UNIT – I- Electrostatics

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INTRODUCTION

VECTOR ALGEBRA

Vector Algebra is a part of algebra that deals with the theory of vectors and vector spaces.

Most of the physical quantities are either scalar or vector quantities.

SCALAR QUANTITY:

Scalar is a number that defines magnitude. Hence a scalar quantity is defined as a quantity that has magnitude only. A scalar quantity does not point to any direction i.e. a scalar quantity has no directional component.

For example when we say, the temperature of the room is 30o C, we don‘t specify the direction.

Hence examples of scalar quantities are mass, temperature, volume, speed etc.

A scalar quantity is represented simply by a letter – A, B, T, V, S.

VECTOR QUANTITY:

A Vector has both a magnitude and a direction. Hence a vector quantity is a quantity that has both magnitude and direction.

Examples of vector quantities are force, displacement, velocity, etc.

$\vec{A}$, $\vec{V}$, $\vec{B}$, $\vec{F}$

A vector quantity is represented by a letter with an arrow over it or a bold letter.

UNIT VECTORS:

When a simple vector is divided by its own magnitude, a new vector is created known as the unit vector. A unit vector has a magnitude of one. Hence the name - unit vector.

A unit vector is always used to describe the direction of respective vector.

$$\mathbf{a}_A = \frac{\vec{A}}{|A|} \Rightarrow \vec{A} = |A|\,\mathbf{a}_A$$

Hence any vector can be written as the product of its magnitude and its unit vector. Unit Vectors along the co-ordinate directions are referred to as the base vectors. For example unit vectors along X, Y and Z directions are ax, ay and az respectively.

Position Vector / Radius Vector ($\overline{OP}$):

A Position Vector / Radius vector define the position of a point(P) in space relative to the origin(O).Hence Position vector is another way to denote a point in space.

$$\overline{OP} = x\bar{a}_x + y\bar{a}_y + z\bar{a}_z$$

Displacement Vector

Displacement Vector is the displacement or the shortest distance from one point to another.

Vector Multiplication

When two vectors are multiplied the result is either a scalar or a vector depending on how they are multiplied. The two important types of vector multiplication are:

- Dot Product/Scalar Product (A.B)
- Cross product (A x B)

1. DOT PRODUCT (A. B):

Dot product of two vectors A and B is defined as:

$$\bar{A}.\bar{B} = |\bar{A}|\,|\bar{B}|\cos\theta_{AB}$$

Where θ_{AB} is the angle formed between A and B.
Also θ_{AB} ranges from 0 to π i.e. $0 \leq \theta_{AB} \leq \pi$
The result of A.B is a scalar, hence dot product is also known as Scalar Product.

Properties of Dot Product:

1. If A = (Ax, Ay, Az) and B = (Bx, By, Bz) then

   $\bar{A}.\bar{B}$= AxBx + AyBy + AzBz

2. $\bar{A}.\bar{B}$= |A| |B|, if $\cos\theta_{AB}$=1 which means $\theta_{AB} = 0^0$

This shows that A and B are in the same direction or we can also say that A and B are parallel to each other.

3. $\bar{A}.\bar{B}$ = - |A| |B|, if $\cos\theta_{AB}$=-1 which means θ_{AB} = 180^0.

This shows that A and B are in the opposite direction or we can also say that A and B are antiparallel to each other.

4. $\bar{A}.\bar{B}$ = 0, if $\cos\theta_{AB}$=0 which means θ_{AB} = 90^0.

This shows that A and B are orthogonal or perpendicular to each other.

5. Since we know the Cartesian base vectors are mutually perpendicular to each other, we have

$$\bar{a}_x.\bar{a}_x = \bar{a}_y.\bar{a}_y = \bar{a}_z.\bar{a}_z = 1$$

$$\bar{a}_x.\bar{a}_y = \bar{a}_y.\bar{a}_z = \bar{a}_z.\bar{a}_x = 0$$

2. Cross Product (A X B):

Cross Product of two vectors A and B is given as:

$$\bar{A}X\bar{B} = |\bar{A}||\bar{B}| \sin\theta_{AB}\, \bar{a}_N$$

Where θ_{AB} is the angle formed between A and B and $\bar{a}_N$ is a unit vector normal to both A and B. Also θ ranges from 0 to π i.e. $0 \leq \theta_{AB} \leq \pi$

The cross product is an operation between two vectors and the output is also a vector.

Properties of Cross Product:

1. If A = (Ax, Ay, Az) and B = (Bx, By, Bz) then,

$$\mathbf{A} * \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \mathbf{A}_x & \mathbf{A}_y & \mathbf{A}_z \\ \mathbf{B}_x & \mathbf{B}_y & \mathbf{B}_z \end{vmatrix}$$

The resultant vector is always normal to both the vector A and B.

2. $\bar{A}X\bar{B}$ = 0, if $\sin\theta_{AB}$ = 0 which means θ_{AB} = 0^0 or 180^0;
This shows that A and B are either parallel or antiparallel to each other.

3. $\bar{A}X\bar{B} = |\bar{A}||\bar{B}|\bar{a}_N$, if $\sin\theta_{AB}$ = 0 which means θ_{AB} = 90^0.
This shows that A and B are orthogonal or perpendicular to each other.

4. Since we know the Cartesian base vectors are mutually perpendicular to each other, we have

$$\bar{a}_x X\, \bar{a}_x = \bar{a}_y\, X\, \bar{a}_y = \bar{a}_z X \bar{a}_z = 0$$
$$\bar{a}_x X\, \bar{a}_y = \bar{a}_z\ ,\bar{a}_y\, X\, \bar{a}_z = \bar{a}_x\ ,\ \bar{a}_z X\, \bar{a}_x = \bar{a}_y$$

CO-ORDINATE SYSTEMS

Co-Ordinate system is a system of representing points in a space of given dimensions by coordinates, such as the Cartesian coordinate system or the system of celestial longitude and latitude.

In order to describe the spatial variations of the quantities, appropriate coordinate system is required. A point or vector can be represented in a curvilinear coordinate system that may be orthogonal or non-orthogonal. An orthogonal system is one in which the coordinates are mutually perpendicular to each other.

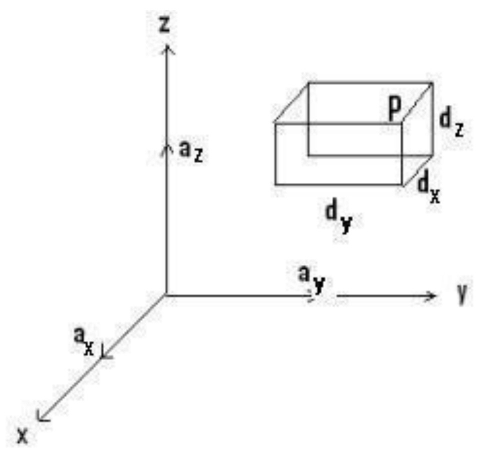
The different co-ordinate system available are:

- Cartesian or Rectangular co-ordinate system.(Example: Cube, Cuboid)
- Circular Cylindrical co-ordinate system.(Example : Cylinder)
- Spherical co-ordinate system. (Example: Sphere)

The choice depends on the geometry of the application.

A set of 3 scalar values that define position and a set of unit vectors that define direction form a co-ordinate system. The 3 scalar values used to define position are called co-ordinates. All coordinates are defined with respect to an arbitrary point called the origin.

1. Cartesian Co-ordinate System / Rectangular Co-ordinate System (x,y,z)



A Vector in Cartesian system is represented as (Ax, Ay, Az) Or

$$\bar{A} = A_x\bar{a}_x + A_y\bar{a}_y + A_z\bar{a}_z$$

Where $\bar{a}_x$, $\bar{a}_y$ and $\bar{a}_z$ are the unit vectors in x, y, z direction respectively.

Range of the variables:

It defines the minimum and the maximum value that x, y and z can have in Cartesian system.
$-\infty \leq x,y,z \leq \infty$

Differential Displacement / Differential Length (dl):

It is given as

$$\overline{dl} = dx\bar{a}_x + dy\bar{a}_y + dz\bar{a}_z$$

Differential length for a line parallel to x, y and z axis are respectively given as:

dl = $dx\bar{a}_x$---(For a line parallel to x-axis).
dl = $dy\bar{a}_y$ ---(For a line Parallel to y-axis).
dl = $dz\bar{a}_z$ ---(For a line parallel to z-axis).

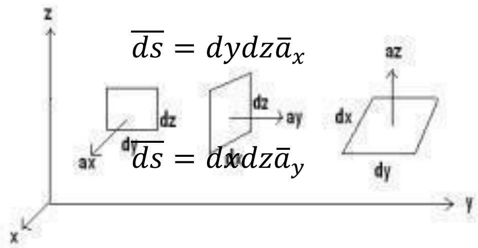
If there is a wire of length L in z-axis, then the differential length is given as dl = dz az. Similarly if the wire is in y-axis then the differential length is given as dl = dy ay.

Differential Normal Surface (ds):

Differential surface is basically a cross product between two parameters of the surface.
The differential surface (area element) is defined as

$$\overline{ds} = ds\bar{a}_N$$

Where$\bar{a}_N$, is the unit vector perpendicular to the surface.



For the 1st figure,

$$\overline{ds} = dydz\bar{a}_x$$

2nd figure,

$$\overline{ds} = dxdz\bar{a}_y$$

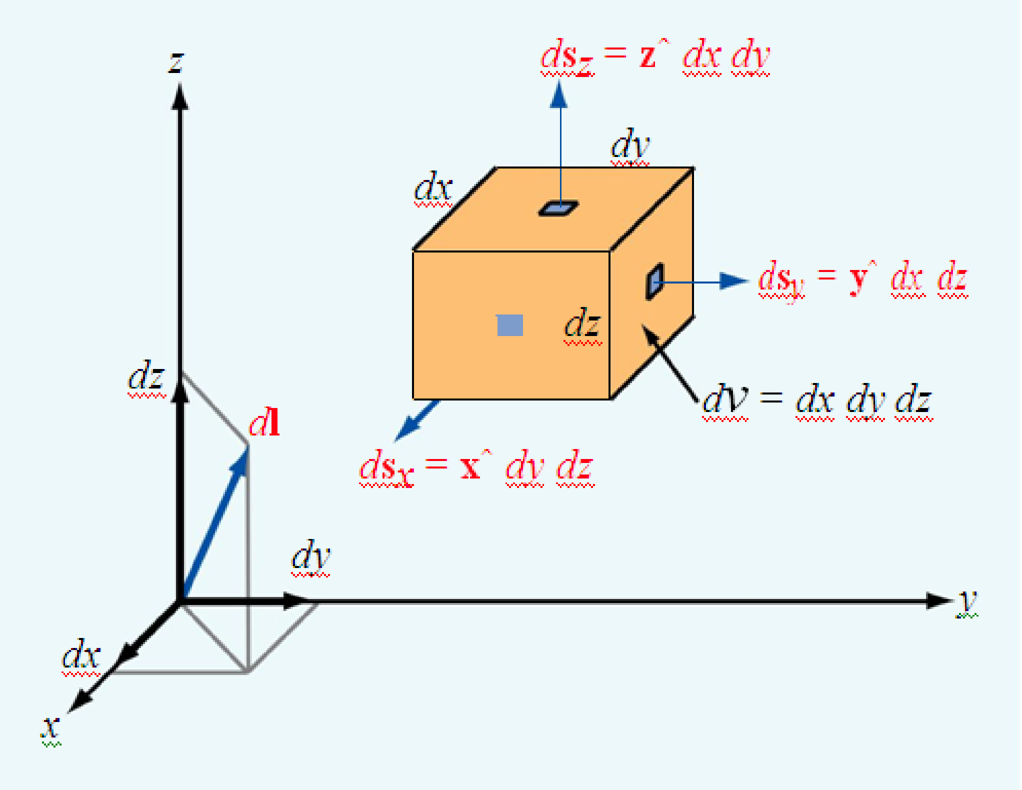
3rd figure,

$$\overline{ds} = dxdy\bar{a}_z$$

Differential Volume:

The differential volume element (dv) can be expressed in terms of the triple product.

$$dv = dxdydz$$



Differential length, area, and volume in Cartesian coordinates.

2. Circular Cylindrical Co-ordinate System

A Vector in Cylindrical system is represented as (A_r, A_ϕ, A_z) or

$$\bar{A} = A_r\bar{a}_r + A_\emptyset\bar{a}_\emptyset + A_z\bar{a}_z$$

Where$\bar{a}_r$, $\bar{a}_\emptyset$ and $\bar{a}_z$ are the unit vectors in r, Φ and z directions respectively.

The physical significance of each parameter of cylindrical coordinates:

1. The value r indicates the distance of the point from the z-axis. It is the radius of the cylinder.
2. The value Φ, also called the azimuthal angle, indicates the rotation angle around the z-axis. It is basically measured from the x axis in the x-y plane. It is measured anti clockwise.
3. The value z indicates the distance of the point from z-axis. It is the same as in the Cartesian system. In short, it is the height of the cylinder.

Range of the variables:

It defines the minimum and the maximum values of r, Φ and z.

$0 \leq r \leq \infty$
$0 \leq \Phi \leq 2\pi$
$-\infty \leq z \leq \infty$

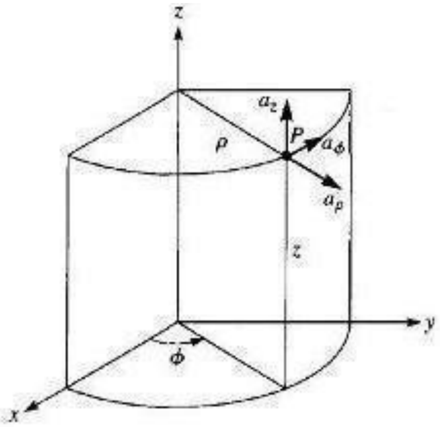


Figure shows Point P and Unit vectors in Cylindrical Co-ordinate System.

Differential Displacement / Differential Length (dl):

It is given as

$$\overline{dl} = dr\bar{a}_r + rd\varphi\bar{a}_\varphi + dz\bar{a}_z$$

Differential length for a line parallel to r, Φ and z axis are respectively given as:

dl = $dr\bar{a}_r$---(For a line parallel to r-direction).
dl = $rd\varphi\bar{a}_\varphi$ ---(For a line Parallel to Φ-direction).
dl = $dz\bar{a}_z$ ---(For a line parallel to z-axis).

Differential Normal Surface (ds):

Differential surface is basically a cross product between two parameters of the surface. The differential surface (area element) is defined as

$$\overline{ds} = ds\bar{a}_N$$

Where$\bar{a}_N$, is the unit vector perpendicular to the surface.

This surface describes a circular disc. Always remember- To define a circular disk we need two parameter one distance measure and one angular measure. An angular parameter will always give a curved line or an arc.

In this case dΦ is measured in terms of change in arc.

Arc is given as:
Arc= radius * angle

$$\overline{ds} = rdrd\varphi\bar{a}_z$$
$$\overline{ds} = drdz\bar{a}_\varphi$$
$$\overline{ds} = rdrd\varphi\bar{a}_r$$

Differential Volume:

The differential volume element (dv) can be expressed in terms of the triple product.

$$dv = rdrd\varphi dz$$

3. Spherical coordinate System:

Spherical coordinates consist of one scalar value (r), with units of distance, while the other two scalarvalues (θ, Φ) have angular units (degrees or radians).

A Vector in Spherical System is represented as (A_r ,A_Θ, A_Φ) or

$$\bar{A} = A_r\bar{a}_r + A_\theta\bar{a}_\theta + A_\varphi\bar{a}_\varphi$$

Where$\bar{a}_r$,$\bar{a}_\theta$ and $\bar{a}_\varphi$ are the unit vectors in r, θ and Φ direction respectively.

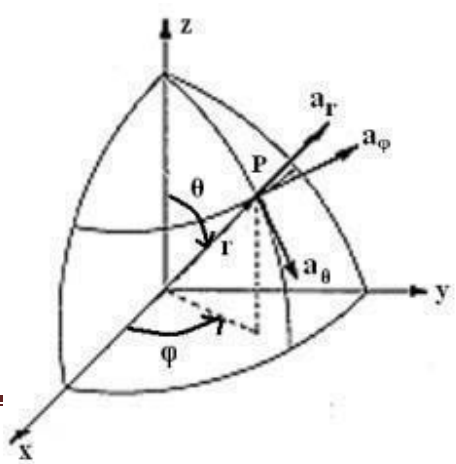
The physical significance of each parameter of spherical coordinates:

1. The value r expresses the distance of the point from origin (i.e. similar to altitude). It is the radius of the sphere.
2. The angle θ is the angle formed with the z- axis (i.e. similar to latitude). It is also called the co-latitude angle. It is measured clockwise.
3. The angle Φ, also called the azimuthal angle, indicates the rotation angle around the z-axis (i.e. similar to longitude). It is basically measured from the x axis in the x-y plane. It is measured counter-clockwise.

Range of the variables:

It defines the minimum and the maximum value that r, θ and υ can have in spherical co-ordinate system.

$0 \le r \le \infty$
$0 \le \theta \le \pi$
$0 \le \Phi \le 2\pi$



Differential length:
It is given as
$\overline{dl} = dr\bar{a}_r + rd\theta\bar{a}_\theta + r\sin\theta\, d\varphi\bar{a}_\varphi$

Differential length for a line parallel to r, θ and Φ axis are respectively given as:

dl = $dr\bar{a}_r$--(For a line parallel to r axis)

dl = $rd\theta\bar{a}_\theta$---(For a line parallel to θ direction)

dl = $r\sin\theta\, d\varphi\bar{a}_\varphi$ --(For a line parallel to Φ direction)

Differential Normal Surface (ds):

Differential surface is basically a cross product between two parameters of the surface.
The differential surface (area element) is defined as

$$\overline{ds} = ds\bar{a}_N$$

Where$\bar{a}_N$, is the unit vector perpendicular to the surface.

$$\overline{ds} = rdrd\theta\bar{a}_\varphi$$
$$\overline{ds} = r^2\sin\theta\, d\varphi d\theta\bar{a}_r$$
$$\overline{ds} = r\sin\theta\, drd\varphi\bar{a}_\theta$$

Differential Volume:

The differential volume element (dv) can be expressed in terms of the triple product.

$$dv = r^2\sin\theta\, drd\varphi d\theta$$

Coordinate transformations:

Coordinate transformations

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt[+]{x^2+y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x\cos\phi + A_y\sin\phi$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r\cos\phi - A_\phi\sin\phi$ $A_y = A_r\sin\phi + A_\phi\cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = \sqrt[+]{x^2+y^2+z^2}$ $\theta = \tan^{-1}[\sqrt[+]{x^2+y^2}/z]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$	$A_R = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$ $A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$
Spherical to Cartesian	$x = R\sin\theta\cos\phi$ $y = R\sin\theta\sin\phi$ $z = R\cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_x = A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$ $A_y = A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$
Cylindrical to spherical	$R = \sqrt[+]{r^2+z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$A_R = A_r\sin\theta + A_z\cos\theta$ $A_\theta = A_r\cos\theta - A_z\sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R\sin\theta$ $\phi = \phi$ $z = R\cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_r = A_R\sin\theta + A_\theta\cos\theta$ $A_\phi = A_\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$

Vector relations in the three common coordinate systems.

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation $\mathbf{A} =$	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of A $\|\mathbf{A}\| =$	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1} =$	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P(x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P(R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}} \cdot \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}} \times \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product $\mathbf{A} \cdot \mathbf{B} =$	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product $\mathbf{A} \times \mathbf{B} =$	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length $d\mathbf{l} =$	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume $dV =$	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

DIVERGENCE THEOREM:

It states that the net outward flux of a vector field A through a closed surface S is equal to the volume integral of the divergence of the field A inside the surface.

STOKES THEOREM:

It states that the circulation of a vector field A around a closed path L is equal to the surface integral of the curl of A over the open surface S bounded by L.

Electrostatics:

Electrostatics is a branch of science that involves the study of various phenomena caused by electric charges that are slow-moving or even stationary. Electric charge is a fundamental property of matter and charge exist in integral multiple of electronic charge. Electrostatics as the study of electric charges at rest.

The two important laws of electrostatics are

- Coulomb‘s Law.
- Gauss‘s Law.

Both these laws are used to find the electric field due to different charge configurations.

Coulomb‘s law is applicable in finding electric field due to any charge configurations where as Gauss‘s law is applicable only when the charge distribution is symmetrical.

Coulomb's Law

Coulomb's Law states that the force between two point charges Q1and Q2 is directly proportional to the product of the charges and inversely proportional to the square of the distance between them.

A point charge is a charge that occupies a region of space which is negligibly small compared to the distance between the point charge and any other object.

Point charge is a hypothetical charge located at a single point in space. It is an idealized model of a particle having an electric charge.

Mathematically, $F = \frac{kQ_1Q_2}{R^2}$, where k is the proportionality constant.

In SI units, Q1 and Q2 are expressed in Coulombs(C) and R is in meters.

Force F is in Newtons (N) and $k = \frac{1}{4\pi\varepsilon_0}$, ε_0 is called the permittivity of free space.

(We are assuming the charges are in free space. If the charges are any other dielectric medium, we will use $\varepsilon = \varepsilon_0 \varepsilon_r$ instead where ε_r is called the relative permittivity or the dielectric constant of the medium).

Therefore $F = \frac{1}{4\pi\varepsilon_0} \frac{Q_1Q_2}{R^2}$ .... (1)

As shown in the Figure 1 let the position vectors of the point charges Q1and Q2 are given by $\vec{r_1}$ and $\vec{r_2}$. Let $\overrightarrow{F_{12}}$ represent the force on Q1 due to charge Q2.

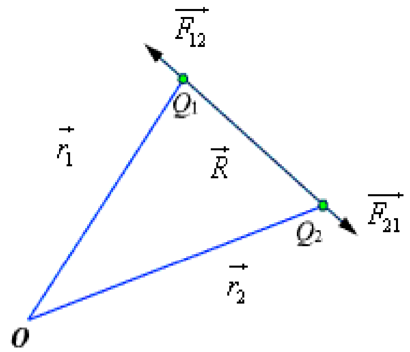


Fig 1: Coulomb's Law

The charges are separated by a distance of $R = \left|\vec{r_1} - \vec{r_2}\right| = \left|\vec{r_2} - \vec{r_1}\right|$. We define the unit vectors as

$\widehat{a_{12}} = \dfrac{\left(\vec{r_2} - \vec{r_1}\right)}{R}$ and $\widehat{a_{21}} = \dfrac{\left(\vec{r_1} - \vec{r_2}\right)}{R}$

$\overrightarrow{F_{12}}$ can be defined as $\overrightarrow{F_{12}} = \dfrac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\widehat{a_{12}} = \dfrac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\dfrac{\left(\vec{r_2} - \vec{r_1}\right)}{\left|\vec{r_2} - \vec{r_1}\right|^3}$.

Similarly the force on Q_1 due to charge Q_2 can be calculated and if $\overrightarrow{F_{21}}$ represents this force then we can write $\overrightarrow{F_{21}} = -\overrightarrow{F_{12}}$

When we have a number of point charges, to determine the force on a particular charge due to all other charges, we apply principle of superposition. If we have N number of charges $Q_1, Q_2, \ldots\ldots\ldots Q_N$ located respectively at the points represented by the position vectors $\vec{r_1}, \vec{r_2}, \ldots\ldots \vec{r_N}$, the force experienced by a charge Q located at $\vec{r}$ is given by,

$$\vec{F} = \frac{Q}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{Q_i\left(\vec{r} - \vec{r_i}\right)}{\left|\vec{r} - \vec{r_i}\right|^3}$$

Electric Field:

Electric field due to a charge is the space around the unit charge in which it experiences a force. Electric field intensity or the electric field strength at a point is defined as the force per unit charge.

Mathematically,

E = F / Q

OR

F = E Q

The force on charge Q is the product of a charge (which is a scalar) and the value of the electric field (which is a vector) at the point where the charge is located. That is

$$\vec{E} = \lim_{Q \to 0} \frac{\vec{F}}{Q} \quad \text{or,} \quad \vec{E} = \frac{\vec{F}}{Q}$$

The electric field intensity *E* at a point *r* (observation point) due a point charge *Q* located at $\vec{r}\,'$ (source point) is given by:

$$\vec{E} = \frac{Q(\vec{r} - \vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}\,'\right|^3}$$

For a collection of *N* point charges $Q_1, Q_2, \ldots\ldots Q_N$ located at $\vec{r}_1, \vec{r}_2, \ldots\ldots \vec{r}_N$, the electric field intensity at point $\vec{r}\,'$ is obtained as

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{Q_k(\vec{r} - \vec{r}_i)}{\left|\vec{r} - \vec{r}_i\right|^3}$$

The expression (6) can be modified suitably to compute the electric filed due to a continuous distribution of charges.

In figure 2 we consider a continuous volume distribution of charge *(t)* in the region denoted as the source region.

For an elementary charge $dQ = \rho(\vec{r}\,')dv'$, i.e. considering this charge as point charge, we can write the field expression as:

$$d\vec{E} = \frac{dQ(\vec{r} - \vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}\,'\right|^3} = \frac{\rho(\vec{r}\,')dv'(\vec{r} - \vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}\,'\right|^3}$$

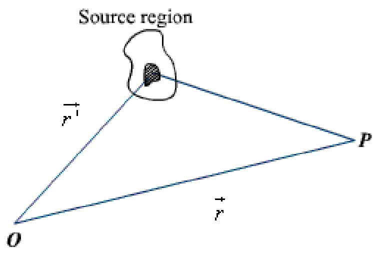


Fig 2: Continuous Volume Distribution of Charge

When this expression is integrated over the source region, we get the electric field at the point *P* due to this distribution of charges. Thus the expression for the electric field at *P* can be written as:

$$\overrightarrow{E(r)} = \int_V \frac{\rho(\vec{r}')(\vec{r} - \vec{r}')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}'\right|^3} dv'$$ ....volume charge...........................

Similar technique can be adopted when the charge distribution is in the form of a line charge density or a surface charge density.

$$\overrightarrow{E(r)} = \int_L \frac{\rho_L(\vec{r}')(\vec{r} - \vec{r}')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}'\right|^3} dl'$$ ....line charge ....

$$\overrightarrow{E(r)} = \int_S \frac{\rho_s(\vec{r}')(\vec{r} - \vec{r}')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}'\right|^3} ds'$$ ....surface charge......................

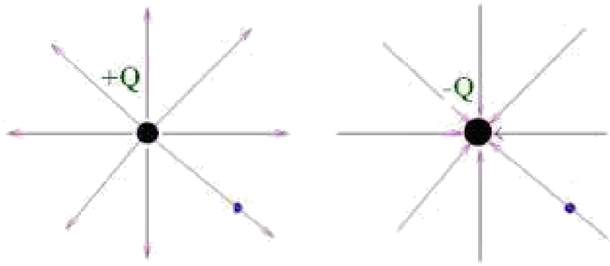
Electric Lines of Forces:

Electric line of force is a pictorial representation of the electric field.

Electric line of force (also called Electric Flux lines or Streamlines) is an imaginary straight or curved path along which a unit positive charge tends to move in an electric field.

Properties Of Electric Lines Of Force:

1. Lines of force start from positive charge and terminate either at negative charge or move to infinity.
2. Similarly lines of force due to a negative charge are assumed to start at infinity and terminate at the negative charge.



3. The number of lines per unit area, through a plane at right angles to the lines, is proportional to the magnitude of E. This means that, where the lines of force are close together, E is large and where they are far apart E is small.
4. If there is no charge in a volume, then each field line which enters it must also leave it.
5. If there is a positive charge in a volume then more field lines leave it than enter it.
6. If there is a negative charge in a volume then more field lines enter it than leave it.
7. Hence we say Positive charges are sources and Negative charges are sinks of the field.
8. These lines are independent on medium.
9. Lines of force never intersect i.e. they do not cross each other.
10. Tangent to a line of force at any point gives the direction of the electric field E at that point.

Electricfluxdensity:

As stated earlier electric field intensity or simply 'Electric field' gives the strength of the field at a particular point. The electric field depends on the material media in which the field is being considered. The flux density vector is defined to be independent of the material media (as we'll see that it relates to the charge that is producing it).For a linear isotropic medium under consideration; the flux density vector is defined as:

$$\vec{D} = \varepsilon \vec{E}$$

We define the electric flux as

$$\psi = \int_S \vec{D} . d\vec{s}$$

Gauss's Law:

Gauss's law is one of the fundamental laws of electromagnetism and it states that the total electric flux through a closed surface is equal to the total charge enclosed by the surface.

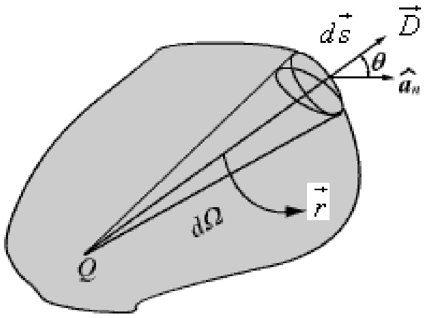


Fig 3: Gauss's Law

Let us consider a point charge *Q* located in an isotropic homogeneous medium of dielectric constant . The flux density at a distance *r* on a surface enclosing the charge is given by

$$\vec{D} = \varepsilon\vec{E} = \frac{Q}{4\pi r^2}\hat{a}_r$$

If we consider an elementary area ***ds***, the amount of flux passing through the elementary area is given by

$$d\psi = \vec{D}.ds = \frac{Q}{4\pi r^2} ds\cos\theta$$

But $\frac{ds\cos\theta}{r^2} = d\Omega$, is the elementary solid angle subtended by the area $d\vec{s}$ at the location of *Q*.

Therefore we can write $d\psi = \frac{Q}{4\pi} d\Omega$

For a closed surface enclosing the charge, we can write $\psi = \oint_s d\psi = \frac{Q}{4\pi}\oint_s d\Omega = Q$

which can seen to be same as what we have stated in the definition of Gauss's Law.

Hence we have,

$$Q_{enc} = \oint_S D \cdot ds = \int_V \rho_V \, dv$$

Applying Divergence theorem we have,

$$\oint_S D \cdot ds = \int_V \nabla \cdot D \, dv$$

Comparing the above two equations, we have

$$\int_V \nabla \cdot D \, dv = \int_V \rho_V \, dv$$

This equation is called the 1st Maxwell's equation of electrostatics.

Application of Gauss's Law:

Gauss's law is particularly useful in computing $\vec{E}$ or $\vec{D}$ where the charge distribution has some symmetry. We shall illustrate the application of Gauss's Law with some examples.

1. **$\vec{E}$ due to an infinite line charge**

As the first example of illustration of use of Gauss's law, let consider the problem of determination of the electric field produced by an infinite line charge of density $_L$C/m. Let us consider a line charge positioned along the *z*-axis as shown in Fig. 4(a) (next slide). Since the line charge is assumed to be infinitely long, the electric field will be of the form as shown in Fig. 4(b) (next slide).

If we consider a close cylindrical surface as shown in Fig. 2.4(a), using Gauss's theorm we can write,

$$\rho_L l = Q = \oint_S \varepsilon_0 \vec{E}.d\vec{s} = \int_{S_1} \varepsilon_0 \vec{E}.d\vec{s} + \int_{S_2} \varepsilon_0 \vec{E}.d\vec{s} + \int_{S_3} \varepsilon_0 \vec{E}.d\vec{s}$$

Considering the fact that the unit normal vector to areas S_1 and S_3 are perpendicular to the electric field, the surface integrals for the top and bottom surfaces evaluates to zero. Hence we can write, $\rho_L l = \varepsilon_0 E.2\pi\rho l$

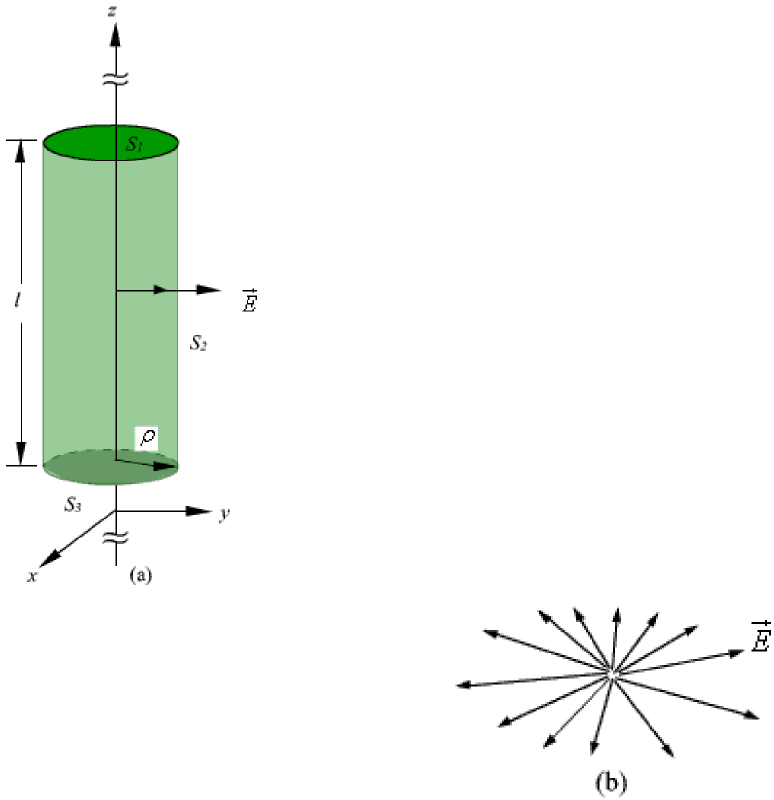


Fig 4: Infinite Line Charge

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\hat{a}_\rho$$

2. Infinite Sheet of Charge

As a second example of application of Gauss's theorem, we consider an infinite charged sheet covering the *x-z* plane as shown in figure 5. Assuming a surface charge density of ρ_s for the infinite surface charge, if we consider a cylindrical volume having sides Δs placed symmetrically as shown in figure 5, we can write:

$$\oint_s \vec{D}\cdot d\vec{s} = 2D\Delta s = \rho_s \Delta s$$

$$\therefore \qquad \vec{E} = \frac{\rho_s}{2\varepsilon_0}\hat{a}_y$$

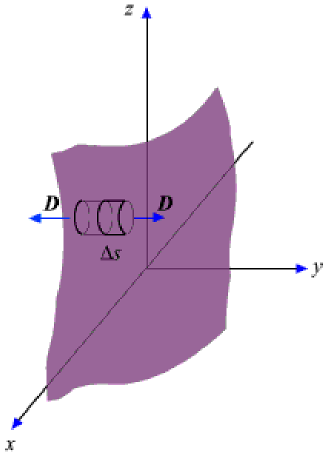


Fig 5: Infinite Sheet of Charge

It may be noted that the electric field strength is independent of distance. This is true for the infinite plane of charge; electric lines of force on either side of the charge will be perpendicular to the sheet and extend to infinity as parallel lines. As number of lines of force per unit area gives the strength of the field, the field becomes independent of distance. For a finite charge sheet, the field will be a function of distance.

3. Uniformly Charged Sphere

Let us consider a sphere of radius r0 having a uniform volume charge density of rv C/m3. To determine $\vec{D}$ everywhere, inside and outside the sphere, we construct Gaussian surfaces of radius r < r0 and r > r0 as shown in Fig. 6 (a) and Fig. 6(b).

For the region $r \leq r_0$; the total enclosed charge will be

$$Q_{en} = \rho_v \frac{4}{3}\pi r^3$$

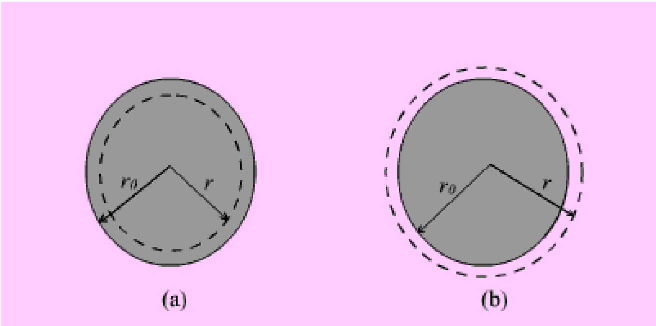


Fig 6: Uniformly Charged Sphere

By applying Gauss's theorem,

$$\oint_s \vec{D}\cdot d\vec{s} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} D_r r^2 \sin\theta d\theta d\phi = 4\pi r^2 D_r = Q_{en}$$

Therefore

$$\vec{D} = \frac{r}{3}\rho_v \hat{a}_r \qquad 0 \le r \le r_0$$

For the region $r \ge r_0$; the total enclosed charge will be

$$Q_{en} = \rho_v \frac{4}{3}\pi r_0^3$$

By applying Gauss's theorem,

$$\vec{D} = \frac{r_0^3}{3r^2}\rho_v \hat{a}_r \qquad r \ge r_0$$

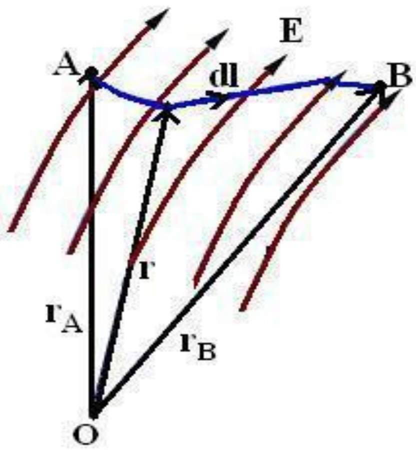
Electric Potential / Electrostatic Potential (V):

If a charge is placed in the vicinity of another charge (or in the field of another charge), it experiences a force. If a field being acted on by a force is moved from one point to another, then work is either said to be done on the system or by the system.

Say a point charge Q is moved from point A to point B in an electric field E, then the work done in moving the point charge is given as:

WA→B = - ∫AB (F . dl) = - Q ∫AB(E . dl)

where the – ve sign indicates that the work is done on the system by an external agent.



The work done per unit charge in moving a test charge from point A to point B is the electrostatic potential difference between the two points(VAB).

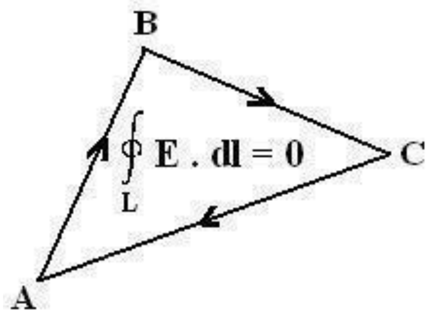
VAB = WA→B / Q

- ∫AB(E . dl)

- ∫InitialFinal (E . dl)

If the potential difference is positive, there is a gain in potential energy in the movement, external agent performs the work against the field. If the sign of the potential difference is negative, work is done by the field.

The electrostatic field is conservative i.e. the value of the line integral depends only on end points and is independent of the path taken.



- Since the electrostatic field is conservative, the electric potential can also be written as:

$$V_{AB} = -\int_{A}^{B} \overline{E}\,.\,\overline{dl}$$

$$V_{AB} = -\int_{A}^{p_0} \overline{E}\,.\,\overline{dl} - \int_{p_0}^{B} \overline{E}\,.\,\overline{dl}$$

$$V_{AB} = -\int_{p_0}^{B} \overline{E}\,.\,\overline{dl} + \int_{p_0}^{A} \overline{E}\,.\,\overline{dl}$$

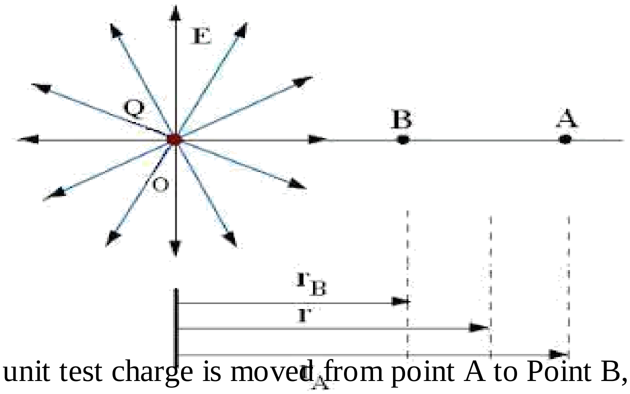
$$V_{AB} = V_B - V_A$$

Thus the potential difference between two points in an electrostatic field is a scalar field that is defined at every point in space and is independent of the path taken.

- The work done in moving a point charge from point A to point B can be written as:

$W_{A\rightarrow B} = -\,Q\,[V_B - V_A] = -Q\int_A^B \bar{E}\,.\,\bar{dl}$

- Consider a point charge Q at origin O.



Now if a unit test charge is moved from point A to Point B, then the potential difference between them is given as:

$$V_{AB} = -\int_A^B \mathbf{E}\,.\,d\mathbf{l} = -\int_{r_A}^{r_B} \mathbf{E}\,.\,d\mathbf{l} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\varepsilon r^2}\,\mathbf{a}_r \cdot dr\,\mathbf{a}_r$$

$$= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{r_B} - \frac{1}{r_A}\right) = V_B - V_A$$

- Electrostatic potential or Scalar Electric potential (V) at any point P is given by:

$$V = -\int_{P_0}^{P} \bar{E}\,.\,\bar{dl}$$

The reference point Po is where the potential is zero (analogues to ground in a circuit). The reference is often taken to be at infinity so that the potential of a point in space is defined as

$$V = -\int_{\infty}^{P} \bar{E}\,.\bar{dl}$$

Basically potential is considered to be zero at infinity. Thus potential at any point (rB = r) due to a point charge Q can be written as the amount of work done in bringing a unit positive charge frominfinity to that point (i.e. rA → ∞)

Electric potential (V) at point r due to a point charge Q located at a point with position vector r1 is given as:

$$V = \frac{Q}{4\pi\varepsilon|r - r_1|}$$

Similarly for N point charges Q1, Q2 ….Qn located at points with position vectors r1, r2, r3…..rn, theelectric potential (V) at point r is given as:

$$V = \frac{1}{4\pi\varepsilon}\sum_{k=1}^{N}\frac{Q_k}{|r - r_k|} \qquad V = \frac{Q}{4\pi\varepsilon r}$$

The charge element dQ and the total charge due to different charge distribution is given as:

dQ = ρldl → Q = ∫L (ρldl) → (Line Charge)

dQ = ρsds → Q = ∫S (ρsds) → (Surface Charge)

dQ = ρvdv → Q = ∫V (ρvdv) → (Volume Charge)

$$V = \int_L \frac{\rho_L\, dl}{4\pi\varepsilon|r - r_1|} \quad \text{(Line Charge)}$$

$$V = \int_S \frac{\rho_S\, ds}{4\pi\varepsilon|r - r_1|} \quad \text{(Surface Charge)}$$

$$V = \int_V \frac{\rho_V\, dv}{4\pi\varepsilon|r - r_1|} \quad \text{(Volume Charge)}$$

Second Maxwell's Equation of Electrostatics:

The work done per unit charge in moving a test charge from point A to point B is the electrostatic potential difference between the two points(V_{AB}).

$V_{AB} = V_B – V_A$

Similarly,

$V_{BA} = V_A – V_B$

Hence it's clear that potential difference is independent of the path taken. Therefore

$V_{AB} = - V_{BA}$

$V_{AB} + V_{BA} = 0$

$\int_{AB} (E . dl) + [- \int_{BA} (E . dl)] = 0$

$$\oint_L \mathbf{E} . \mathbf{dl} = 0$$

The above equation is called the second Maxwell's Equation of Electrostatics in integral form.. The above equation shows that the line integral of Electric field intensity (E) along a closed path is equal to zero.

In simple words—No work is done in moving a charge along a closed path in an electrostatic field.

Applying Stokes' Theorem to the above Equation, we have:

$$\oint_L \mathbf{E} . \mathbf{dl} = \int_S (\nabla \times \mathbf{E}) \cdot \mathbf{ds} = 0$$

$$\text{--->} \ \nabla \times \mathbf{E} = 0$$

If the Curl of any vector field is equal to zero, then such a vector field is called an Irrotational or Conservative Field. Hence an electrostatic field is also called a conservative field.

The above equation is called the second Maxwell's Equation of Electrostatics in differential form.

Relationship Between Electric Field Intensity (E) and Electric Potential (V):

Since Electric potential is a scalar quantity, hence dV (as a function of x, y and z variables) can be written as:

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

$$\left(\frac{\partial V}{\partial x}a_x + \frac{\partial V}{\partial y}a_y + \frac{\partial V}{\partial z}a_z\right)\cdot\left(dx\,a_x + dy\,a_y + dz\,a_z\right) = -\,E\,.\,dl$$

$$\nabla V\,.\,dl = -\,E\,.\,dl \quad \text{--->} \quad \boxed{E = -\nabla V}$$

Hence the Electric field intensity (E) is the negative gradient of Electric potential (V).
The negative sign shows that E is directed from higher to lower values of V i.e. E is opposite to the direction in which V increases.

Energy Density In Electrostatic Field / Work Done To Assemble Charges:

In case, if we wish to assemble a number of charges in an empty system, work is required to do so. Also electrostatic energy is said to be stored in such a collection.

Let us build up a system in which we position three point charges Q1, Q2 and Q3 at position r1, r2 and r3 respectively in an initially empty system.

Consider a point charge Q1 transferred from infinity to position r1 in the system. It takes no work to bring the first charge from infinity since there is no electric field to fight against (as the system is empty i.e. charge free).

Hence, W1 = 0 J

Now bring in another point charge Q2 from infinity to position r2 in the system. In this case we have to do work against the electric field generated by the first charge Q1.

Hence, W2 = Q2 V21

where V21 is the electrostatic potential at point r2 due to Q1.

- Work done W2 is also given as:

$$W_2 = \frac{Q_2 Q_1}{4\pi\varepsilon\,|r_2 - r_1|}$$

Now bring in another point charge Q3 from infinity to position r3 in the system. In this case we have to do work against the electric field generated by Q1 and Q2.

Hence, W3 = Q3 V31 + Q3 V32 = Q3 (V31 + V32)

where V31 and V32 are electrostatic potential at point r3 due to Q1 and Q2 respectively.

The work done is simply the sum of the work done against the electric field generated by point charge Q1 and Q2 taken in isolation:

$$W_3 = \frac{Q_3 Q_1}{4\pi\varepsilon\,|r_3 - r_1|} + \frac{Q_3 Q_2}{4\pi\varepsilon\,|r_3 - r_2|}$$

- Thus the total work done in assembling the three charges is given as:

WE = W1 + W2 + W3

0 + Q2 V21 + Q3 (V31 + V32)

Also total work done (WE) is given as:

$$W_E = \frac{1}{4\pi\varepsilon}\left(\frac{Q_2 Q_1}{|r_2 - r_1|} + \frac{Q_3 Q_1}{|r_3 - r_1|} + \frac{Q_3 Q_2}{|r_3 - r_2|}\right)$$

If the charges were positioned in reverse order, then the total work done in assembling them is given as:

WE = W3 + W2+ W1
= 0 + Q2V23 + Q3(V12+ V13)

Where V23 is the electrostatic potential at point r2 due to Q3 and V12 and V13 are electrostatic potential at point r1 due to Q2 and Q3 respectively.

- Adding the above two equations we have,

2WE = Q1 (V12 + V13) + Q2 (V21 + V23) + Q3 (V31 + V32)

= Q1 V1 + Q2 V2 + Q3 V3

Hence

WE =1 / 2 [Q1V1 + Q2V2 + Q3V3]

where V1, V2 and V3 are total potentials at position r1, r2 and r3 respectively.

- The result can be generalized for N point charges as:

$$W_E = \frac{1}{2} \sum_{k=1}^{N} Q_k V_k$$

The above equation has three interpretation: This equation represents the potential energy of the system.This is the work done in bringing the static charges from infinity and assembling them in the required system. This is the kinetic energy which would be released if the system gets dissolved i.e. the charges returns back to infinity.

In place of point charge, if the system has continuous charge distribution (line, surface or volume charge), then the total work done in assembling them is given as:

$$W_E = \frac{1}{2} \int_L \rho_L V \, dl \quad \text{(Line Charge)}$$

$$W_E = \frac{1}{2} \int_S \rho_S V \, ds \quad \text{(Surface Charge)}$$

$$W_E = \frac{1}{2} \int_V \rho_V V \, dv \quad \text{(Volume Charge)}$$

Since $\rho v = \nabla \cdot D$ and $E = -\nabla V$,

Substituting the values in the above equation, work done in assembling a volume charge distribution in terms of electric field and flux density is given as:

$$W_E = \frac{1}{2}\int_V D \cdot E \, dv = \frac{1}{2}\int_V \varepsilon E^2 \, dv$$

The above equation tells us that the potential energy of a continuous charge distribution is stored in an electric field.

The electrostatic energy density w_E is defined as:

$$w_E = \frac{1}{2}\varepsilon E^2 \quad ; \quad W_E = \int_V w_E \, dv$$

ELECTROSTATICS-II

Properties of Materials and Steady Electric Current:

Electric field can not only exist in free space and vacuum but also in any material medium. When an electric field is applied to the material, the material will modify the electric field either by strengthening it or weakening it, depending on what kind of material it is.

Materials are classified into 3 groups based on conductivity / electrical property:

- Conductors (Metals like Copper, Aluminum, etc.) have high conductivity ($\sigma >> 1$).
- Insulators / Dielectric (Vacuum, Glass, Rubber, etc.) have low conductivity ($\sigma << 1$).
- Semiconductors (Silicon, Germanium, etc.) have intermediate conductivity.

Conductivity (σ) is a measure of the ability of the material to conduct electricity. It is the reciprocal of resistivity (ρ). Units of conductivity are Siemens/meter and mho.

The basic difference between a conductor and an insulator lies in the amount of free electrons available for conduction of current. Conductors have a large amount of free electrons where as insulators have only a few number ofelectrons for conduction of current. Most of the conductors obey ohm‘s law. Such conductors are also called ohmic conductors.

Due to the movement of free charges, several types of electric current can be caused. The different types of electric current are:

- Conduction Current.
- Convection Current.
- Displacement Current.

Electric current:

Electric current (I) defines the rate at which the net charge passes through a wire of cross sectional surface area S.

Mathematically,

If a net charge ΔQ moves across surface S in some small amount of time Δt, electric current(I) is defined as:

$$I = \lim_{\Delta t \to 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}$$

How fast or how speed the charges will move depends on the nature of the material medium.

Current density:

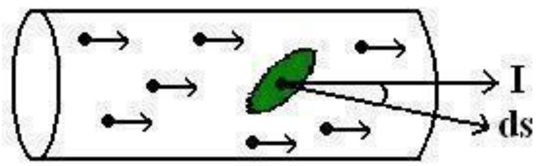
Current density (J) is defined as current ΔI flowing through surface ΔS.

Imagine surface area ΔS inside a conductor at right angles to the flow of current. As the area approaches zero, the current density at a point is defined as:

$$J = \lim_{\Delta s \to 0} \frac{\Delta I}{\Delta S}$$

The above equation is applicable only when current density (J) is normal to the surface.

In case if current density(J) is not perpendicular to the surface, consider a small area ds of the conductor at an angle θ to the flow of current as shown:



In this case current flowing through the area is given as:

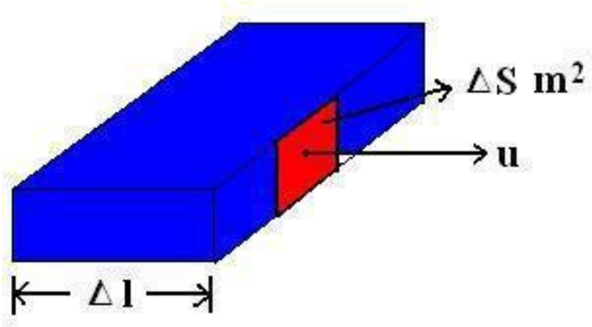
$dI = J\ dS \cos\theta = J\ .\ dS$ and $I = \int_S \bar{J}.\overline{ds}$

Where angle θ is the angle between the normal to the area and direction of the current.

From the above equation it‘s clear that electric current is a scalar quantity.

CONVECTION CURRENT DENSITY:

Convection current occurs in insulators or dielectrics such as liquid, vacuum and rarified gas. Convection current results from motion of electrons or ions in an insulating medium. Since convection current doesn‘t involve conductors, hence it does not satisfy ohm‘s law. Consider a filament where there is a flow of charge ρv at a velocity u = uy ay.



- Hence the current is given as:

$$\Delta I = \frac{\Delta Q}{\Delta t}$$

But we know $\boxed{\Delta Q = \rho_V \Delta V}$

Hence

$$\Delta I = \frac{\Delta Q}{\Delta t} = \frac{\rho_V \Delta V}{\Delta t} = \rho_V \Delta S \frac{\Delta l}{\Delta t}$$

$$= \rho_V \Delta S\, u_y$$

Again, we also know that $\boxed{J_y = \frac{\Delta I}{\Delta S}}$

Hence $$J_y = \frac{\Delta I}{\Delta S} = \rho_V u_y$$

Where uy is the velocity of the moving electron or ion and ρ_v is the free volume charge density.

- Hence the convection current density in general is given as:

$$J = \rho_v u$$

Conduction Current Density:

Conduction current occurs in conductors where there are a large number of free electrons. Conduction current occurs due to the drift motion of electrons (charge carriers). Conduction current obeys ohm‘s law.

When an external electric field is applied to a metallic conductor, conduction current occurs due to the drift of electrons.

The charge inside the conductor experiences a force due to the electric field and hence should accelerate but due to continuous collision with atomic lattice, their velocity is reduced. The net effect is that the electrons moves or drifts with an average velocity called the drift velocity (υd) which is proportional to the applied electric field (E).

Hence according to Newton‘s law, if an electron with a mass m is moving in an electric field E with anaverage drift velocity υd, the the average change in momentum of the free electron must be equal to the applied force (F = - e E).

$$\frac{m\upsilon_d}{\tau} = -eE$$

where τ is the average time interval between collision.

$$\upsilon_d = \left(-\frac{e\tau}{m}\right)E$$

The drift velocity per unit applied electric field is called the mobility of electrons (μe).

$\upsilon d = -\mu e E$

where μe is defined as:

$$\mu_e = \left(-\frac{e\tau}{m}\right)$$

Consider a conducting wire in which charges subjected to an electric field are moving with drift velocity υd.
Say there are Ne free electrons per cubic meter of conductor, then the free volume charge density(ρv)within the wire is

$\rho_v = -\, e\, Ne$

The charge ΔQ is given as:

$\Delta Q = \rho_v\, \Delta V = -\, e\, Ne\, \Delta S\, \Delta l = -\, e\, Ne\, \Delta S\, \upsilon d\, \Delta t$

- The incremental current is thus given as:

$$\Delta I = \frac{\Delta Q}{\Delta t} = -N_e\, e\, \Delta S\, \upsilon_d$$

Now since $\boxed{\upsilon_d = -\mu_e E}$

Therefore

$$\Delta I = N_e\, e\, \Delta S\, \mu_e E$$

The conduction current density is thus defined as:

$$J_c = \frac{\Delta I}{\Delta S} = N_e\, e\, \mu_e E = \sigma E$$

where σ is the conductivity of the material.

The above equation is known as the Ohm's law in point form and is valid at every point in space.

In a semiconductor, current flow is due to the movement of both electrons and holes, hence conductivity is given as:

$\sigma = (\, Ne\, \mu e + Nh\, \mu h\,)e$

DIELECTRC CONSTANT:

It is also known as Relative permittivity.

If two charges q 1 and q 2 are separated from each other by a small distance r. Then by using the coulombs law of forces the equation formed will be

$$\mathbf{F}_0 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{\mathbf{r}^2}$$

In the above equation ε_0 is the electrical permittivity or you can say it, Dielectric constant.

If we repeat the above case with only one change i.e. only change in the separation medium between the charges. Here some material medium must be used. Then the equation formed will be.

$$\mathbf{F}_\mathrm{m} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{\mathbf{r}^2}$$

Now after division of above two equations

$$\frac{\mathbf{F}_0}{\mathbf{F}_\mathrm{m}} = \frac{\varepsilon}{\varepsilon_0} = \varepsilon_\mathrm{r} \text{ Or k}$$

In the above figure

$\varepsilon_\mathbf{r}$ is the Relative Permittivity. Again one thing to notice is that the dielectric constant is represented by the symbol (K) but permittivity by the symbol $\varepsilon_\mathbf{r}$

CONTINUITY EQUATION:

The continuity equation is derived from two of Maxwell's equations. It states that the divergence of the current density is equal to the negative rate of change of the charge density,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.$$

Derivation

One of Maxwell's equations, Ampère's law, states that

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

Taking the divergence of both sides results in

$$\nabla \cdot \nabla \times \mathbf{H} = \nabla \cdot \mathbf{J} + \frac{\partial \nabla \cdot \mathbf{D}}{\partial t},$$

but the divergence of a curl is zero, so that

$$\nabla \cdot \mathbf{J} + \frac{\partial \nabla \cdot \mathbf{D}}{\partial t} = 0. \qquad (1)$$

Another one of Maxwell's equations, Gauss's law, states that

$$\nabla \cdot \mathbf{D} = \rho.$$

Substitute this into equation (1) to obtain

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0,$$

which is the continuity equation.

1.13 RELAXATION TIME:

- Let us consider that a charge is introduced at some interior point of a given material (conductor or dielectric).
- From, continuity of current equation, we have

$$\overline{J} = \frac{-rf_v}{rt} ----(1)$$

- We have, the point form of Ohm's law as,

$$\overline{J} = 6\overline{E} --- (2)$$

- From Gauss's law, we have,

$$\nabla\overline{D} = f_v \Rightarrow \in \nabla.\overline{E} = f_v \left[\because \overline{D} = \in .\overline{E}\right]$$

$$\therefore \nabla.\overline{E} = \frac{f_v}{\in} ------(1)$$

- Substitute equations (2) and (3) in equation (1), we get

$$\nabla.6\overline{E}f = 6.\nabla.\overline{E} = 6.\frac{f_v}{\in} = \frac{-\partial f_v}{\partial t}$$

$$\Rightarrow \frac{\partial f_v}{\partial t} + \frac{6}{\in}.f_v = 0 ------(4)$$

- The above equation is a homogeneous linear ordinary differential equation. By separating variable in eq (4), we get,

$$\frac{\partial f_v}{\partial t} = \frac{-6}{\in}.f_v$$

$$\Rightarrow \frac{\partial f_v}{\partial t} = \frac{-6}{\in}.\partial t$$

- Now integrate on both sides of above equation

$$\int\frac{\partial f_v}{\partial t} = -\frac{6}{\in}.\int\partial t$$

$$\Rightarrow \ln f_v = -\frac{6}{\in}.t + \ln f_{v0}$$

Where ln p_{v0} is a constant of integration.

Thus,

$$f_v = f_{v0}e^{-t/Tr} \quad -----(5)$$

$$T_r = \frac{\in}{6}$$

- In eq (5), f_{v0} is the initial charge density (i.e fv at t=0).
- We can see from the equation that as a result of introducing charge at some interior point of the material there is a decay of volume charge density f_v.
- The time constant "T_r" is known as the relaxation time or rearrangement time.
- Relaxation time is the time it takes a charge placed in the interior of a material to drop to e^{-1} = 36.8 percent f its initial value.
- The relation time is short for good conductors and long for good dielectrics.

LAPLACE'S AND POISSON'S EQUATIONS:

A useful approach to the calculation of electric potentials is to relate that potential to the charge density which gives rise to it. The electric field is related to the charge density by the divergence relationship

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$$

E = electric field
ρ = charge density
ε_0 = permittivity

and the electric field is related to the electric potential by a gradient relationship

$$E = -\nabla V$$

Therefore the potential is related to the charge density by Poisson's equation

$$\nabla \cdot \nabla V = \nabla^2 V = \frac{-\rho}{\varepsilon_0}$$

In a charge-free region of space, this becomes LaPlace's equation

$$\nabla^2 V = 0$$

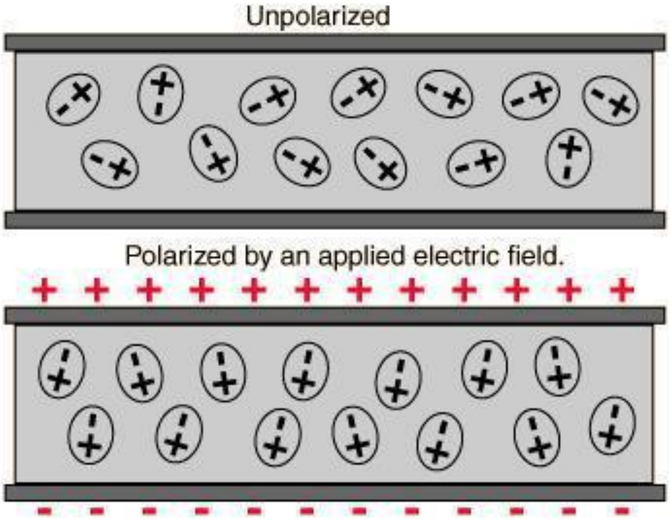
This mathematical operation, the divergence of the gradient of a function, is called the LaPlacian. Expressing the LaPlacian in different coordinate systems to take advantage of the symmetry of a charge distribution helps in the solution for the electric potential V. For example, if the charge distribution has spherical symmetry, you use the LaPlacian in spherical polar coordinates.

Since the potential is a scalar function, this approach has advantages over trying to calculate the electric field directly. Once the potential has been calculated, the electric field can be computed by taking the gradient of the potential.

Polarization of Dielectric:

If a material contains polar molecules, they will generally be in random orientations when no electric field is applied. An applied electric field will polarize the material by orienting the dipole moments of polar molecules.

This decreases the effective electric field between the plates and will increase the capacitance of the parallel plate structure. The dielectric must be a good electric insulator so as to minimize any DC leakage current through a capacitor.



The presence of the dielectric decreases the electric field produced by a given charge density.

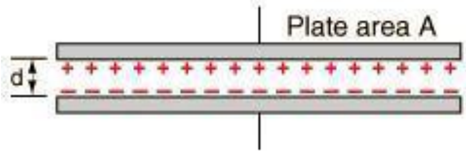
$$E_{effective} = E - E_{polarization} = \frac{\sigma}{k\varepsilon_0}$$

The factor k by which the effective field is decreased by the polarization of the dielectric is called the dielectric constant of the material.

Capacitance:

The capacitance of a set of charged parallel plates is increased by the insertion of adielectric material. The capacitance is inversely proportional to the electric field between the plates, and the presence of the dielectric reduces the effective electric field. The dielectric is characterized by a dielectric constant k, and the capacitance is multiplied by that factor.

Parallel Plate Capacitor



$$C = \frac{\varepsilon A}{d} = \frac{k\varepsilon_0 A}{d}$$

Show

The capacitance of flat, parallel metallic plates of area A and separation d is given by the expression above where:

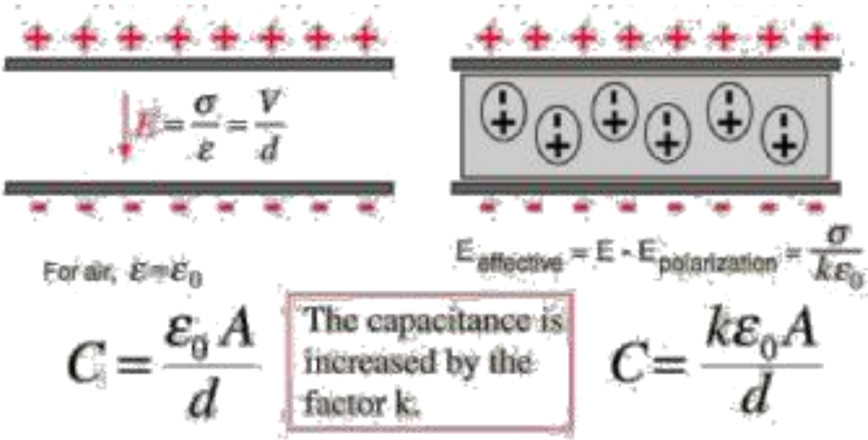
$$\varepsilon_0 = 8.854 \ x \ 10^{-12} \ F/m$$

= permittivity of space and

k = relative permittivity of the dielectric material between the plates.

k=1 for free space, k>1 for all media, approximately =1 for air.

The Farad, F, is the SI unit for capacitance and from the definition of capacitance is seen to be equal to a Coulomb/Volt.



Series and parallel Connection of capacitors

Capacitors are connected in various manners in electrical circuits; series and parallel connections are the two basic ways of connecting capacitors. We compute the equivalent capacitance for such connections.

Series Case: Series connection of two capacitors is shown in the figure 1. For this case we can write,

$$V = V_1 + V_2 = \frac{Q}{C_1} + \frac{Q}{C_2}$$

$$\frac{V}{Q} = \frac{1}{C_{eqs}} = \frac{1}{C_1} + \frac{1}{C_2} \quad \text{.......................(1)}$$

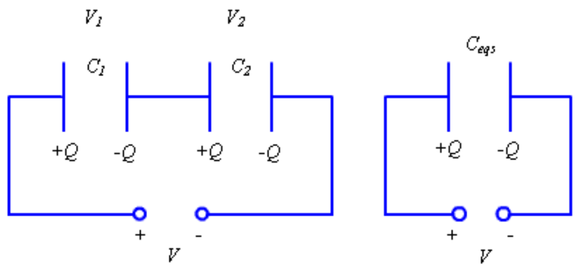


Fig 1.: Series Connection of Capacitors

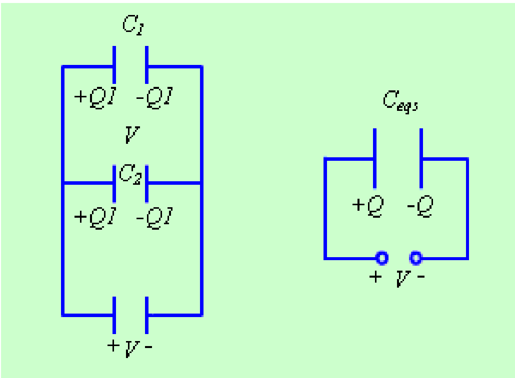


Fig 2: Parallel Connection of Capacitors

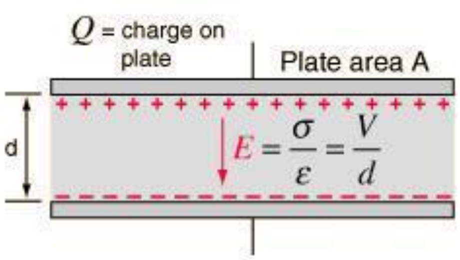
The same approach may be extended to more than two capacitors connected in series.

Parallel Case: For the parallel case, the voltages across the capacitors are the same.

The total charge $Q = Q_1 + Q_2 = C_1V + C_2V$

Therefore, $$C_{eqp} = \frac{Q}{V} = C_1 + C_2$$

Capacitance of Parallel Plates:



The electric field between two large parallel plates is given by

$$E = \frac{\sigma}{\varepsilon} \quad \text{where} \quad \sigma = \text{charge density}, \quad \varepsilon = \text{permittivity}$$

$$\text{and} \quad \sigma = \frac{Q}{A}$$

The voltage difference between the two plates can be expressed in terms of the work done on a positive test charge q when it moves from the positive to the negative plate.

$$V = \frac{work\ \ done}{charge} = \frac{Fd}{q} = Ed$$

It then follows from the definition of capacitance that

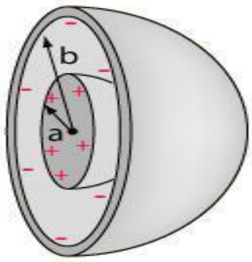
$$C = \frac{Q}{V} = \frac{Q}{Ed} = \frac{Q\varepsilon}{\sigma d} = \frac{QA\varepsilon}{Qd} = \frac{A\varepsilon}{d}$$

Spherical Capacitor:

The capacitance for spherical or cylindrical conductors can be obtained by evaluating the voltage difference between the conductors for a given charge on each.

By applying Gauss' law to an charged conducting sphere, the electric field outside it is **found to be**

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$



The voltage between the spheres can be found by integrating the electric field along a radial line:

$$\Delta V = \frac{Q}{4\pi\varepsilon_0}\int_a^b \frac{1}{r^2}dr = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]$$

From the definition of capacitance, the capacitance is

$$C = \frac{Q}{\Delta V} = \frac{4\pi\varepsilon_0}{\left[\frac{1}{a} - \frac{1}{b}\right]}$$

Isolated Sphere Capacitor:

An isolated charged conducting sphere has capacitance. Applications for such a capacitor may not be immediately evident, but it does illustrate that a charged sphere has stored some energy as a result of being charged. Taking the concentric sphere capacitance expression:

$$C = \frac{4\pi\varepsilon_0}{\left[\frac{1}{a} - \frac{1}{b}\right]}$$

$$C = 4\pi\varepsilon_0 R$$

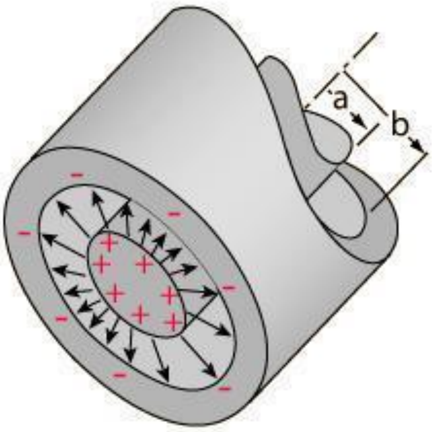
and taking the limits $a \to R$ and $b \to \infty$ gives

Further confirmation of this comes from examining the potential of a charged conducting sphere:

$$V = \frac{Q}{4\pi\varepsilon_0 r} \quad \text{so at the surface } C = \frac{Q}{V} = 4\pi\varepsilon_0 R$$

Cylindrical Capacitor:

For a cylindrical geometry like a coaxial cable, the capacitance is usually stated as a capacitance per unit length. The charge resides on the outer surface of the inner conductor and the inner wall of the outer conductor. The capacitance expression is



$$\frac{C}{L} = \frac{2\pi k\varepsilon_0}{\ln\left[\frac{b}{a}\right]}$$

The capacitance for cylindrical orspherical conductors can be obtained by evaluating the voltage difference between the conductors for a given charge on each. By applying Gauss' law to an infinite cylinder in a vacuum, the electric field outside a charged cylinder is found to be

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

The voltage between the cylinders can be found by integrating the electric field along a radial line:

$$\Delta V = \frac{\lambda}{2\pi\varepsilon_0}\int_a^b \frac{1}{r}dr = \frac{\lambda}{2\pi\varepsilon_0}\ln\left[\frac{b}{a}\right] \qquad \frac{C}{L} = \frac{\lambda}{\Delta V} = \frac{2\pi k\varepsilon_0}{\ln\left[\frac{b}{a}\right]}$$

From the definition of capacitance and including the case where the volume is filled by a dielectric of dielectric constant k, the capacitance per unit length is defined above.

Solved problems:

Problem1:

Find the charge in the volume defined by $0 \le x \le 1$ m, $0 \le y \le 1$ m, and $0 \le z \le 1$ m if $\rho = 30x^2y$ (μC/m^3). What change occurs for the limits $-1 \le y \le 0$ m?

Since $dQ = \rho\, dv$,

$$Q = \int_0^1 \int_0^1 \int_0^1 30x^2y\, dx\, dy\, dz = 5\ \mu\text{C}$$

For the change in limits on y,

$$Q = \int_0^1 \int_{-1}^0 \int_0^1 30x^2y\, dx\, dy\, dz = -5\ \mu\text{C}$$

Problem-2

Three point charges, $Q_1 = 30$ nC, $Q_2 = 150$ nC, and $Q_3 = -70$ nC, are enclosed by surface S. What net flux crosses S?

Since electric flux was defined as originating on positive charge and terminating on negative charge, part of the flux from the positive charges terminates on the negative charge.

$$\Psi_{net} = Q_{net} = 30 + 150 - 70 = 110\ \text{nC}$$

Problem-3

A point charge, $Q = 30$ nC, is located at the origin in cartesian coordinates. Find the electric flux density $\mathbf{D}$ at (1, 3, −4) m.

Referring to Fig. 3.12,

$$\mathbf{D} = \frac{Q}{4\pi R^2}\mathbf{a}_R$$

$$= \frac{30 \times 10^{-9}}{4\pi(26)}\left(\frac{\mathbf{a}_x + 3\mathbf{a}_y - 4\mathbf{a}_z}{\sqrt{26}}\right)$$

$$= (9.18 \times 10^{-11})\left(\frac{\mathbf{a}_x + 3\mathbf{a}_y - 4\mathbf{a}_z}{\sqrt{26}}\right)\ \text{C/m}^2$$

or, more conveniently, $D = 91.8$ pC/m^2.

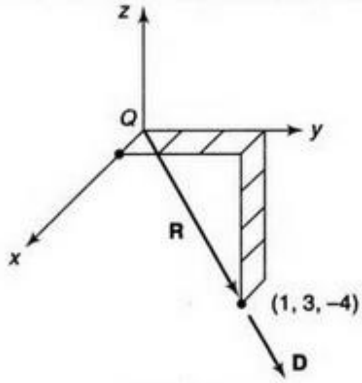


Fig. 3.12

Problem-4

Given that $\mathbf{D} = 10x\mathbf{a}_x$ (C/m^2), determine the flux crossing a 1-m^2 area that is normal to the x axis at $x = 3$ m.

Since $\mathbf{D}$ is constant over the area and perpendicular to it,

$$\Psi = DA = (30\ \text{C/m}^2)(1\ \text{m}^2) = 30\ \text{C}$$

Problem-5

Given the vector field $\mathbf{A} = 5x^2\left(\sin\frac{\pi x}{2}\right)\mathbf{a}_x$, find div $\mathbf{A}$ at $x = 1$.

$$\text{div } \mathbf{A} = \frac{\partial}{\partial x}\left(5x^2 \sin\frac{\pi x}{2}\right) = 5x^2\left(\cos\frac{\pi x}{2}\right)\frac{\pi}{2} + 10x\sin\frac{\pi x}{2} = \frac{5}{2}\pi x^2\cos\frac{\pi x}{2} + 10x\sin\frac{\pi x}{2}$$

and $\quad \text{div } \mathbf{A}|_{x=1} = 10.$

Problem-6

Given that $\mathbf{D} = (10r^3/4)\mathbf{a}_r$ (C/m^2) in the region $0 < r \le 3$ m in cylindrical coordinates and $\mathbf{D} = (810/4r)\mathbf{a}_r$ (C/m^2) elsewhere, find the charge density.

For $\quad 0 < r \le 3$ m,

$$\rho = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{10r^4}{4}\right) = 10r^2 \text{ (C/m}^3\text{)}$$

and for $\quad r > 3$ m,

$$\rho = \frac{1}{r}\frac{\partial}{\partial r}(810/4) = 0$$

Problem-7

An electrostatic field is given by $\mathbf{E} = (x/2 + 2y)\mathbf{a}_x + 2x\mathbf{a}_y$ (V/m). Find the work done in moving a point charge $Q = -20\ \mu$C (a) from the origin to (4, 0, 0) m, and (b) from (4, 0, 0) m to (4, 2, 0) m.

(a) The first path is along the x axis, so that $d\mathbf{l} = dx\,\mathbf{a}_x$.

$$dW = -Q\mathbf{E}\cdot d\mathbf{l} = (20\times10^{-6})\left(\frac{x}{2} + 2y\right)dx$$

$$W = (20\times10^{-6})\int_0^4\left(\frac{x}{2} + 2y\right)dx = 80\ \mu\text{J}$$

(b) The second path is in the $\mathbf{a}_y$ direction, so that $d\mathbf{l} = dy\,\mathbf{a}_y$.

$$W = (20\times10^{-6})\int_0^2 2x\,dy = 320\ \mu\text{J}$$

Problem-8

What electric field intensity and current density correspond to a drift velocity of 6.0×10^{-4} m/s in a silver conductor?

For silver $\quad \sigma = 61.7$ MS/m $\quad$ and $\quad \mu = 5.6\times10^{-3}$ m^2/V · s.

$$E = \frac{U}{\mu} = \frac{6.0\times10^{-4}}{5.6\times10^{-3}} = 1.07\times10^{-1}\text{ V/m}$$

$$J = \sigma E = 6.61\times10^{6}\text{ A/m}^2$$

Problem-9

Find the current in the circular wire shown in Fig. 6.6 if the current density is $\mathbf{J} = 15(1 - e^{-1000r})\mathbf{a}_z$ (A/m^2). The radius of the wire is 2 mm.

A cross section of the wire is chosen for S. Then

$$dI = \mathbf{J} \cdot d\mathbf{S}$$
$$= 15(1 - e^{-1000r})\mathbf{a}_z \cdot r\,dr\,d\phi\,\mathbf{a}_z$$

and

$$I = \int_0^{2\pi}\int_0^{0.002} 15(1 - e^{-1000r})r\,dr\,d\phi$$
$$= 1.33 \times 10^{-4}\text{ A} = 0.133\text{ mA}$$

Any surface S which has a perimeter that meets the outer surface of the conductor all the way around will have the same total current, $I = 0.133$ mA, crossing it.

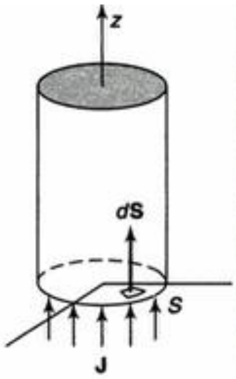


Fig. 6.6

Problem-10

Determine the relaxation time for silver, given that $\sigma = 6.17 \times 10^7$ S/m. If charge of density ρ_0 is placed within a silver block, find ρ after one, and also after five, time constants.

Since $\varepsilon \approx \varepsilon_0$,

$$\tau = \frac{\varepsilon}{\sigma} = \frac{10^{-9}36\pi}{6.17 \times 10^7} = 1.43 \times 10^{-19}\text{ s}$$

Therefore

$$\text{at } t = \tau: \quad \rho = \rho_0 e^{-1} = 0.368\rho_0$$
$$\text{at } t = 5\tau: \quad \rho = \rho_0 e^{-5} = 6.74 \times 10^{-3}\rho_0$$

Problem-11

Find the magnitudes of $\mathbf{D}$ and $\mathbf{P}$ for a dielectric material in which $E = 0.15$ MV/m and $\chi_e = 4.25$.

Since $\varepsilon_r = \chi_e + 1 = 5.25$,

$$D = \varepsilon_0\varepsilon_r E = \frac{10^{-9}}{36\pi}(5.25)(0.15 \times 10^6) = 6.96\ \mu\text{C/m}^2$$

$$P = \chi_e\varepsilon_0 E = \frac{10^{-9}}{36\pi}(4.25)(0.15 \times 10^6) = 5.64\ \mu\text{C/m}^2$$

Problem-12

In order to illustrate the application of (13) or (14), let us find **E** at $P(1, 1, 1)$ caused by four identical 3-nC (nanocoulomb) charges located at $P_1(1, 1, 0)$, $P_2(-1, 1, 0)$, $P_3(-1, -1, 0)$, and $P_4(1, -1, 0)$, as shown in Fig. 2.4.

Solution. We find that $\mathbf{r} = \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$, $\mathbf{r}_1 = \mathbf{a}_x + \mathbf{a}_y$, and thus $\mathbf{r} - \mathbf{r}_1 = \mathbf{a}_z$. The magnitudes are: $|\mathbf{r} - \mathbf{r}_1| = 1$, $|\mathbf{r} - \mathbf{r}_2| = \sqrt{5}$, $|\mathbf{r} - \mathbf{r}_3| = 3$, and $|\mathbf{r} - \mathbf{r}_4| = \sqrt{5}$. Since $Q/4\pi\epsilon_0 = 3 \times 10^{-9}/(4\pi \times 8.854 \times 10^{-12}) = 26.96\ \mathrm{V \cdot m}$, we may now use (13) or (14) to obtain

$$\mathbf{E} = 26.96\left[\frac{\mathbf{a}_z}{1}\frac{1}{1^2} + \frac{2\mathbf{a}_x + \mathbf{a}_z}{\sqrt{5}}\frac{1}{(\sqrt{5})^2} + \frac{2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z}{3}\frac{1}{3^2} + \frac{2\mathbf{a}_y + \mathbf{a}_z}{\sqrt{5}}\frac{1}{(\sqrt{5})^2}\right]$$

or

$$\mathbf{E} = 6.82\mathbf{a}_x + 6.82\mathbf{a}_y + 32.8\mathbf{a}_z\ \mathrm{V/m}$$

Problem-13

Ex. *A charge $Q_1 = -20\mu C$ is located at P (– 6, 4, 6) and a charge $Q_2 = 50\mu C$ is located at R (5, 8, – 2) in a free space. Find the force exerted on Q_2 by Q_1 in vector form. The distances given are in metres.*

Sol. : From the co-ordinates of P and R , the respective position vectors are –

$$\bar{P} = -6\bar{a}_x + 4\bar{a}_y + 6\bar{a}_z$$

and

$$\bar{R} = 5\bar{a}_x + 8\bar{a}_y - 2\bar{a}_z$$

The force on Q_2 is given by,

$$\bar{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}\bar{a}_{12}$$

$$\bar{R}_{12} = \bar{R}_{PR} = \bar{R} - \bar{P} = [5-(-6)]\,\bar{a}_x + (8-4)\,\bar{a}_y + [-2-(6)\bar{a}_z]$$

$$= 11\bar{a}_x + 4\bar{a}_y - 8\bar{a}_z$$

$$\therefore \quad |R_{12}| = \sqrt{(11)^2 + (4)^2 + (-8)^2} = 14.1774$$

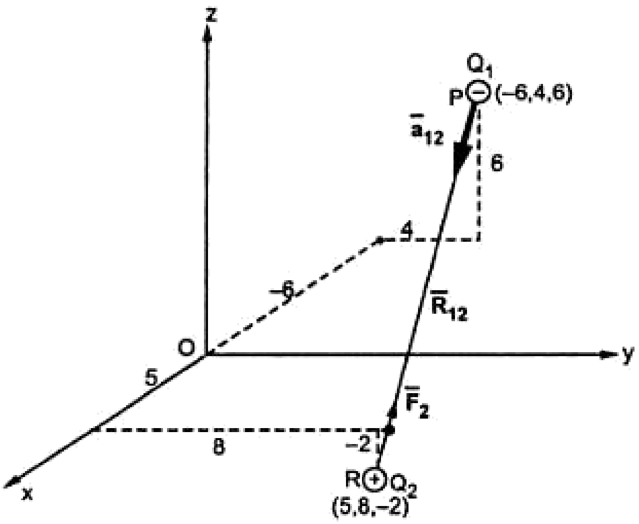


Fig. 2.5

$$\therefore \quad \bar{a}_{12} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|} = \frac{11\bar{a}_x + 4\bar{a}_y - 8\bar{a}_z}{14.1774}$$

$$\therefore \quad \bar{a}_{12} = 0.7758\,\bar{a}_x + 0.2821\,\bar{a}_y - 0.5642\,\bar{a}_z$$

$$\therefore \quad \bar{F}_2 = \frac{-20\times 10^{-6}\times 50\times 10^{-6}}{4\pi\times 8.854\times 10^{-12}\times (14.1774)^2}[\bar{a}_{12}]$$

$$= -0.0447\left[0.7758\,\bar{a}_x + 0.2821\,\bar{a}_y - 0.5642\,\bar{a}_z\right] \quad \text{... (A)}$$

$$= -0.0346\,\bar{a}_x - 0.01261\,\bar{a}_y + 0.02522\,\bar{a}_z \text{ N} \quad \text{... (B)}$$

This is the required force exerted on Q_2 by Q_1.