UNIT-III

MAXWELL’S EQUATIONS (Time varying Fields)

Introduction:

In our study of static fields so far, we have observed that static electric fields are produced by electric charges, static magnetic fields are produced by charges in motion or by steady current. Further, static electric field is a conservative field and has no curl, the static magnetic field is continuous and its divergence is zero. The fundamental relationships for static electric fields among the field quantities can be summarized as:

$$\nabla \times \vec{E} = 0 \quad (1)$$

$$\nabla \cdot \vec{D} = \rho_v \quad (2)$$

For a linear and isotropic medium,

$$\vec{D} = \varepsilon \vec{E} \quad (3)$$

Similarly for the magnetostatic case

$$\nabla . \vec{B} = 0 \quad (4)$$

$$\nabla \times \vec{H} = \vec{J} \quad (5)$$

$$\nabla \times \vec{H} = \vec{J} \quad (6)$$

It can be seen that for static case, the electric field vectors $\vec{E}$ and $\vec{D}$ and magnetic field vectors $\vec{B}$ and $\vec{H}$ form separate pairs.

In this chapter we will consider the time varying scenario. In the time varying case we will observe that a changing magnetic field will produce a changing electric field and vice versa.

We begin our discussion with Faraday's Law of electromagnetic induction and then present the Maxwell's equations which form the foundation for the electromagnetic theory.

Maxwell's equations represent one of the most elegant and concise ways to state the fundamentals of electricity and magnetism. From them one can develop most of the working relationships in the field. Because of their concise statement, they embody a high level of mathematical sophistication and are therefore not generally introduced in an introductory treatment of the subject, except perhaps as summary relationships.

These basic equations of electricity and magnetism can be used as a starting point for advanced courses, but are usually first encountered as unifying equations after the study of electrical and magnetic phenomena.

Symbols Used		
E = Electric field	ρ = charge density	i = electric current
B = Magnetic field	ε0 = permittivity	J = current density
D = Electric displacement	μ0 = permeability	c = speed of light
H = Magnetic field strength	M = Magnetization	P = Polarization

Integral form in the absence of magnetic or polarizable media:

I. Gauss' law for electricity $\oint \vec{E} \cdot d\vec{A} = \frac{q}{\varepsilon_0}$

Gauss' law for magnetism $\oint \vec{B} \cdot d\vec{A} = 0$

III. Faraday's law of induction $\oint \vec{E} \cdot \vec{ds} = -\frac{d\Phi_B}{dt}$

IV. Ampere's law

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i + \frac{1}{c^2} \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{A}$$

Differential form in the absence of magnetic or polarizable media:

I. Gauss' law for electricity $\nabla \cdot E = \frac{\rho}{\varepsilon_0} = 4\pi k \rho$

Gauss' law for magnetism $\nabla \cdot B = 0$

III. Faraday's law of induction $\nabla x\, E = -\frac{\partial B}{\partial t}$

IV. Ampere's law

$$\nabla\, x\; B = \frac{4\pi k}{c^2} J + \frac{1}{c^2} \frac{\partial E}{\partial t} = \frac{J}{\varepsilon_0 c^2} + \frac{1}{c^2} \frac{\partial E}{\partial t}$$

$k = \frac{1}{4\pi\varepsilon_0} =$ *Coulomb's constant* $\qquad c^2 = \frac{1}{\mu_0 \varepsilon_0}$

Differential form with magnetic and/or polarizable media:

I. Gauss' law for electricity $\nabla \cdot D = \rho$

$$D = \varepsilon_0 E + P$$ General case

$$D = \varepsilon_0 E$$ *Free space*

$$D = \varepsilon E$$ *Isotropic linear dielectric*

118

II. Gauss' law for magnetism $\nabla \cdot B = 0$

III. Faraday's law of induction $\nabla x\, E = -\dfrac{\partial B}{\partial t}$

IV. Ampere's law $\nabla \, x \, H = J + \dfrac{\partial D}{\partial t}$

$$B = \mu_0 (H + M)$$ General case

$$B = \mu_0 H$$ *Free space*

$$B = \mu H$$ *Isotropic linear magnetic medium*

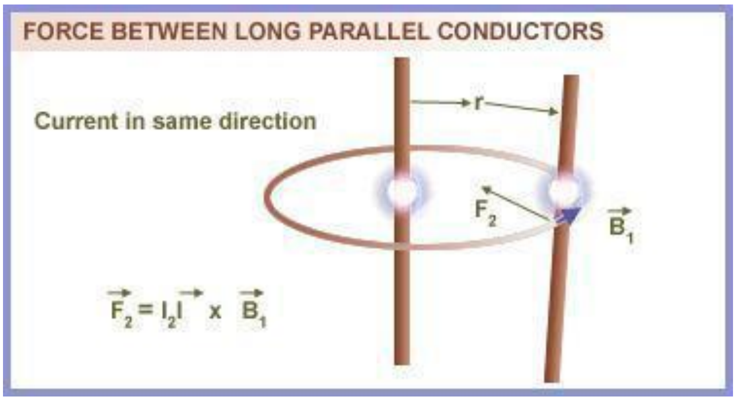
Faraday's Law:

Michael Faraday, in 1831 discovered experimentally that a current was induced in a conducting loop when the magnetic flux linking the loop changed. In terms of fields, we can say that a time varying magnetic field produces an electromotive force (emf) which causes a current in a closed circuit. The quantitative relation between the induced emf (the voltage that arises from conductors moving in a magnetic field or from changing magnetic fields) and the rate of change of flux linkage developed based on experimental observation is known as Faraday's law.

Any change in the magnetic environment of a coil of wire will cause a voltage (emf) to be "induced" in the coil. No matter how the change is produced, the voltage will be generated. The change could be produced by changing the magnetic field strength, moving a magnet toward or away from the coil, moving the coil into or out of the magnetic field, rotating the coil relative to the magnet, etc.

Faraday's law is a fundamental relationship which comes from Maxwell's equations. It serves as a succinct summary of the ways a voltage (or emf) may be generated by a changing magnetic environment. The induced emf in a coil is equal to the negative of the rate of change of magnetic flux times the number of turns in the coil. It involves the interaction of charge with magnetic field.

When two current carrying conductors are placed next to each other, we notice that each induces a force on the other. Each conductor produces a magnetic field around itself (Biot– Savart law) and the second experiences a force that is given by the Lorentz force.



Mathematically, the induced emf can be written as

$$\text{Emf} = -\frac{d\phi}{dt} \text{ Volts}$$

where ϕ is the flux linkage over the closed path.

A non zero $\frac{d\phi}{dt}$ may result due to any of the following:

(a) time changing flux linkage a stationary closed path.

(b) relative motion between a steady flux a closed path.

(c) a combination of the above two cases.

The negative sign in equation (7) was introduced by Lenz in order to comply with the polarity of the induced emf. The negative sign implies that the induced emf will cause a current flow in the closed loop in such a direction so as to oppose the change in the linking magnetic flux which produces it. (It may be noted that as far as the induced emf is concerned, the closed path forming a loop does not necessarily have to be conductive).

If the closed path is in the form of N tightly wound turns of a coil, the change in the magnetic flux linking the coil induces an emf in each turn of the coil and total emf is the sum of the induced emfs of the individual turns, i.e.,

$$\text{Emf} = -N\frac{d\phi}{dt} \text{ Volts}$$

By defining the total flux linkage as

$$\lambda = N\phi$$

The emf can be written as

Emf = $-\frac{d\lambda}{dt}$

Continuing with equation (3), over a closed contour 'C' we can write

Emf = $\oint_C \vec{E}.d\vec{l}$

where $\vec{E}$ is the induced electric field on the conductor to sustain the current.

Further, total flux enclosed by the contour 'C ' is given by

$$\phi = \int_s \vec{B}.d\vec{s}$$

Where S is the surface for which 'C' is the contour.

From (11) and using (12) in (3) we can write

$$\oint_C \vec{E}.d\vec{l} = -\frac{\partial}{\partial t}\oint_s \vec{B}.d\vec{s}$$

By applying stokes theorem

$$\int_s \nabla\times\vec{E}.d\vec{s} = -\int_s \frac{\partial\vec{B}}{\partial t}.d\vec{s}$$

Therefore, we can write

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

which is the Faraday's law in the point form

We have said that non zero $\frac{d\phi}{dt}$ can be produced in a several ways. One particular case is when a time varying flux linking a stationary closed path induces an emf. The emf induced in a stationary closed path by a time varying magnetic field is called a transformer emf .

Inconsistency of amperes law

Ampere's circuit law states that the line integral of tangential component of H around a closed path is same as the net current Ienc enclosed by the path.

i.e.

$$\int H.dl = I_{enc}$$

By applying stoke's theorem,

$$\int H.dl \; becomes \int_s J.ds$$

$\therefore$ Therefore, $\Delta \times H = J$ ______ (3.14)

This is true in case of static EM fields.

But in case of time-varying fields, the above Ampere's law shows same inconsistency.

The inconsistency of ampere law for time varying fields is shown in two cases:

1. For static EM fields, we have

$$\Delta \times H = J$$

Applying divergence on both sides, we get,

$$\Delta.(\Delta \times H) = \Delta.J$$

But divergence of curl of a vector field is always zero.

Therefore,

$$\Delta.(\Delta \times H) = 0 = \Delta.J$$

The continuity of current equation is given by

$$\Delta.J = \frac{-dp_v}{dt}$$

Where J = Current density
e_v = Charge density

For static fields, no current is produced, therefore, $e_v = 0 \Rightarrow \Delta.J = 0$

Implies eq. 3.15 is satisfied but for time varying fields, current is produced and therefore,

$$\nabla . J = \frac{-de_v}{dt} \# 0 \_\_\_\_\_\_ \_\_\_\_\_\_\_\_\_\_\_\_\_\_ (3.16)$$

Eq. (3.15) and eq. (3.16) are contradicting each other.

This is an inconsistency of ampere's law and the Ampere's law must be modified for time varying fields.

2. Consider the typical example of where the surface passes between the capacitor plates.

The figure is shown below.

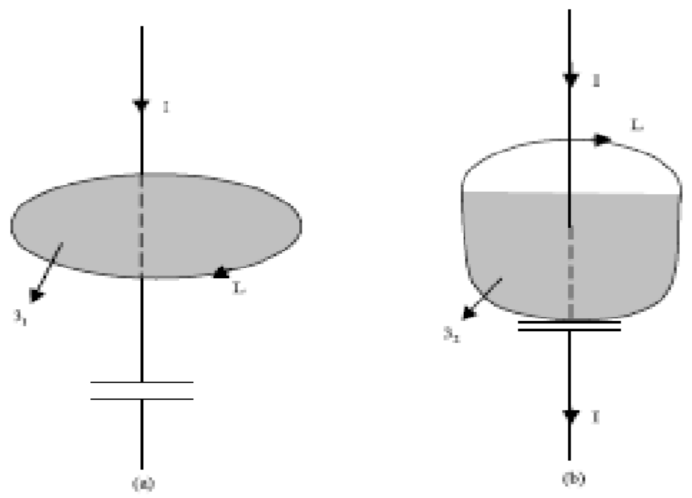


Fig 3.3 (a): Two surfaces of integration which explain the inconsistency of Ampere's law

In fig 3.3(a),

Based on Ampere's circuit law we get figure

$$\oint_L H.dl = \int_{S_1} J.ds = I_{enc} = I \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (3.17)$$

In fig 3.3(b), based the ampere's circuit law, we get,

$$\oint_L H.dl = \int_{3_2} J.ds = I_{enc} = 0 \quad \_\_\_\_\_\_\_\_\_\_ \quad (3.18)$$

Because no conduction current flows through 3_2

i.e. J=0

in both (a) and (b), same closed path is used, but equations 3.17 and 3.18 are different.

This is an inconsistency of Ampere's circuit law.

This inconsistency of Ampere's circuit law in both cases (1) and (2) can be resolved by including displacement current in Ampere's circuit law.

Substituting in (3.19), we get,

$$\Delta \times H = J + \frac{dD}{dt} \quad \_\_\_\_\_\_\_\_\_\_ \quad (3.21)$$

This is the Maxwell equation (based on ampere's circuit Law) for tiem varying fields.

In equation (3.21),

J_d = Displacement current density

$J =$ Conduction current density,

The conduction current density J involves flow of charges. The displacement current density J_d does not involve flow of charges. Displacement current,

$$I_d = \int Jd.ds = \int \frac{do}{dt}.ds \quad \_\_\_\_\_\_\_\_\_\_ \quad (3.22)$$

Displacement Current Density:

The equation

$\Delta \times H = J$ For static EM fields is modified to Modified to

$\Delta \times H = J + J_d$ ________________ (3.19)

To make the Ampere's law compatible for varying fields.

Now, applying divergence, we get

$$\Delta.(\Delta \times H) = 0 = \Delta.J + \Delta J_d$$

$$\Delta.J_d = -\Delta.J = \frac{de_v}{dt}$$

From Gauss Law, we have

$$e_v = \Delta.D$$

Therefore,

$$\Delta.J_d = \frac{d(\Delta.D)}{dt} = \Delta.\frac{dD}{dt}$$

$\Rightarrow J_d = \frac{dD}{dt}$ ____________ (3.20)

Boundary Condition for Magnetic Fields:

Similar to the boundary conditions in the electro static fields, here we will consider the behavior of $\vec{B}$ and $\vec{H}$ at the interface of two different media. In particular, we determine how the tangential and normal components of magnetic fields behave at the boundary of two regions having different permeabilities.

The figure 4.9 shows the interface between two media having permeabities μ_1 and μ_2, $\hat{a}_n$ being the normal vector from medium 2 to medium 1.

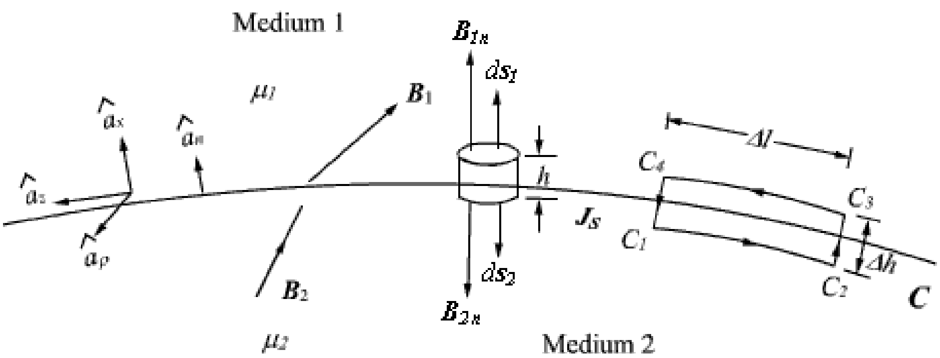


Figure 4.9: Interface between two magnetic media

o determine the condition for the normal component of the flux density vector $\vec{B}$, we consider a small pill box P with vanishingly small thickness *h* and having an elementary area ΔS for the faces. Over the pill box, we can write

$$\oint_S \vec{B}.d\vec{s} = 0 \quad \text{...(4.36)}$$

Since h --> 0, we can neglect the flux through the sidewall of the pill box.

$$\therefore \int_{\Delta S} \vec{B}_1.d\vec{S}_1 + \int_{\Delta S} \vec{B}_2.d\vec{S}_2 = 0 \quad \text{...........................(4.37)}$$

$$d\vec{S}_1 = dS\hat{a}_n \text{ and } d\vec{S}_2 = dS\left(-\hat{a}_n\right) \quad \text{..................(4.38)}$$

$$\therefore \int_{\Delta S} B_{1n} dS - \int_{\Delta S} B_{2n} dS = 0$$

where

Since ΔS is small, we can write

$$(B_{1n} - B_{2n})\Delta S = 0$$

or, $B_{1n} = B_{2n}$ ....(4.40)

That is, the normal component of the magnetic flux density vector is continuous across the interface.

In vector form,

$$\hat{a}_n \cdot \left(\vec{B}_1 - \vec{B}_2\right) = 0$$ ....(4.41)

To determine the condition for the tangential component for the magnetic field, we consider a closed path C as shown in figure 4.8. By applying Ampere's law we can write

Since h -->0 , $\int_{c_1-c_2} \vec{H}.d\vec{l} + \int_{c_3-c_4} \vec{H}.d\vec{l} = I$

We have shown in figure 4.8, a set of three unit vectors $\hat{a}_n$, $\hat{a}_t$ and $\hat{a}_\rho$ such that they satisfy $\hat{a}_t = \hat{a}_\rho \times \hat{a}_n$ (R.H. rule). Here $\hat{a}_t$ is tangential to the interface and $\hat{a}_\rho$ is the vector perpendicular to the surface enclosed by C at the interface.

$$\oint \vec{H}.d\vec{l} = I$$

if $J_s = 0$, the tangential magnetic field is also continuous. If one of the medium is a perfect conductor J_s exists on the surface of the perfect conductor.

In vector form we can write,

$$\left(\vec{H}_1 - \vec{H}_2\right).\hat{a}_t \,\Delta l$$

$$= \left(\vec{H}_1 - \vec{H}_2\right).\left(\hat{a}_\rho \times \hat{a}_n\right)\Delta l$$

$$= J_{Sn\Delta l} = \vec{J}_S.\hat{a}_\rho \,\Delta l$$

Therefore,

$$\hat{a}_n \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_S$$

Solved problems:

Problem1:

(a) In a cylindrical conductor to the region $0.01 \le r \le 0.02$, $0 < z < 1$ m and the current density is given by,

$$\vec{J} = 10e^{-100r}\hat{a}_\phi \text{ A/m}^2$$

Find the total current crossing the extential of this region with φ = constant plane.

(b) Find the total current in a circular conductor of 4 mm radius if the current density varies according to $J = \frac{10^4}{r}$ A/m^2.

Solution

(a) Total current in the wire is given as,

$$I = \int_S \vec{J}.d\vec{S} = \int_{r=0.01}^{0.02}\int_{z=0}^{1}\left[10e^{-100r}\hat{a}_\phi\right].\left[rdrdz\hat{a}_\phi\right]$$

$$= \int_{r=0.01}^{0.02}\int_{z=0}^{1} 10re^{-100r}drdz$$

$$= 10\int_{r=0.01}^{0.02} re^{-100r}dr$$

$$I = 10\left[\frac{re^{-100r}}{-100}\Bigg|_{0.01}^{0.02} - \int_{r=0.01}^{0.02}\frac{e^{-100r}}{-100}dr\right]$$

$$= 10\left[-\frac{1}{100}(0.02e^{-2} - 0.01e^{-1}) + \frac{e^{-100r}}{-100\times 100}\Bigg|_{0.01}^{0.02}\right]$$

$$= 2\times 10^{-3}e^{-1}$$

$$= 310^{-3}e^{-2}$$

(b) Total current is given as,

$$I = \int_S \vec{J}.d\vec{S} = \int_{\phi=0}^{2\pi}\int_{r=0}^{0.004}\frac{10^4}{r}rdrd\phi = 2\pi\times 10^4\int_{r=0}^{0.004}dr = 2\pi\times 10^4\times 0.004 = 80\pi \text{ A}$$

Problem2:

If $\vec{J} = \frac{1}{r^3}(2\cos\theta\, \hat{a}_r + \sin\theta\, \hat{a}_\theta)$ A/m^2, calculate the current passing through

(a) A hemispherical shell of 20 cm radius

(b) A spherical shell of 10 cm radius

Solution

Total current is given as $I = \int \vec{J}.d\vec{S}$

Here, $d\vec{S} = r^2 \sin\theta\, d\phi\, d\theta\, \hat{a}_r$

(a) Total current passing through a hemispherical shell of 20 cm radius is,

$$I = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \frac{1}{r^3}(2\cos\theta\, \hat{a}_r + \sin\theta\, \hat{a}_\theta).(r^2 \sin\theta\, d\phi\, d\theta\, \hat{a}_r)\Bigg|_{r=0.2}$$

$$= \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \frac{1}{r^3} 2\cos\theta r^2 \sin\theta\, d\phi\, d\theta\Bigg|_{r=0.2}$$

$$= 2\pi \times \frac{2}{r} \int_{\theta=0}^{\pi/2} \sin\theta d(\sin\theta)\Bigg|_{r=0.2}$$

$$= \frac{4\pi}{0.2}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = 10\pi = 31.42 \text{ A}$$

(b) Total current passing through a spherical shell of 10 cm radius is,

$$I = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{1}{r^3}(2\cos\theta\, \hat{a}_r + \sin\theta\, \hat{a}_\theta).(r^2 \sin\theta\, d\phi\, d\theta\, \hat{a}_r)\Bigg|_{r=0.1}$$

$$= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{1}{r^3} 2\cos\theta r^2 \sin\theta\, d\phi\, d\theta\Bigg|_{r=0.1}$$

$$= 2\pi \times \frac{2}{r} \int_{\theta=0}^{\pi} \sin\theta d(\sin\theta)\Bigg|_{r=0.1}$$

$$= \frac{4\pi}{0.1}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi}$$

$$= 0$$

Problem3:

For the current density, $\vec{J} = 10z\sin^2\phi\,\hat{a}_r$ A/m^2, find the current through the cylindrical surface of $r = 2$, $1 \le z \le 5$ m.

Solution

Total current passing through the cylindrical surface is,

$$I = \int \vec{J}.d\vec{S} = \int_{z=1}^{5}\int_{\phi=0}^{2\pi} (10z\sin^2\phi\,\hat{a}_r).(rd\phi dz\hat{a}_r)\Big|_{r=2} = 10r\left[\frac{z^2}{2}\right]_1^5 \int_{\phi=0}^{2\pi} \sin^2\phi\, d\phi\Big|_{r=2}$$

$$= 10\times 2\times\frac{24}{2}\times\frac{2\pi}{2} = 240\pi = 754 \text{ A}$$

Problem4:

Determine the current density function $\vec{J}$ associated with the magnetic field defined by

(a) $\vec{H} = 3\hat{i} + 7\hat{j} + 2x\hat{k}$ A/m (Cartesian)

(b) $\vec{H} = 6r\hat{a}_r + 2r\hat{a}_\phi + 5\hat{a}_z$ A/m (Cylindrical)

(c) $\vec{H} = 2\rho\hat{a}_\rho + 3\hat{a}_\theta + \cos\theta\,\hat{a}_\phi$ A/m (Spherical)

(a) $\vec{H} = 3\hat{i} + 7\hat{j} + 2x\hat{k}$

By Ampere's law in Cartesian coordinates,

$$\vec{J} = \nabla\times\vec{H} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3 & 7 & 2x \end{vmatrix} = -2\hat{a}_y \text{ A/m}^2$$

(b) By Ampere's law in cylindrical coordinates,

$$\vec{J} = \nabla\times\vec{H} = \begin{vmatrix} \frac{1}{r}\hat{a}_r & \hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ H_r & rH_\phi & H_z \end{vmatrix}$$

$$= \left[\frac{1}{r}\frac{\partial H_z}{\partial\phi} - \frac{\partial H_\phi}{\partial z}\right]\hat{a}_r + \left[\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right]\hat{a}_\phi + \frac{1}{r}\left[\frac{\partial(rH_\phi)}{\partial r} - \frac{\partial H_r}{\partial\phi}\right]\hat{a}_z$$

$$= \left[\frac{1}{r}\frac{\partial}{\partial\phi}(5) - \frac{\partial}{\partial z}(2r)\right]\hat{a}_r + \left[\frac{\partial}{\partial z}(6r) - \frac{\partial}{\partial r}(5)\right]\hat{a}_\phi + \left(\frac{1}{r}\right)\left[\frac{\partial}{\partial r}(r2r) - \frac{\partial}{\partial\phi}(6r)\right]\hat{a}_z$$

$$= \left(\frac{1}{r}\right)\times 4r\hat{a}_z$$

$$= 4\hat{a}_z \text{ A/m}^2$$

(c) $\vec{H} = 2\rho\hat{a}_\rho + 3\hat{a}_\theta + \cos\theta\,\hat{a}_\phi$

By Ampere's law in spherical coordinates,

$$\vec{J} = \nabla \times \vec{H} = \frac{1}{\rho^2 \sin\theta} \begin{vmatrix} \hat{a}_\rho & \hat{a}_\theta & \hat{a}_\phi \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ H_\rho & \rho H_\theta & \rho \sin\theta H_\phi \end{vmatrix}$$

$$= \frac{1}{\rho \sin\theta}\left[\frac{\partial}{\partial \theta}(H_\phi \sin\theta) - \frac{\partial H_\theta}{\partial \phi}\right]\hat{a}_\rho + \left(\frac{1}{\rho}\right)\left[\frac{1}{\sin\theta}\frac{\partial H_\rho}{\partial \phi} - \frac{\partial}{\partial \rho}(\rho H_\phi)\right]\hat{a}_\theta + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho H_\theta) - \frac{\partial H_\rho}{\partial \theta}\right]\hat{a}_\phi$$

$$= \frac{1}{\rho \sin\theta}\left[\frac{\partial}{\partial \theta}(\cos\theta \sin\theta) - \frac{\partial}{\partial \phi}(3)\right]\hat{a}_\rho + \left(\frac{1}{\rho}\right)\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}(2\rho) - \frac{\partial}{\partial \rho}(\rho \cos\theta)\right]\hat{a}_\theta + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho 3) - \frac{\partial}{\partial \theta}(2\rho)\right]\hat{a}_\phi$$

$$= \frac{1}{\rho}\left(\frac{\cos 2\theta}{\sin\theta}\right)\hat{a}_\rho - \frac{1}{\rho}\cos\theta\, \hat{a}_\theta + \frac{3}{\rho}\hat{a}_\phi \text{ A/m}^2$$

Problem5:

An infinitely long conductor of radius a is placed such that its axis is along the *z*-axis. The vector magnetic potential, due to a direct current I_0 flowing along $\hat{a}_z$ in the conductor is given by

$$\vec{A} = -\frac{I_0}{4\pi a^2}\mu_0 (x^2 + y^2)\hat{a}_z \text{ Wb/m}$$

Find the corresponding $\vec{H}$. Also confirm the result using Ampere's law.

Solution

The magnetic flux density is given as,

$$\vec{B} = \nabla \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & -\frac{I_0}{4\pi a^2}\mu_0 (x^2 + y^2) \end{vmatrix} = -\frac{I_0}{2\pi a^2}\mu_0 (y\hat{a}_x - x\hat{a}_y)$$

So, the magnetic field intensity is given as,

$$\vec{H} = \frac{\vec{B}}{\mu_0} = -\frac{I_0}{2\pi a^2}(y\hat{a}_x - x\hat{a}_y)$$

We calculate the closed line integral of this field as follows.

$$\oint_L \vec{H}.d\vec{l} = -\frac{I_0}{2\pi a^2}\oint_L (y\hat{a}_x - x\hat{a}_y).(ad\phi\hat{a}_\phi) = -\frac{I_0}{2\pi a^2}\oint_L ad\phi(y\hat{a}_x - x\hat{a}_y).(\hat{a}_\phi)$$

$$= -\frac{I_0}{2\pi a^2}\oint_L ad\phi(y\hat{a}_x - x\hat{a}_y).(-\sin\phi\hat{a}_x + \cos\phi\hat{a}_y)$$

$$= -\frac{I_0}{2\pi a^2}\oint_L ad\phi(-y\sin\phi - x\cos\phi)$$

$$= \frac{I_0}{2\pi a^2}\oint_L ad\phi(a\sin^2\phi + a\cos^2\phi) \qquad \{\because x = r\cos\phi \quad \text{and} \quad y = r\sin\phi\}$$

$$= \frac{I_0}{2\pi}\oint_L d\phi(\sin^2\phi + \cos^2\phi)$$

$$= \frac{I_0}{2\pi}\oint_L d\phi = \frac{I_0}{2\pi} \times 2\pi = I_0$$

Since $\oint_L \vec{H}.d\vec{l} = I_0$, Ampere's law is verified.

Problem6:

Obtain an expression for the self-inductance of a toroid of circular section with '*N*' closely spaced turns.

Solution

Let,

r = Mean radius of the toroid

N = Number of turns

S = Radius of the coil

We have the magnetic field,

$$H = \frac{NI}{2\pi r}$$

total flux linkage per turn is, $\phi = BA = \mu HA = \mu\frac{NI}{2\pi r}\pi S^2 = \frac{\mu NI}{2r}S^2$

Hence, the self-inductance of the toroid is $L = \frac{N\phi}{I} = \frac{\mu N^2 S^2}{2r}$

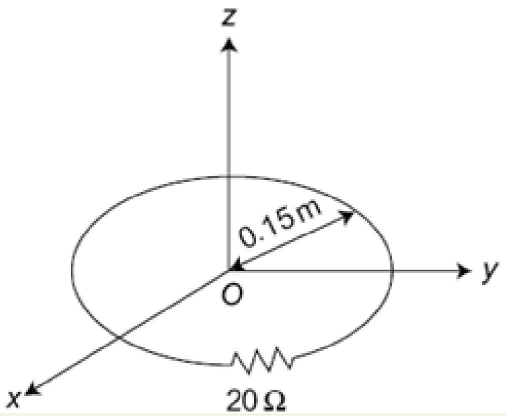
$$\boxed{L = \frac{\mu N^2 S^2}{2r}}$$

Problem7:

The circular loop conductor having a radius of 0.15 m is placed in the *xy* plane. This loop consists of a resistance of 20 Ω as shown in Fig. If the magnetic flux density is

$\vec{B} = 0.5 \sin 10^3 t \, \hat{a}_z$ **T**

Find the current flowing through the loop.



Circular loop conductor

Solution

Here since the loop is stationary and the magnetic field is time varying, only the transformer emf is induced.

Transformer emf induced is,

$$V_s = -\iint_S \frac{\partial \vec{B}}{\partial t} . d\vec{S} = -\iint_S \frac{\partial}{\partial t} (0.5 \sin 10^3 t \hat{a}_z).(r dr \, d\phi \, \hat{a}_z)$$

$$= -0.5 \times 10^3 \cos 10^3 t \int_{r=0}^{0.15} \int_{\phi=0}^{2\pi} r dr \, d\phi$$

$$= -0.5 \times 2\pi \times 10^3 \cos 10^3 t \left[\frac{r^2}{2} \right]_0^{0.15}$$

$$= -10^3 \pi \cos 10^3 t \times 0.01125$$

$$= -35.34 \cos 10^3 t \text{ V}$$

Problem8:

(a) In free space, $\vec{D} = D_m \sin(\omega t + \beta z)\hat{a}_x$. Using Maxwell's equations, show that

$$\vec{B} = -\frac{\omega\mu_0 D_m}{\beta}\sin(\omega t + \beta z)\hat{a}_y$$

(b) In free space, $\vec{B} = B_m e^{j(\omega t+\beta z)}\hat{a}_y$. Using Maxwell's equations, show that

$$\vec{E} = -\frac{\omega B_m}{\beta} e^{j(\omega t+\beta z)}\hat{a}_x$$

Solution

(a) By Maxwell's equation,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \text{ and } \vec{D} = \varepsilon_0 \vec{E} \quad \text{or,} \quad \vec{E} = \frac{\vec{D}}{\varepsilon_0} \text{ for free space}$$

$$-\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{D_m}{\varepsilon_0}\sin(\omega t + \beta z) & 0 & 0 \end{vmatrix} = \frac{D_m}{\varepsilon_0}\frac{\partial}{\partial z}[\sin(\omega t + \beta z)]\hat{a}_y = \frac{D_m \beta}{\varepsilon_0}\cos(\omega t + \beta z)\hat{a}_y$$

$$\vec{B} = -\frac{D_m \beta}{\varepsilon_0}\int \cos(\omega t + \beta z)\hat{a}_y dt = -\frac{D_m \beta}{\omega\varepsilon_0}\sin(\omega t + \beta z)\hat{a}_y$$

or,

Also, for free space,

$$\frac{\omega}{\beta} = v = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \quad \Rightarrow \quad \frac{1}{\varepsilon_0} = \mu_0\left(\frac{\omega}{\beta}\right)^2$$

$$\vec{B} = -\frac{D_m \beta}{\omega\varepsilon_0}\sin(\omega t + \beta z)\hat{a}_y = -\frac{D_m \beta}{\omega} \times \mu_0\left(\frac{\omega}{\beta}\right)^2 \sin(\omega t + \beta z)\hat{a}_y = -\frac{\omega\mu_0 D_m}{\beta}\sin(\omega t + \beta z)\hat{a}_y$$

$$\boxed{\vec{B} = -\frac{\omega\mu_0 D_m}{\beta}\sin(\omega t + \beta z)\hat{a}_y}$$

(b) By Maxwell's equation,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t} B_m e^{j(\omega t+\beta z)} \hat{a}_y$$

or, $$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -B_m j\omega e^{j(\omega t+\beta z)} \hat{a}_y$$

Comparing both sides, we get,

$$\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) \hat{a}_y = -B_m j\omega e^{j(\omega t+\beta z)} \hat{a}_y$$

$$\frac{\partial E_x}{\partial z} = -B_m j\omega e^{j(\omega t+\beta z)} \qquad (\because E_z \text{ is not a function of } x)$$

$$E_x = \int -B_m j\omega e^{j(\omega t+\beta z)} dz = -B_m j\omega \frac{1}{j\beta} e^{j(\omega t+\beta z)} = -\frac{B_m \omega}{\beta} e^{j(\omega t+\beta z)}$$

$$\boxed{\vec{E} = -\frac{\omega B_m}{\beta} e^{j(\omega t+\beta z)} \hat{a}_x}$$