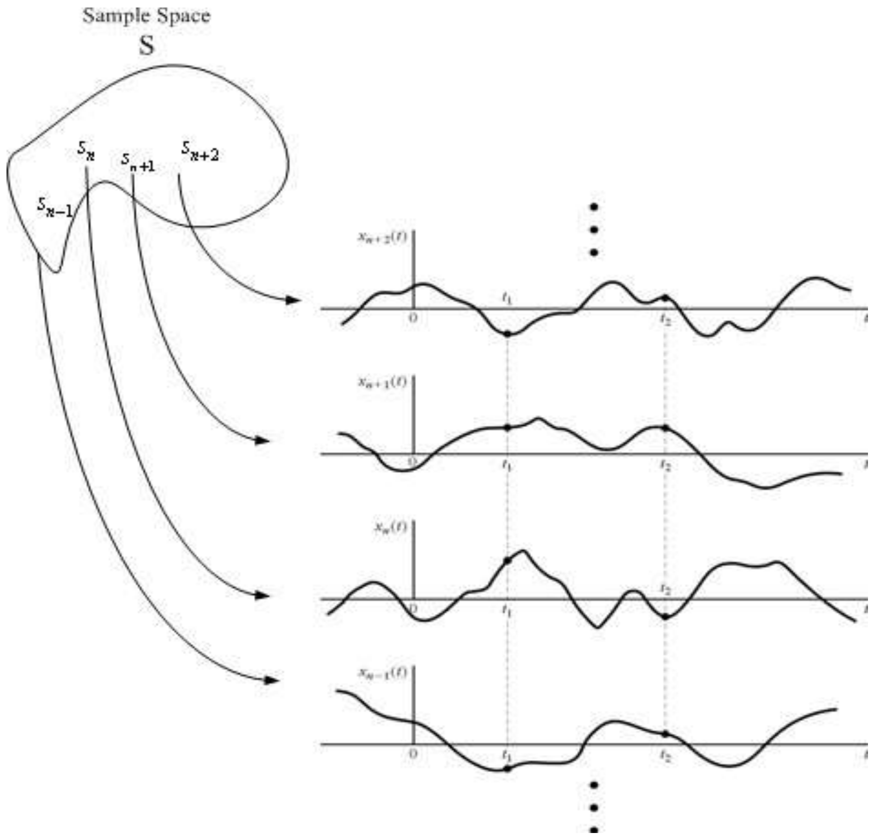
RANDOM PROCESS– TEMPORAL & SPECTRAL CHARACTERISTIC

- Random Process – Concept, Classification
- Distribution and Density Function, Stationarity and Statistical Independence
- Time Averages and Ergodicity, Auto Correlation, Cross Correlation - Properties
- Covariance Functions, Gaussian Random Processes, Poisson Random Process
- The Power Density Spectrum and its Properties
- Relationship between Power Spectrum and Autocorrelation Function
- The Cross-Power Density Spectrum and its Properties
- Relationship between Cross-Power Spectrum and Cross-Correlation Function

RANDOM PROCESS CONCEPT

- A random process is a function of both sample space and time variables. It is represented as $X(t,s)$ or simply $X(t)$. The random processes are also called as stochastic processes which deal with randomly varying time wave forms such as any message signals and noise.

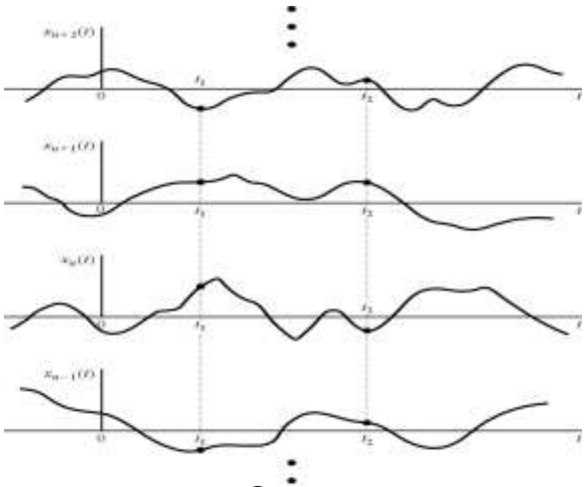
- ❑ The concept of random process can be extend to include time and the outcome will be random functions of time as shown beside $x(t,s)$, Where s is the outcome of anexperiment.
- ❑ Thefunctions $\ldots x_{n+2}(t), x_{n+1}(t), x_n(t), x_{n-1}(t) \ldots$ are one realizations of many of the random process $X(t)$.

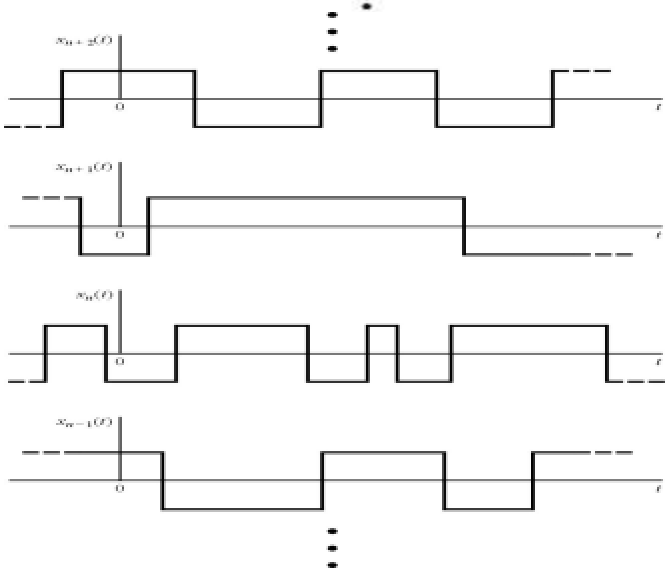


- ❑ A random process also represents a random variable when time is fixed. $X(t1)$ is a random variable.

CLASSIFICATION OF RANDOM PROCESS

- Continuous Random Process
- Discrete Random Process
- Continuous Random Sequence
- Discrete Random Sequence





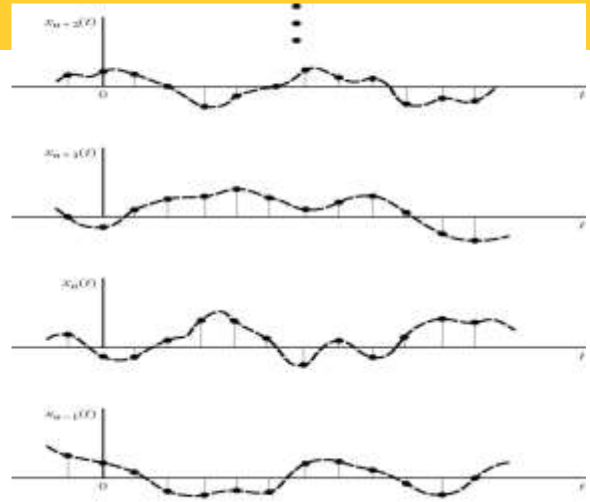
Continuous RandomProcess:

A random process is said to be continuous if both the random variable X and time t are continuous.

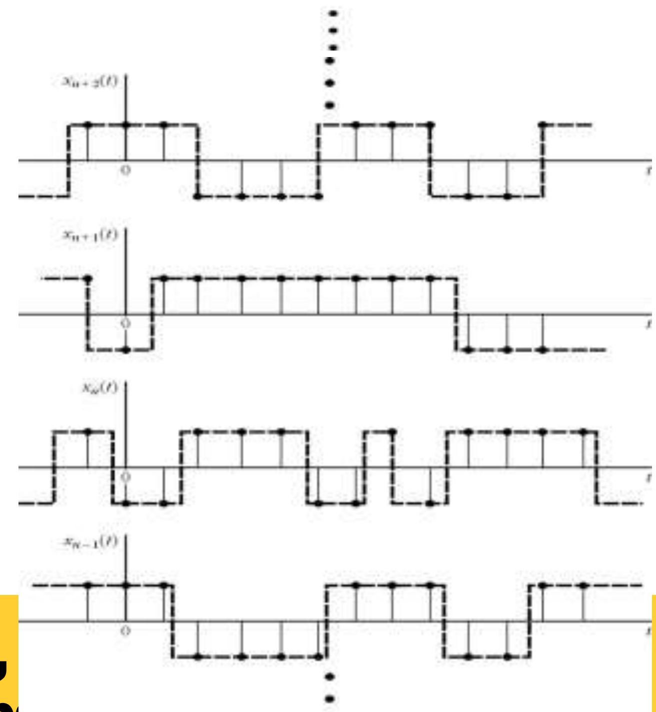
Discrete Random Process: In discrete random process, the random variable X has only discrete values while time, t is continuous.

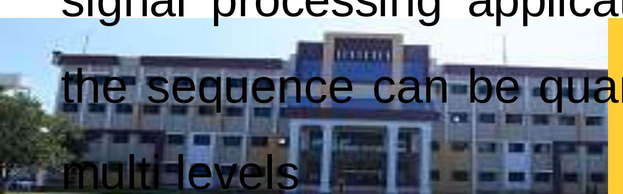
CLASSIFICATION OF RANDOM PROCESS

Continuous Random Sequence: A random process for which the random variable X is continuous but t has discrete values is called continuous random sequence. A continuous random signal is defined only at discrete (sample) time intervals. It is also called as a discrete time random process and can be represented as a set of random variables $\{X(t)\}$ for samples $t_k, k = 0, 1, 2, \ldots$



Discrete Random Sequence: In discrete random sequence both random variable X and time t are discrete. It can be obtained by sampling and quantizing a random signal. This is called the random process and is mostly used in digital signal processing applications. The amplitude of the sequence can be quantized into two levels or, multi levels





DETERMINISTIC AND NON-DETERMINISTIC PROCESSES

- Deterministic processes:A process is called as deterministic random process if future values of any sample function can be predicted from its past values.
- For example, $X(t) = A\sin(\omega_0 t + \Theta)$, where the parameters A, ω_0 and Θ may be random variables, is deterministic random process because the future values of the sample function can be detected from its known shape.
- Non-Determinist processes: If future values of a sample function cannot be detected from observed past values, the process is called non-deterministic process.

DISTRIBUTION AND DENSITY FUNCTION

- If $X(t)$ is a stochastic process, then for fixed t, $X(t)$ represents a random variable. Its distribution function is given by $F_X(x,t) = P\{X(t) \leq x\}$
- Notice that $F_X(x,t)$ depends on t, since for a different t, we obtain a different random variable. Further

$$f_X(x,t) = \frac{dF_X(x,t)}{dx}$$

- represents the first-order probability density function of the process $X(t)$.
- For $t = t_1$ and $t = t_2$, $X(t)$ represents two different random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$ respectively. Their joint distribution is given by

$$F_X(x_1,x_2,t_1,t_2) = P\{X(t_1) \leq x_1, X(t_2) \leq x_2\}$$

- and

$$f_X(x_1,x_2,t_1,t_2) = \frac{\partial^2 F_X(x_1,x_2,t_1,t_2)}{\partial x_1\, \partial x_2}$$

- represents the second-order density function of the process $X(t)$.

STATIONARITY AND STATISTICAL INDEPENDENCE

- Similarly $f_X(x_1, x_2, \cdots x_n,\ t_1, t_2 \cdots, t_n)$ represents the n[th] order density function of the process $X(t)$. Complete specification of the stochastic process $X(t)$ requires the knowledge of $f_X(x_1, x_2, \cdots x_n,\ t_1, t_2 \cdots, t_n) \forall t_i, i = 1,2,3 \ldots$, and for all n.

  Stationary RandomProcess: all its statistical properties do not change withtime

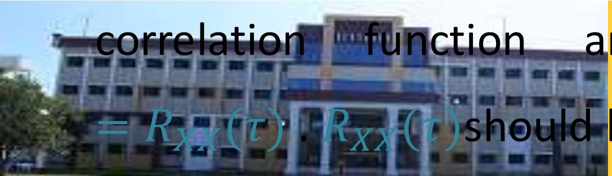
Non Stationary Random Process: not stationary

- Statistical Independence: Two Processes $X(t)$and $Y(t)$are statistically independent if the random variable group $X(t_1), X(t_2), \ldots, X(t_N)$ is independent of the group $Y(t_1'), Y(t_2'), \ldots, Y(t_M')$ for any choice of times. Independence requires that the joint density be factorable by groups:

- $$f_{X,Y}(x_1, x_2, \cdots x_N, y_1, y_2, \cdots y_M; t_1, t_2 \cdots, t_n, t_1', t_2', \ldots, t_M')$$
  $$= f_X(x_1, x_2, \cdots x_N,; t_1, t_2 \cdots, t_n)\ f_Y(y_1, y_2, \cdots y_M; t_1', t_2', \ldots, t_M')$$

FIRST ORDER STATIONARY, SECOND ORDER STATIONARY

- A random process is classified as first-order stationary if its first-order probability density function remains equal regardless of any shift in time to its timeorigin.
- In other words, $f_X(x_1, t_1) = f_X(x_1, t_1 + \Delta)$must be true for any t_1 and any real number Δ, if $X(t)$ to be a first order stationary.
- Therefore the condition for a process to be a first order stationary random process is that its mean value must be constant at any time instant. i.e. $E[X(t)] = \bar{X} = constant.$
- A random process is said to be stationary to order two or second order stationary if its second order joint density function does not change with time or shift in time value i.e. $f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta, t_2 + \Delta)$ for all t_1,t_2 and Δ.
- It is a function of time difference $(t_2, , t_1)$ and not absolute time t. Note that a second order stationary process is also a first order stationary process. The condition for a process to be a second order stationary is that its autocorrelation should depend only on time differences and not on absolute time. i.e. If $R_{XX}(t_1, t_2) = E[X(t_1) X(t_2)]$ is auto correlation function and $\tau = t_2 - t_1$ then $R_{XX}(t_1, t_1 + \tau) = E[X(t_1) X(t_1 + \tau)] = R_{XX}(\tau)$. $R_{XX}(\tau)$should be independent of time t.



WIDE-SENSE STATIONARY (WSS)

A process that satisfies thefollowing:

The mean is a constant and the autocorrelation function depends only on the difference between the timeindices

- $E[X(t)] = \bar{X} = constant$
- $E[X(t)X(t+\tau)] = R_{XX}(\tau)$

- is a Wide-Sense Stationary(WSS)
- Second-orderstationary → Wide-SenseStationary
- The converse is not true ingeneral.

- If they are jointly WSS, then the cross correlation function of $X(t)$ and $Y(t)$ is a function of time difference $\tau = t_2 - t_1$ only and not absolute time. i.e. $R_{XY}(t_1, t_2) = E[X(t_1)\,Y(t_2)]$.
- If $\tau = t_2 - t_1, t_1 = t$ then $R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)] = R_{XY}(\tau)$.

Therefore the conditions for a process to be joint wide sense stationary are

$E[X(t)] = \bar{X} = constant$, $E[Y(t)] = \bar{Y} = constant$ and

$E[X(t)\,Y(t+\tau)] = R_{XY}(\tau)$ is independent of time t.

STRICT SENSE STATIONARY (SSS) PROCESSES

- In strict terms, the statistical properties are governed by the joint probability density function. Hence a process is n^th-order Strict-Sense Stationary (S.S.S) if

$$f_X(x_1, x_2, \cdots x_n,\ t_1, t_2 \cdots, t_n) \equiv f_X(x_1, x_2, \cdots x_n,\ t_1 + c, t_2 + c \cdots, t_n + c)$$

- for *any c*, where the left side represents the joint density function of the random variables $X_1 = X(t_1),\ X_2 = X(t_2), \dots,\ X_n = X(t_n)$ and the right side corresponds to the joint density function of the random variables

$$X_1' = X(t_1 + c),\ X_2' = X(t_2 + c),\ \cdots,\ X_n' = X(t_n + c).$$

- A process *X*(*t*) is said to be strict-sense stationary if the above equation is true for all

$$t_i,\ i = 1, 2, \cdots, n;\ n = 1, 2, \cdots \ \text{and}\ any\ \ c.$$

TIME AVERAGES & ERGODICITY

Time Average Function: Consider a random process X(t). Let x(t) be a sample function which exists for all time at a fixed value in the given sample space S. The average value of x(t) taken over all times is called the time average of x(t). It is also called mean value of x(t). It can be expressed as $\bar{x} = A[x(t)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)dt.$

Time autocorrelation function: Consider a random process X(t). The time average of the product X(t) and X(t+ τ) is called time average autocorrelation function of x(t) and is denoted as $\mathcal{R}_{xx}(\tau) = A[X(t)\,X(t+\tau)]$ or $\mathcal{R}_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt.$

Time mean square function: If τ = 0, the time average of $x^2(t)$ is called time mean square value of x(t) defined as $= A[X^2(t)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x^2(t)dt.$

Time cross correlation function: Let X(t) and Y(t) be two random processes with sample functions x(t) and y(t) respectively. The time average of the product of x(t) y(t+ τ) is called time cross correlation function of x(t) and y(t). Denoted as

$\mathcal{R}_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)y(t+\tau)dt.$

- Ergodic => $\bar{x} = \bar{X}$, $\Re_{xx}(\tau) = R_{XX}(\tau)$
- Jointly Ergodic => Ergodic X(t) and Y(t), $\Re_{xy}(\tau) = R_{XY}(\tau)$

AUTO CORRELATION

- Autocorrelation occurs in time-series studies when the errors associated with a given time period carry over into future time periods.
- For example, if we are predicting the growth of stock dividends, an overestimate in one year is likely to lead to overestimates in succeeding years.
- Times series data follow a natural ordering over time.
- It is likely that such data exhibit intercorrelation, especially if the time interval between successive observations is short, such as weeks or days.
- We expect stock market prices to move or move down for several days in succession.
- We experience autocorrelation when $E(u_i u_j) \neq 0$
- Tintner defines autocorrelation as 'lag correlation of a given series within itself, lagged by a number of times units' whereas serial correlation is the 'lag correlation between two different series'.

AUTO CORRELATION

The autocorrelation function of a random process X(t) is thecorrelation $E[X_1X_2]$ of tworandom variables $X_1 = X(t_1)\, and\ X_2 = X(t_2)$ by the process at times t_1 and t_2

- $R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$

- Assuming a second-order stationaryprocess

- $R_{XX}(t, t + \tau) = E[X(t)X(t, t + \tau)] = R_{XX}(\tau)$

PROPERTIES OF ACF

(1) $R_{XX}(\tau) \leq R_{XX}(0)$

(2) $R_{XX}(-\tau) = R_{XX}(\tau)$

(3) $R_{XX}(0) = E[X^2(t)]$

(4) $X(t)$ is stationary & ergodic with no periodic components $\Rightarrow \lim_{|\tau|\to\infty} R_{XX}(\tau) = X^2$

(5) If $X(t)$ is stationary and has a periodic component

$\Rightarrow$ $R_{XX}(\tau)$ has a periodic component with the same period.

(6) The autocorrelation function of random process $R_{XX}(\tau)$ cannot have any arbitrary shape.

Cross Correlation - Properties

- Properties of cross-correlation function of jointly WSS random processes

$$(1) \quad R_{XY}(-\tau) = R_{YX}(\tau)$$

$$(2) \quad |R_{XY}(\tau)| \le \sqrt{R_{XX}(0)R_{YY}(0)}$$

$$(3) \quad |R_{XY}(\tau)| \le \frac{1}{2}\left[R_{XX}(0) + R_{YY}(0)\right]$$

$$E[\{Y(t+\tau) + \alpha X(t)\}^2] \ge 0, \quad \forall \alpha$$

$$\sqrt{R_{XX}(0)R_{YY}(0)} \le \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$$

AUTO COVARIANCE

Auto Covariance function: Consider two random processes X(t) and X(t+ τ) at two time intervals t and t+ τ. The auto covariance function can be expressed as

$C_{XX}(t, t+\tau) = E[(X(t)-E[X(t)]) ((X(t+\tau) - E[X(t+\tau)])]$ or

$C_{XX}(t, t+\tau) = R_{XX}(t, t+\tau) - E[(X(t) E[X(t+\tau)]$

If X(t) is WSS, then $C_{XX}(\tau) = R_{XX}(\tau) - \bar{X}^2$. At $\tau = 0$, $C_{XX}(0) = R_{XX}(0) - \bar{X}^2 = E[X^2] - \bar{X}^2 = \sigma X^2$

Therefore at τ = 0, the auto covariance function becomes the Variance of the random process. The autocorrelation coefficient of the random process, X(t) is defined as

$\rho_{XX}(t, t+\tau) = \frac{C_{XX}(t,t+\tau)}{\sqrt{C_{XX}(t,t)C_{XX}(t+\tau,t+\tau)}}$ if $\tau = 0$,

$\rho_{XX}(0) = \frac{C_{XX}(t,t)}{C_{XX}(t,t)} = 1$. Also $|\rho_{XX}(t, t+\tau)| \leq 1$.

CROSS COVARIANCE

Cross Covariance Function: If two random processes X(t) and Y(t) have random variables X(t) and Y(t+ τ), then the cross covariance function can be defined as

$C_{XY}(t, t+\tau) = E[(X(t)-E[X(t)]) ((Y(t+\tau) – E[Y(t+\tau)])]$ or

$C_{XY}(t, t+\tau) = R_{XY}(t, t+\tau) - E[(X(t) E[Y(t+\tau)]$. If X(t) and Y(t) are jointly WSS, then

$C_{XY}(\tau) = R_{XY}(\tau) -\bar{X}\,\bar{Y}$. If X(t) and Y(t) are Uncorrelated then

$C_{XY}(t, t+\tau) =0$.

The cross correlation coefficient of random processes X(t) and Y(t) is defined as

$$\rho_{XY}(t, t+\tau) = \frac{C_{XY}(t,t+\tau)}{\sqrt{C_{XX}(t,t)C_{YY}(t+\tau,t+\tau)}} \text{ if } \tau = 0,$$

$$\rho_{XY}(0) = \frac{C_{XY}(0)}{\sqrt{C_{XX}(0)C_{YY}(0)}} = \frac{C_{XY}(0)}{\sigma_X \sigma_Y}.$$

GAUSSIAN & POISSON RANDOM PROCESS

Gaussian Random Process: Consider a continuous random process X(t). Let N random variables $X_1=X(t_1), X_2=X(t_2), \ldots, X_N=X(t_N)$ be defined at time intervals $t_1, t_2, \ldots t_N$ respectively. If random variables are jointly Gaussian for any N=1,2,…. And at any time instants $t_1, t_2, \ldots t_N$. Then the random process X(t) is called Gaussian random process. The Gaussian density function is given as

$$f_X(x_1, x_2 \ldots x_N; t_1, t_2 \ldots t_N) = \frac{1}{(2\pi)^{N/2} |[C_{XX}]|^{1/2}} exp(-[X-\overline{X}]^T [C_{XX}]^{-1} [X-\overline{X}])/2$$

where C_{XX} is a covariance matrix.

Poisson's random process: The Poisson process X(t) is a discrete random process which represents the number of times that some event has occurred as a function of time. If the number of occurrences of an event in any finite time interval is described by a Poisson distribution with the average rate of occurrence is λ, then the probability of exactly occurrences over a time interval (0,t) is

$$P[X(t)=K] = \frac{(\lambda t)^K e^{-\lambda t}}{k!}, \quad K=0,1,2, \ldots$$

And the probability density function is

$$f_X(x) = \sum_{k=0}^{\infty} \frac{(\lambda t)^K e^{-\lambda t}}{k!} \delta(x-k).$$

RANDOM PROCESSES – SPECTRAL CHARACTERISTICS

- Consider a random process X (t). The amplitude of the random process, when it varies randomly with time, does not satisfy Dirichlet's conditions.
- Therefore it is not possible to apply the Fourier transform directly on the random process for a frequency domain analysis. Thus the autocorrelation function of a WSS random process is used to study spectral characteristics such as power density spectrum or power spectral density (psd

- ❑ Fourier integral

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty}x(\tau)e^{-j\omega\tau}d\tau]e^{j\omega t}d\omega$$

- ❑ Fourier transform

$$X(\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$$

- ❑ Inverse Fourier transform

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)e^{j\omega t}d\omega$$

INTRODUCTION

$$x_T(t) = \begin{cases} x(t), & -T < t < T \\ 0, & o/w \end{cases}$$

Assume $\int_{-T}^{T} |x_T(t)|\,dt < \infty$, for all finite T.

$$X_T(\omega) = \int_{-\infty}^{\infty} x_T(t) e^{-j\omega t}\,dt = \int_{-T}^{T} x(t) e^{-j\omega t}\,dt$$

- Energy contained in x(t) in the interval (-T,T)

$$E(T) = \int_{-\infty}^{\infty} x_T(t)^2\,dt = \int_{-T}^{T} x(t)^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_T(\omega)|^2\,d\omega$$

POWER SPECTRAL DENSITY

- Average power in x(t) in the interval (-T,T)

$$P(T) = \frac{1}{2T}\int_{-T}^{T} x(t)^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|X_T(\omega)|^2}{2T}\,d\omega$$

$$x(t) \to X(t), \quad \text{take expectation,} \quad \text{let } T \to \infty.$$

- Average power in random process x(t)

$$P_{XX} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} E[X(t)^2]\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty}\frac{E[|X_T(\omega)|^2]}{2T}\,d\omega$$

$$P_{XX} = A\{E[X(t)^2]\} \qquad P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\,d\omega$$

$$S_{XX} = \lim_{T\to\infty}\frac{E[|X_T(\omega)|^2]}{2T} \quad \text{power density spectrum}$$

PROPERTIES OF PSD

(1) $S_{XX}(\omega) \geq 0$

(2) $X(t)$ real $\Rightarrow$ $S_{XX}(-\omega) = S_{XX}(\omega)$

(3) $S_{XX}(\omega)$ is real

$$S_{XX}(\omega) = \lim_{T \to \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

(4) $\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)d\omega = A\{E[X(t)^2]\}$

PF of (2): $X_T(\omega) = \int_{-T}^{T} X(t)e^{-j\omega t}dt$

$$X_T(\omega)^* = \int_{-T}^{T} X(t)^* e^{j\omega t}dt = \int_{-T}^{T} X(t)e^{j\omega t}dt = X_T(-\omega)$$

$$S_{XX}(-\omega) = \lim_{T \to \infty} \frac{E[X_T(-\omega)X_T(-\omega)^*]}{2T} = \lim_{T \to \infty} \frac{E[X_T(\omega)^* X_T(\omega)]}{2T} = S_{XX}(\omega)$$

PROPERTIES OF PSD

(5) $S_{\dot{X}\dot{X}}(\omega) = \omega^2 S_{XX}(\omega)$ $\qquad$ $\frac{d}{dt}X(t) = \lim_{\varepsilon \to 0} \frac{X(t+\varepsilon) - X(t)}{\varepsilon}$

PF of (5):

$$\dot{X}_T(t) = \begin{cases} \lim_{\varepsilon \to 0} \frac{X(t+\varepsilon) - X(t)}{\varepsilon}, & -T < t < T \\ 0, & \text{o/w} \end{cases}$$

$$f(t-a) \xleftrightarrow{FT} F(\omega)e^{-j\omega a}$$

$$\dot{X}_T(t) \xleftrightarrow{\text{FT}} \lim_{\varepsilon \to 0} \frac{X_T(\omega)e^{j\omega\varepsilon} - X_T(\omega)}{\varepsilon} = j\omega X_T(\omega)$$

$$S_{\dot{X}\dot{X}}(\omega) = \lim_{T \to \infty} \frac{E[|\dot{X}_T(\omega)|^2]}{2T} = \lim_{T \to \infty} \frac{E[|j\omega X_T(\omega)|^2]}{2T} = \omega^2 \lim_{T \to \infty} \frac{E[|X_T(\omega)|^2]}{2T} = \omega^2 S_{XX}(\omega)$$

BANDWIDTH

Bandwidth of the power density spectrum

$$X(t) \text{ real} \Rightarrow S_{XX}(\omega) \text{ even}$$

$S_{XX}(\omega)$ lowpass form $\Rightarrow$

$$W_{\text{rms}}^2 = \frac{\int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega)d\omega}{\int_{-\infty}^{\infty} S_{XX}(\omega)d\omega}$$

root mean square Bandwidth

$S_{XX}(\omega)$ bandpass form $\Rightarrow$ $\bar{\omega}_0 = \frac{\int_0^{\infty} \omega S_{XX}(\omega)d\omega}{\int_0^{\infty} S_{XX}(\omega)d\omega}$ mean frequency

$$W_{\text{rms}}^2 = \frac{4\int_0^{\infty} (\omega - \bar{\omega}_0)^2 S_{XX}(\omega)d\omega}{\int_0^{\infty} S_{XX}(\omega)d\omega}$$ rms BW

RELATIONSHIP BETWEEN PSD AND ACF

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{j\omega\tau}d\omega = A[R_{XX}(t,t+\tau)]$$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} A[R_{XX}(t,t+\tau)]e^{-j\omega\tau}d\tau$$

$$S_{XX}(\omega) = \lim_{T\to\infty}\frac{E[X_T(\omega)^* X_T(\omega)]}{2T} = \lim_{T\to\infty}\frac{1}{2T}E[\int_{-T}^{T} X(t_1)e^{j\omega t_1}dt_1 \int_{-T}^{T} X(t_2)e^{-j\omega t_2}dt_2]$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} E[X(t_1)X(t_2)]e^{j\omega(t_1-t_2)}dt_2 dt_1$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2)e^{j\omega(t_1-t_2)}dt_2 dt_1$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{j\omega\tau}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2)e^{j\omega(t_1-t_2)}dt_2 dt_1 \; e^{j\omega\tau}d\omega$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2)\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(\tau+t_1-t_2)}d\omega\, dt_2 dt_1$$

RELATIONSHIP BETWEEN PSD AND ACF

$$\delta(t) \xleftrightarrow{FT} 1$$

$$\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} d\omega$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_{XX}(t_1, t_2) \delta(\tau + t_1 - t_2) dt_2 dt_1$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{XX}(t_1, t_1 + \tau) dt_1 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{XX}(t, t + \tau) dt$$

$$= A[R_{XX}(t, t + \tau)]$$

$$A[R_{XX}(t, t + \tau)] \xleftrightarrow{FT} S_{XX}(\omega)$$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} A[R_{XX}(t, t + \tau)] e^{-j\omega\tau} d\tau$$

RELATIONSHIP BETWEEN PSD AND ACF

$$X(t) \quad \text{w.s.s.} \quad \Rightarrow \quad A[R_{XX}(t, t+\tau)] = R_{XX}(\tau)$$

$$R_{XX}(\tau) \quad \xleftrightarrow{FT} \quad S_{XX}(\omega)$$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} \, d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} \, d\omega$$

EXAMPLE

$$X(t) = A\cos(\omega_0 t + \Theta)$$

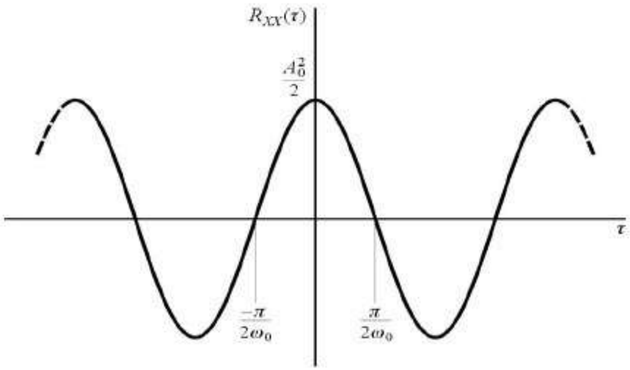
$$R_{XX}(\tau) = \frac{A_0^2}{2}\cos(\omega_0 \tau)$$

$$R_{XX}(\tau) = \frac{A_0^2}{4}(e^{j\omega_0\tau} + e^{-j\omega_0\tau})$$

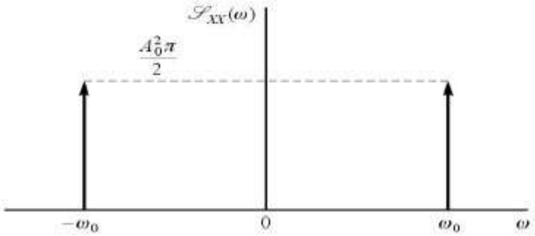
$$x(t)e^{j\alpha t} \quad \overset{FT}{\longleftrightarrow} \quad X(\omega - \alpha)$$

$$1 \quad \overset{FT}{\longleftrightarrow} \quad 2\pi\delta(\omega)$$

$$S_{XX}(\omega) = \frac{A_0^2\pi}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$



(a)



(b)

CROSS POWER SPECTRAL DENSITY

$$W(t) = X(t) + Y(t)$$

$$R_{WW}(t,t+\tau) = E[W(t)W(t+\tau)] = E\{[X(t)+Y(t)][X(t+\tau)+Y(t+\tau)]\}$$
$$= R_{XX}(t,t+\tau) + R_{YY}(t,t+\tau) + R_{XY}(t,t+\tau) + R_{YX}(t,t+\tau)$$

$$S_{WW}(\omega) = S_{XX}(\omega) + S_{YY}(\omega) + F\{A[R_{XY}(t,t+\tau)]\} + F\{A[R_{YX}(t,t+\tau)]\}$$

CROSS POWER SPECTRAL DENSITY

$$x_T(t) = \begin{cases} x(t), & -T < t < T \\ 0, & o/w \end{cases} \qquad y_T(t) = \begin{cases} y(t), & -T < t < T \\ 0, & o/w \end{cases}$$

Assume $\int_{-T}^{T} |x_T(t)| dt < \infty$ & $\int_{-T}^{T} |y_T(t)| dt < \infty$, for all finite T.

$$x_T(t) \xleftrightarrow{\text{FT}} X_T(\omega) \qquad y_T(t) \xleftrightarrow{\text{FT}} Y_T(\omega)$$

Cross Power contained in $x(t), y(t)$ in the interval $(-T, T)$

$$P_{XY}(T) = \frac{1}{2T}\int_{-\infty}^{\infty} x_T(t) y_T(t) dt = \frac{1}{2T}\int_{-T}^{T} x(t) y(t) dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{X_T(\omega)^* Y_T(\omega)}{2T} d\omega$$

Parseval's theorem

CROSS POWER SPECTRAL DENSITY

average Cross Power contained in $X(t), Y(t)$ in the interval $(-T, T)$

$$\bar{P}_{XY}(T) = \frac{1}{2T}\int_{-T}^{T} R_{XY}(t,t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{E[X_T(\omega)^* Y_T(\omega)]}{2T}\,d\omega$$

total average Cross Power contained in $X(t), Y(t)$

$$P_{XY} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_{XY}(t,t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty}\frac{E[X_T(\omega)^* Y_T(\omega)]}{2T}\,d\omega$$

cross-power density spectrum $\qquad S_{XY}(\omega) = \lim_{T\to\infty}\frac{E[X_T(\omega)^* Y_T(\omega)]}{2T}$

CROSS POWER SPECTRAL DENSITY

$$P_{XY} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)d\omega$$

$$S_{YX}(\omega) = \lim_{T\to\infty} \frac{E[Y_T(\omega)^* X_T(\omega)]}{2T}$$

$$P_{YX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YX}(\omega)d\omega = P_{XY}^*$$

$$\text{Total cross power} = P_{XY} + P_{YX}$$

$$X(t), Y(t) \text{ orthogonal} \Rightarrow P_{XY} = P_{YX} = 0$$

PROPERTIES OF CROSS POWER SPECTRAL DENSITY (CPSD)

(1) $S_{XY}(\omega) = S_{YX}(-\omega) = S_{YX}(\omega)^*$

PF of (1): $X_T(\omega) = \int_{-T}^{T} X(t) e^{-j\omega t} dt$

$$X_T(\omega)^* = \int_{-T}^{T} X(t)^* e^{j\omega t} dt = \int_{-T}^{T} X(t) e^{j\omega t} dt = X_T(-\omega)$$

$$S_{YX}(-\omega) = \lim_{T\to\infty} \frac{E[Y_T(-\omega)^* X_T(-\omega)]}{2T} = \lim_{T\to\infty} \frac{E[Y_T(\omega) X_T(\omega)^*]}{2T} = S_{XY}(\omega)$$

$$S_{YX}(-\omega) = \lim_{T\to\infty} \frac{E[Y_T(-\omega)^* X_T(-\omega)]}{2T} = \lim_{T\to\infty} \frac{E[Y_T(\omega) X_T(\omega)^*]}{2T} = S_{YX}(\omega)^*$$

(2) $\mathrm{Re}[S_{XY}(\omega)]$ & $\mathrm{Re}[S_{YX}(\omega)]$ -- even

(3) $\mathrm{Im}[S_{XY}(\omega)]$ & $\mathrm{Im}[S_{YX}(\omega)]$ -- odd

$$A[R_{XY}(t, t+\tau)] \xleftrightarrow{FT} S_{XY}(\omega)$$

$$A[R_{YX}(t, t+\tau)] \xleftrightarrow{FT} S_{YX}(\omega)$$

PROPERTIES OF CROSS POWER SPECTRAL DENSITY (CPSD)

$$(4)\ \ X(t)\ \&\ Y(t)\ \ \text{orthogonal} \ \Rightarrow \ S_{XY}(\omega) = S_{YX}(\omega) = 0$$

$$X(t)\ \&\ Y(t)\ \ \text{orthogonal} \ \Rightarrow \ R_{XY}(t, t+\tau) = 0 \ \Rightarrow \ \mathrm{A}[R_{XY}(t, t+\tau)] = 0$$

$$(5)\ \ X(t)\ \&\ Y(t)\ \ \text{uncorrelated \& have constant mean } \overline{X}, \overline{Y}$$

$$\Rightarrow \ S_{XY}(\omega) = S_{YX}(\omega) = 2\pi \overline{X}\,\overline{Y}\delta(\omega)$$

$$\text{PF of (5):} \quad R_{XY}(t, t+\tau) = \overline{X}\,\overline{Y} \ \Rightarrow \ A[R_{XY}(t, t+\tau)] = \overline{X}\,\overline{Y}$$

$$\Rightarrow \ S_{XY}(\omega) = 2\pi \overline{X}\,\overline{Y}\delta(\omega) = S_{YX}(\omega)^*$$

$$X(t), Y(t) \text{ -- jointly w.s.s.} \ \Rightarrow \ R_{XY}(\tau) \xleftrightarrow{\text{FT}} S_{XY}(\omega)$$

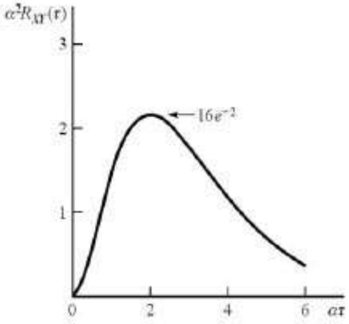
$$R_{YX}(\tau) \xleftrightarrow{\text{FT}} S_{YX}(\omega)$$

$X(t), Y(t)$ -- jointly w.s.s.

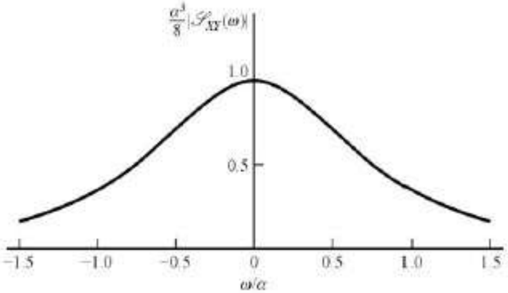
$$S_{XY}(\omega) = \frac{8}{(\alpha + j\omega)^3} \qquad \alpha > 0$$

$$u(\tau)\tau^2 e^{-\alpha\tau} \overset{FT}{\longleftrightarrow} \frac{2}{(\alpha + j\omega)^3}$$

$$R_{XY}(\tau) = 4u(\tau)\tau^2 e^{-\alpha\tau}$$



(a)



(b)

RELATIONSHIP BETWEEN CPSD AND CCF

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)e^{j\omega\tau}\,d\omega = A[R_{XY}(t,t+\tau)]$$

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} A[R_{XY}(t,t+\tau)]e^{-j\omega\tau}\,d\tau$$

$$S_{XY}(\omega) = \lim_{T\to\infty}\frac{E[X_T(\omega)^{*}Y_T(\omega)]}{2T} = \lim_{T\to\infty}\frac{1}{2T}E[\int_{-T}^{T} X(t_1)e^{j\omega t_1}dt_1 \int_{-T}^{T} Y(t_2)e^{-j\omega t_2}dt_2]$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} E[X(t_1)Y(t_2)]e^{j\omega(t_1-t_2)}dt_2dt_1$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XY}(t_1,t_2)e^{j\omega(t_1-t_2)}dt_2dt_1$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)e^{j\omega\tau}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XY}(t_1,t_2)e^{j\omega(t_1-t_2)}dt_2dt_1\; e^{j\omega\tau}d\omega$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XY}(t_1,t_2)\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(\tau+t_1-t_2)}d\omega dt_2 dt_1$$

RELATIONSHIP BETWEEN CPSD AND CCF

$$\delta(t) \xleftrightarrow{FT} 1$$

$$\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} d\omega$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) e^{j\omega\tau} d\omega = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_{XY}(t_1, t_2) \delta(\tau + t_1 - t_2) dt_2 dt_1$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{XY}(t_1, t_1 + \tau) dt_1 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{XY}(t, t + \tau) dt$$

$$= A[R_{XY}(t, t+\tau)]$$

$$A[R_{XY}(t, t+\tau)] \xleftrightarrow{FT} S_{XY}(\omega)$$

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} A[R_{XY}(t, t+\tau)] e^{-j\omega\tau} d\tau$$

EXAMPLE

$$R_{XY}(t,t+\tau) = \frac{AB}{2}\{\sin(\omega_0\tau) + \cos[\omega_0(2t+\tau)]\}$$

$$A[R_{XY}(t,t+\tau)] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_{XY}(t,t+\tau)dt$$

$$= \frac{AB}{2}\sin(\omega_0\tau) + \frac{AB}{2}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\cos[\omega_0(2t+\tau)]dt$$

$$= \frac{AB}{2}\sin(\omega_0\tau) = \frac{AB}{4j}[e^{j\omega_0\tau} - e^{-j\omega_0\tau}]$$

$$S_{XY}(\omega) = \frac{AB}{4j}[2\pi\delta(\omega-\omega_0) - 2\pi\delta(\omega+\omega_0)] = \frac{-j\pi AB}{2}[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)]$$

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THANK YOU