 2:

Use the Smith chart to find the impedance of a short-circuited section of a lossless 50 Ω co-axial transmission line that is 100 mm long. The transmission line has a dielectric of relative permittivity $\varepsilon_r = 9$ between the inner and outer conductor, and the frequency under consideration is 100 MHz.

For the transmission line, we find that $\beta = \omega\sqrt{\mu_0 \varepsilon_0 \varepsilon_r} = 6.2875$ rad/m and $\lambda = 2\pi / \beta = 0.9993 \approx 1$ m. The transmission line of length $z' = 100$ mm is therefore $z'/\lambda = 0.1$ wavelengths long.

- Since $z_L = 0$, enter the Smith chart at a point P_{sc}.
- Move along the perimeter of the chart ($|\Gamma| = 1$) by 0.1 "wavelengths towards the generator" in a clockwise direction to point P_1.
- At P_1, read $r = 0$ and $x \approx 0.725$, or $z_i = j0.725$. Then $Z_i = j0.725 \times 50 = j36.3\,\Omega$.

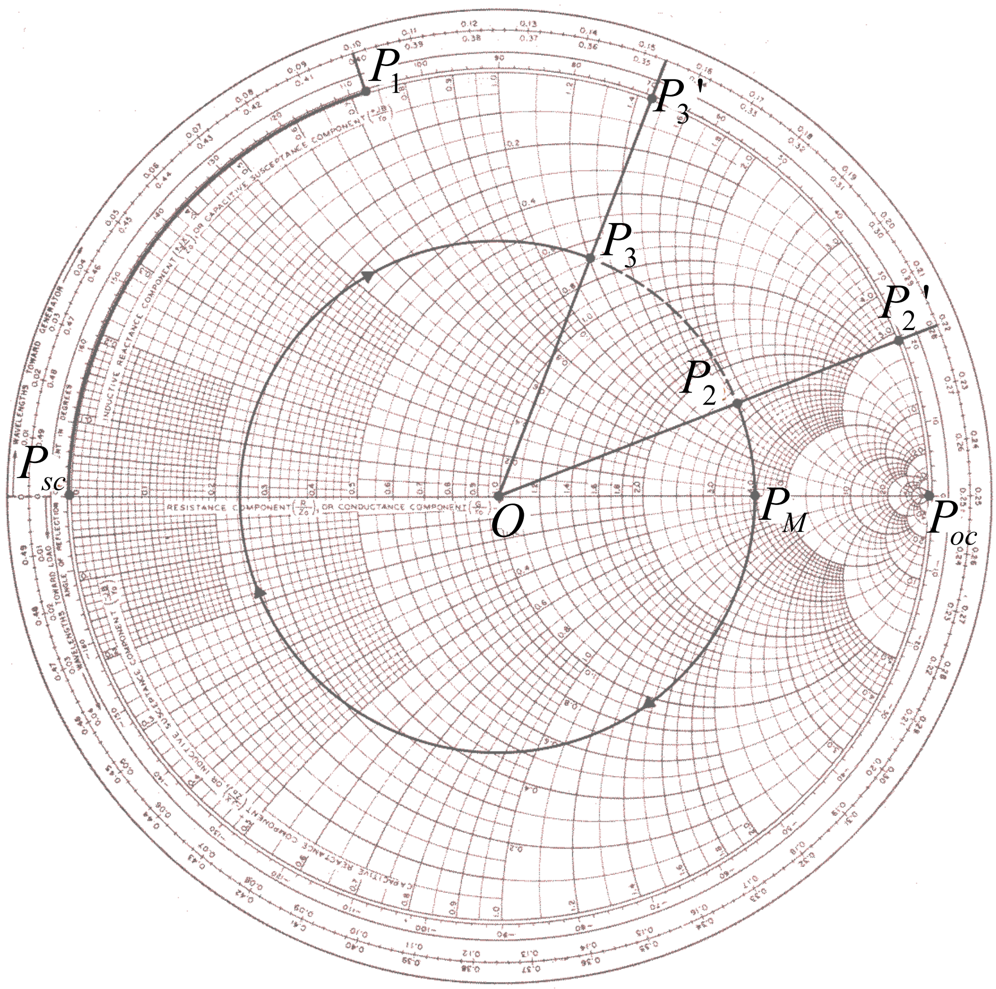


Figure 10: Smith chart calculations for Example 2 and Example 3.

Example 3: A lossless transmission line of length 0.434λ and characteristic impedance 100 Ω is terminated in an impedance 260 + j180 Ω. Find the voltage reflection coefficient, the standing-wave ratio, the input impedance, and the location of a voltage maximum on the line.

Given $z' = 0.434\lambda$, $Z_0 = 100\,\Omega$ and $Z_L = 260 + j180\,\Omega$. Then

- Enter the Smith chart at $z_L = Z_L / Z_0 = 2.6 + j1.8$ shown as point P_2 in Figure 10.
- With the centre at the origin, draw a circle of radius $\overline{OP_2} = |\Gamma_L| = 0.6$.
- Draw the straight line OP_2 and extend it to P_2' on the periphery. Read 0.220 on "wavelengths towards generator" scale. The phase angle θ of the load reflection may either be read directly from the Smith chart as 21° on the "Angle of Reflection Coefficient" scale. Therefore $\Gamma_L = 0.6\, e^{j21\pi/180} = 0.6\, e^{j0.12\pi}$.
- The $|\Gamma| = 0.6$ circle intersects the positive real axis OP_{sc} at $r = S = 4$. Therefore the voltage standing-wave ratio is 4.
- The find the input impedance, move P_2' at 0.220 by a total of 0.434 "wavelengths toward the generator" first to 0.500 (same as 0.000) and then further to 0.434–(0.500–0.220)=0.154 to P_3'.

- Join O and P_3' by a straight line which intersects the $|\Gamma| = 0.6$ circle at P_3. Here $r = 0.69$ and $x = 1.2$, or $z_i = 0.69 + j1.2$. Then $Z_i = (0.69 + j1.2) \times 100 = 69 + j120\,\Omega$.
- In going from P_2 to P_3, the $|\Gamma| = 0.6$ circle intersects the positive real axis at P_M where there is a voltage maximum. Thus the voltage maximum appears at 0.250–0.220=0.030 wavelengths from the load.

3 Transmission line impedance matching.

Transmission lines are often used for the transmission of power and information. For RF power transmission, it is highly desirable that as much power as possible is transmitted from the generator to the load and that as little power as possible is lost on the line itself. This will require that the load be matched to the characteristic impedance of the line, so that the standing wave ratio on the line is as close to unity as possible. For information transmission it is essential that the lines be matched, because mismatched loads and junctions will result in echoes that distort the information-carrying signal.

Impedance matching by quarter-wave transformer

For a lossless transmission line of length *l*, characteristic impedance of $Z_0 = R_0$ and terminated in a load impedance Z_L, the input impedance is given by

$$\begin{aligned} Z_i &= R_0 \frac{Z_L + jR_0 \tan \beta l}{R_0 + jZ_L \tan \beta l} \\ &= R_0 \frac{Z_L + jR_0 \tan(2\pi l/\lambda)}{R_0 + jZ_L \tan(2\pi l/\lambda)}. \end{aligned} \tag{15}$$

If the transmission line has a length of $l = \lambda/4$, this reduces to

$$\begin{aligned} Z_i &= R_0 \frac{Z_L + jR_0 \tan(\pi/2)}{R_0 + jZ_L \tan(\pi/2)} \\ &= R_0 \frac{Z_L/\tan(\pi/2) + jR_0}{R_0/\tan(\pi/2) + jZ_L} \\ &= R_0 \frac{0 + jR_0}{0 + jZ_L} \\ &= \frac{(R_0)^2}{Z_L}. \end{aligned} \tag{16}$$

This presents us with a simple way of matching a resistive load $Z_L = R_L$ to a real-valued input impedance of $Z_i = R_i$: insert a quarter-wave transformer with characteristic impedance R_0. From (16), we have $R_i = (R_0)^2 / R_L$, or

$$R_0 = \sqrt{R_i R_L}\,. \tag{17}$$

Note that the length of the transmission line has to be chosen to be equal to a quarter of a transmission line wavelength at the frequency where matching is desired. This matching method is therefore frequency sensitive, since the transmission line section will no longer be a quarter of a wavelength long at other frequencies. Also note that since the load is usually matched to a purely real impedance $Z_i = R_i$, this method of impedance matching can only be applied to resistive loads $Z_L = R_L$, and is not useful for matching complex load impedances to a lossless (or low-loss) transmission line.

Example 4

A signal generator has an internal impedance of 50 Ω. It needs to feed equal power through a lossless 50 Ω transmission line with a phase velocity of 0.5*c* to two separate resistive loads of

64 Ω and 25 Ω at a frequency of 10 MHz. Quarter-wave transformers are used to match the loads to the 50 Ω line, as shown in Figure 11.
(a) Determine the required characteristic impedances and physical lengths of the quarter-wavelength lines.
(b) Find the standing-wave ratios on the matching line sections.

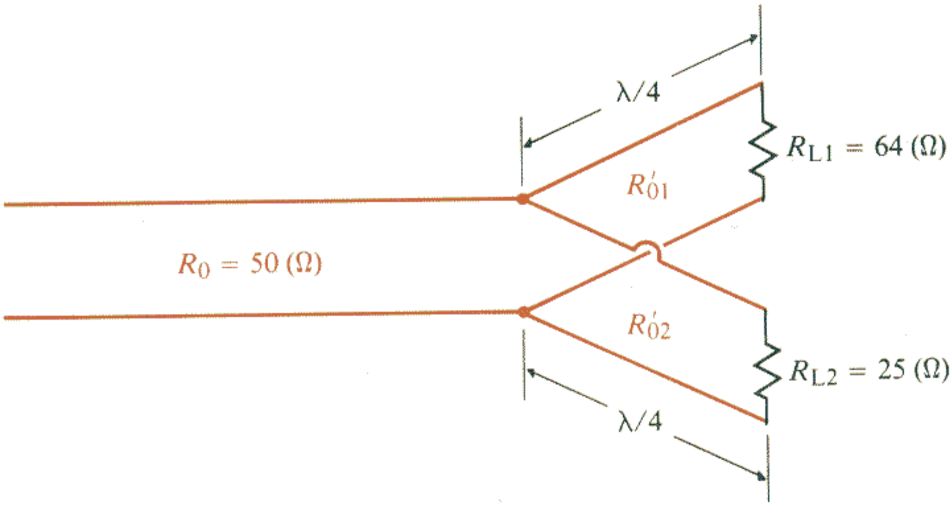


Figure 11: Impedance matching by quarter-wave transformers (Example 4).

(a) To feed equal power to the two loads, the input resistance at the junction with the main line looking toward each load must be

$$R_{i1} = 2R_0 = 100\,\Omega \qquad \text{and} \qquad R_{i2} = 2R_0 = 100\,\Omega$$

Therefore

$$R'_{01} = \sqrt{R_{i1}\,R_{L1}} = 80\,\Omega$$

$$R'_{02} = \sqrt{R_{i2}\,R_{L2}} = 50\,\Omega$$

Assume that the matching sections use the same dielectric as the main line. We know that

$$u_p = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\varepsilon_0\varepsilon_r}} = \frac{c}{2}.$$

We can therefore deduce that it uses a dielectric with a relative permittivity of $\varepsilon_r = 4$.

$$\lambda = \frac{u_p}{f} = \frac{2\pi}{k} = 15 \text{ m.}$$

The length of each transmission line section is therefore $l = \lambda/4 = 3.75$ m.

(b) Under matched conditions, there are no standing waves on the main transmission line, i.e. $S = 1$. The standing wave ratios on the two matching line sections are as follows:
Matching section No. 1:

$$\Gamma_{L1} = \frac{R_{L1} - R'_{01}}{R_{L1} + R'_{01}} = \frac{64-80}{64+80} = -0.11$$

$$S_1 = \frac{1+|\Gamma_{L1}|}{1-|\Gamma_{L1}|} = \frac{1+0.11}{1-0.11} = 1.25$$

Matching section No. 2:

$$\Gamma_{L2} = \frac{R_{L2} - R'_{02}}{R_{L2} + R'_{02}} = \frac{25 - 50}{25 + 50} = -0.33$$

$$S_2 = \frac{1 + |\Gamma_{L2}|}{1 - |\Gamma_{L2}|} = \frac{1 + 0.33}{1 - 0.33} = 1.99$$

Single stub matching

In matching of impedances, we are only allowed to use reactive components (i.e. equivalent to inductors and capacitors – no resistors). Recall that for short-circuited and open-circuited lossless transmission line sections of length *l*, the input impedance was given by

$$Z_{i,s} = jZ_0 \tan\beta l = jZ_0 \tan(2\pi\, l/\lambda)\,, \tag{18}$$

and

$$Z_{i,o} = -jZ_0 \cot\beta l = -jZ_0 \cot(2\pi\, l/\lambda)\,, \tag{19}$$

where $Z_0 = R_0$ is purely real. The impedances in (18) and (19) are purely reactive (imaginary), and therefore these transmission line sections act as inductors or capacitors, depending on the line length. We are going to make use of these elements (called transmission line *stubs*) to design matching circuits. In practice, it is more convenient to use short-circuited stubs. Short-circuited stubs are usually used in preference to open-circuited stubs because an infinite terminating impedance is more difficult to realise than a zero terminating impedance. Radiation from the open end of a stub makes it appear longer than it is, and compensation for these effects makes the use of open-circuited stubs more cumbersome. A short-circuited stub of an adjustable length is much easier to construct than an open-circuited stub.

It is also more common to connect these stubs in parallel with the main line. For parallel connections, it is convenient to use admittances rather than impedances. In thee cases, we use the Smith chart as an admittance chart to design the matching networks.

A single-stub matching circuit is depicted in Figure 12. Note that the short-circuited stub is connected in parallel with the main line. In order to match the complex load impedance Z_L to the characteristic impedance of the lossless main line, $Z_0 = R_0$, we need to determine the lengths *d* and *l*.

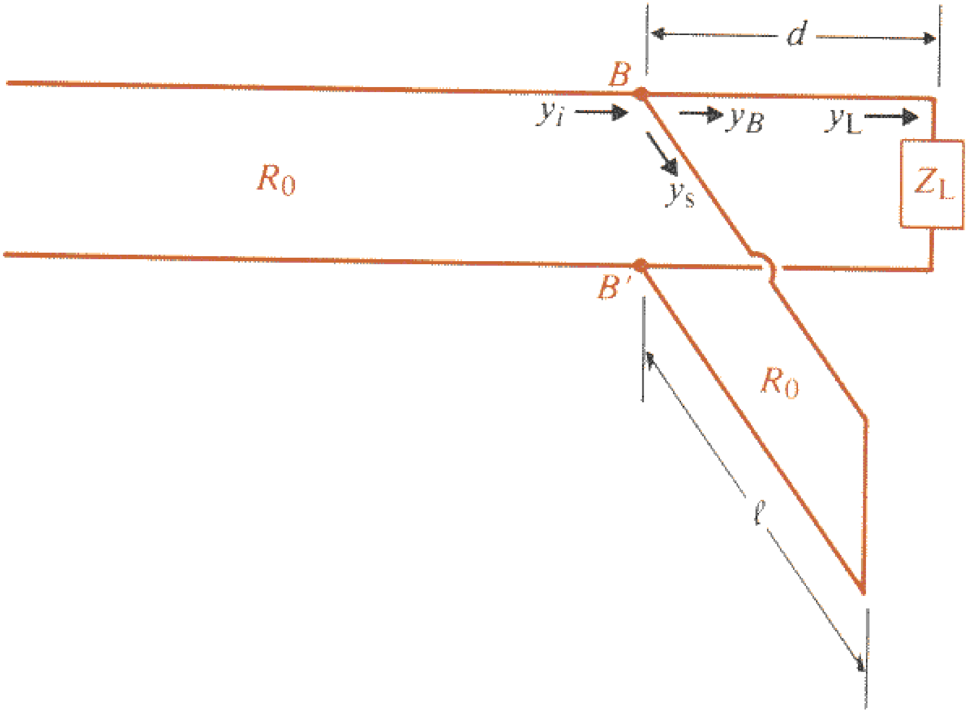


Figure 12: Impedance matching by single stub method.

For the transmission line to be matched at the point $B-B'$, the basic requirement is

$$Y_i = Y_B + Y_s = Y_0 = \frac{1}{R_0}. \tag{20}$$

In terms of normalised admittances, (23) becomes

$$y_i = y_B + y_s = 1. \tag{21}$$

where $y_B = g_B + jb_B = Y_B / Y_0$ for the load section and $y_s = Y_s / Y_0$ for the short-circuited stub. Note that $y_s = -j\cot(2\pi l/\lambda)$ is purely imaginary. It can therefore only contribute to the imaginary part of y_i. The position of $B-B'$ (or, in other words, the length *d*) must be chosen such that $g_B = 1$, i.e.

$$y_B = 1 + jb_B. \tag{22}$$

Next, the length *l* is chosen such that

$$y_s = -jb_B, \tag{23}$$

which yields $y_i = y_B + y_s = (1 + jb_B) + (-jb_B) = 1$. The circuit is therefore matched at $B-B'$, and at any point left of $B-B'$ as well.

If we use the Smith chart, we would rotate on a $|\Gamma|$-circle in a clockwise direction (towards the generator) when transforming the normalised load admittance to the admittance y_B. However, according to (23), y_B must also be located on the $g=1$ circle.

The use of the Smith chart for the purpose of designing a single-stub matching network is best illustrated by means of an example.

Example 5: A 50 Ω transmission line is connected to a load impedance $Z_L = 35 - j37.5\,\Omega$. Find the position and length of a short-circuited stub required to match the load at a frequency

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of 200 MHz. Assume that the transmission line is a co-axial line with a dielectric for which $\varepsilon_r = 9$.

Given $Z_0 = R_0 = 50\,\Omega$ and $Z_L = 35 - j47.5\,\Omega$. Therefore $z_L = Z_L / Z_0 = 0.7 - j0.95$.

- Enter the Smith chart at z_L shown as point P_1 in Figure 13.
- Draw a $|\Gamma|$-circle centred at O with radius $\overline{OP_1}$.
- Draw a straight line from P_1 through O to point P_2' on the perimeter, intersecting the $|\Gamma|$-circle at P_2 , which represents y_L . Note 0.109 at P_2' on the "wavelengths toward generator" scale.
- Note the two points of intersection of the $|\Gamma|$-circle with the $g = 1$ circle:
    - At P_3: $y_{B1} = 1 + j1.2 = 1 + jb_{B1}$
    - At P_4: $y_{B2} = 1 - j1.2 = 1 + jb_{B2}$
- Solutions for the position of the stub:
    - For P_3 (from P_2' to P_3') $d_1 = (0.168 - 0.109)\lambda = 0.059\lambda$
    - For P_4 (from P_2' to P_4') $d_2 = (0.332 - 0.109)\lambda = 0.223\lambda$
- Solutions for the length of the short-circuited stub to provide $y_s = -jb_B$:
    - For P_3 (from P_{sc} on the extreme right of the admittance chart to P_3'', which represents $y_s = -jb_{B1} = -j1.2$): $l_1 = (0.361 - 0.250)\lambda = 0.111\lambda$
    - For P_4 (from P_{sc} on the extreme right of the admittance chart to P_4'', which represents $y_s = -jb_{B2} = j1.2$): $l_2 = (0.139 + 0.250)\lambda = 0.389\lambda$

To compute the physical lengths of the transmission line sections, we need to calculate the wavelength on the transmission line. Therefore

$$\lambda = \frac{u_p}{f} = \frac{1/\sqrt{\mu\varepsilon}}{f} = \frac{c/\sqrt{\varepsilon_r}}{f} \approx 0.5\text{ m}.$$

Thus:

$d_1 = 0.059\lambda = 29.5$ mm	$l_1 = 0.111\lambda = 55.5$ mm
$d_2 = 0.223\lambda = 111.5$ mm	$l_2 = 0.389\lambda = 194.5$ mm

Note that either of these two sets of solutions would match the load. In fact, there is a whole range of possible solutions. For example, when calculating d_1, instead of going straight from P_2' to P_3', we could have started at P_2', rotated clockwise around the Smith chart n times (representing an additional length of $n\lambda/2$) and continued on to P_3' , yielding $d_1 = 0.059\lambda + n\lambda/2$, $n = 0, 1, 2, \ldots$ The same argument applies for d_2, l_1 and l_2.

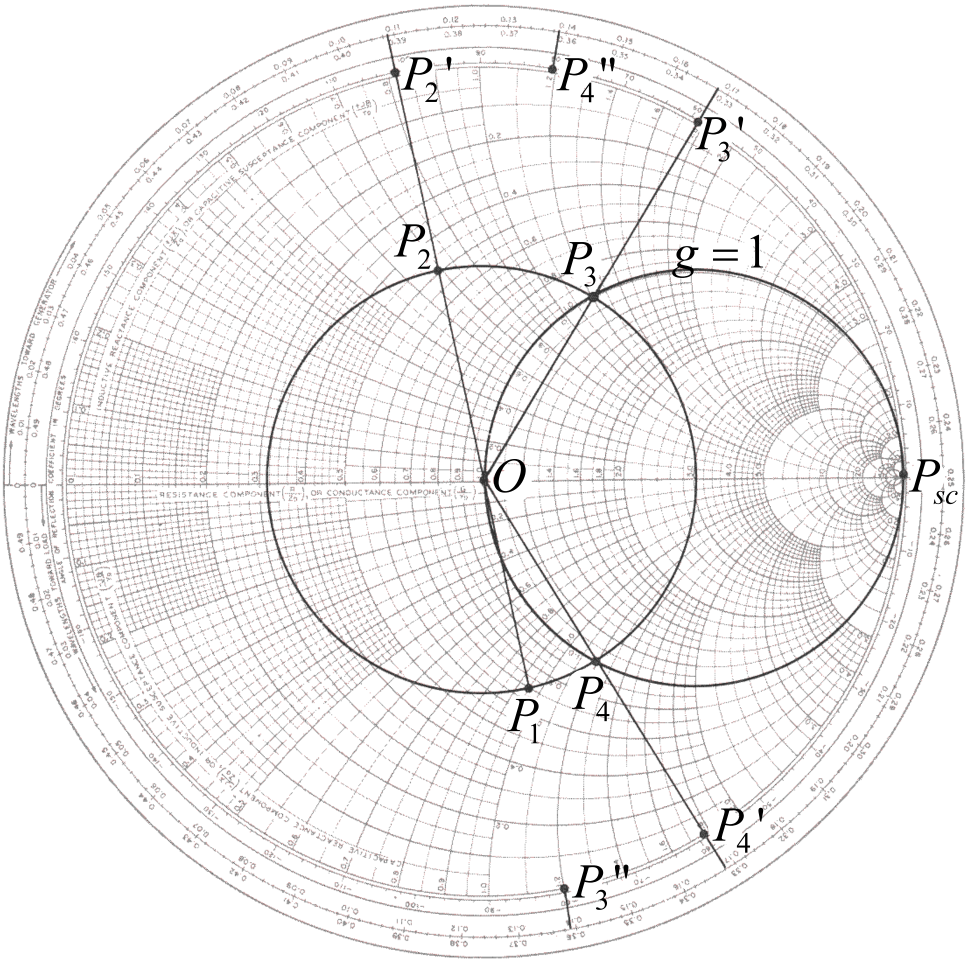

Figure 13: Single-stub matching on an admittance chart (Example 5).