DIGITAL SIGNAL PROCESSING
UNIT-4

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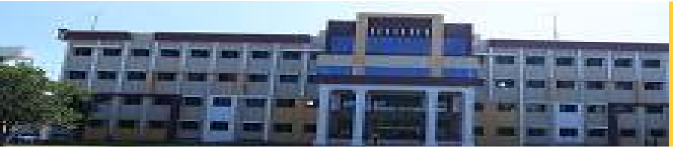
SUBJECT CODE:15A04603

B.TECH III.II SEM



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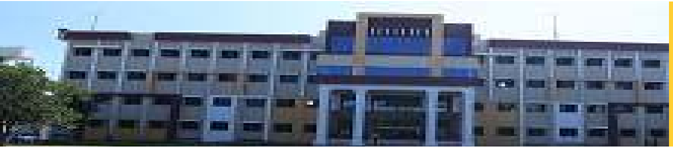
UNIT-4

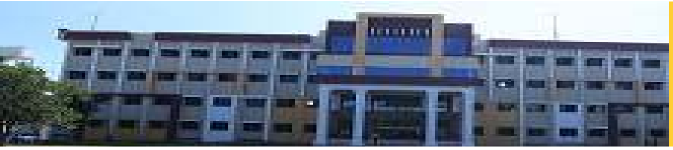
IIR AND FIR Filter DESIGN



Contents

IIR DIGITAL FILTERS:

- *Analog filter approximations – Butter worth and Chebyshev*
- *Design of IIR Digital filters from analog filters*
- *Design Examples: Analog-Digital transformations*
    - *Impulse invariant method*
    - *Bilinear transformation*



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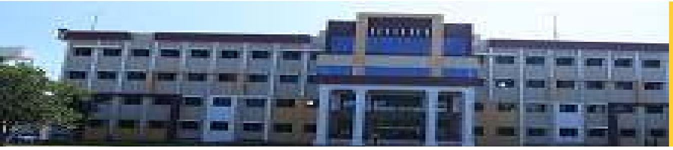
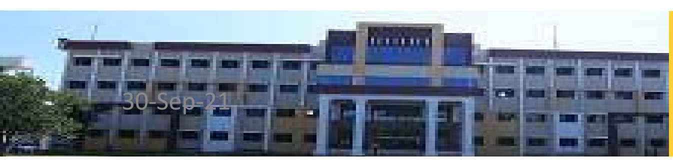
FIR DIGITAL FILTERS:

- *Characteristics of FIR Digital Filters,*
- *Frequency response.*
- *Design of FIR Digital Filters using Window Techniques,*
- *Frequency Sampling technique*
- *Comparison of IIR & FIR filters.*
- ***Applications: Design of IIR/FIR digital filter conforming to given specifications.***

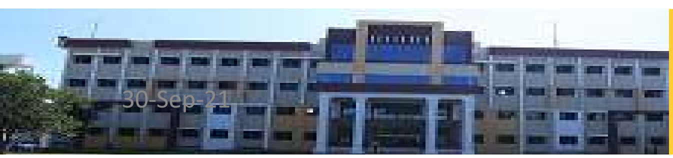


IIR DIGITAL FILTERS

- Introduction
  - Filtering of signals
  - Classification of filters
    - **Analog and digital**
    - **Based on frequency response**
- Practical analog filter specifications
  - LPF. HPF, BPF and BSF.
- Analog filters approximation
  - Butterworth and Chebyshev
  - Finding Order and normalized Stable filter
  - Design examples
- Analog to analog transformations
- Design of IIR Digital filters from analog filters
  - Impulse invariance, step invariance and bilinear transformations
  - Design examples
- Analog to digital transformations



- Difference between IIR System and FIR System
- Definition of filter , pass band and stop band
- Difference between analog filter and digital filter
- Advantages of digital filter
- Disadvantages of digital filter



What is meant by a filter!

The DTFT is remembered again:

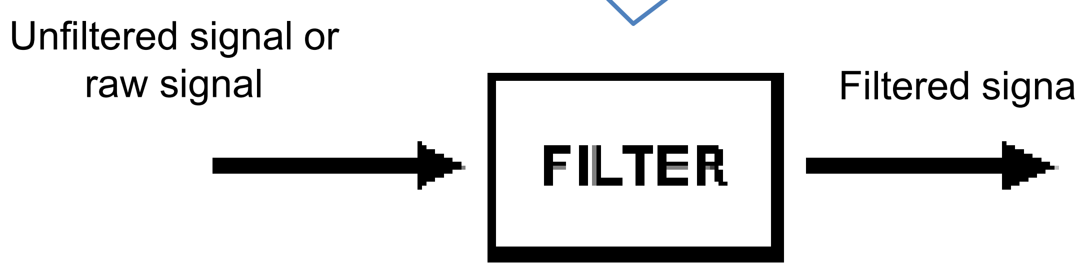
$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw})e^{jwn}dw$$

- x[n] is expressed as a summation of sinusoids with scaled amplitude.
- Using a system with a frequency selective to these inputs, then it is possible to pass some frequencies and attenuate the others.
- **Such a system is called a Filter**.

What is meant by a filter!

The function of a *filter* is to remove unwanted parts of the signal, such as random noise, or to extract useful parts of the signal, such as the components lying within a certain frequency range.



Example.1

Choose frequency response of a system such that

$$\left|H(e^{jw})\right| = \begin{cases} 1 & for\ |\omega| \le \omega_c \\ 0 & for\ \omega_c \le |\omega| \le \pi \end{cases}$$

Almost = 0

If $x[n] = A\cos(\omega_1 n) + B\cos(\omega_2 n)\ \ for\ \ 0 < \omega_1 < \omega_c < \omega_2 < \pi$

$$y[n] = A\left|H(e^{j\omega_1})\right|\cos\left(\omega_1 n + \theta(\omega_1)\right) + B\left|H(e^{j\omega_2})\right|\cos\left(\omega_2 n + \theta(\omega_2)\right)$$

$$y[n] = A\left|H(e^{j\omega_1})\right|\cos\left(\omega_1 n + \theta(\omega_1)\right)$$

Which indicating the LPF effect of the LTI system

Example.2

Design a HP digital filter that passes the 0.4rad/sec, and stops the 0.1 rad/sec frequency.

h[n]=[a $\underset{\uparrow}{b}$ a]

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} = a + be^{-j\omega} + ae^{-j2\omega}$$

$$= a(1+e^{-j2\omega}) + be^{-j\omega} = 2a\left(\frac{e^{j\omega}+e^{-j\omega}}{2}\right)e^{-j\omega} + be^{-j\omega}$$

$$= (2a\cos\omega + b)e^{-j\omega}$$

$$\left|H(e^{j\omega})\right| = 2a\cos\omega + b$$

$\Theta(\omega) = -\omega$

$$\left|H(e^{j0.1})\right| = 2a\cos(0.1) + b = 0$$

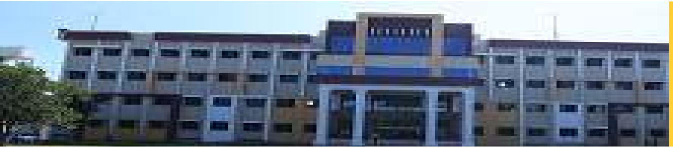
$$\left|H(e^{j0.4})\right| = 2a\cos(0.4) + b = 1$$

Solving for the two equations gives a= -6.76195,

b=13.456335

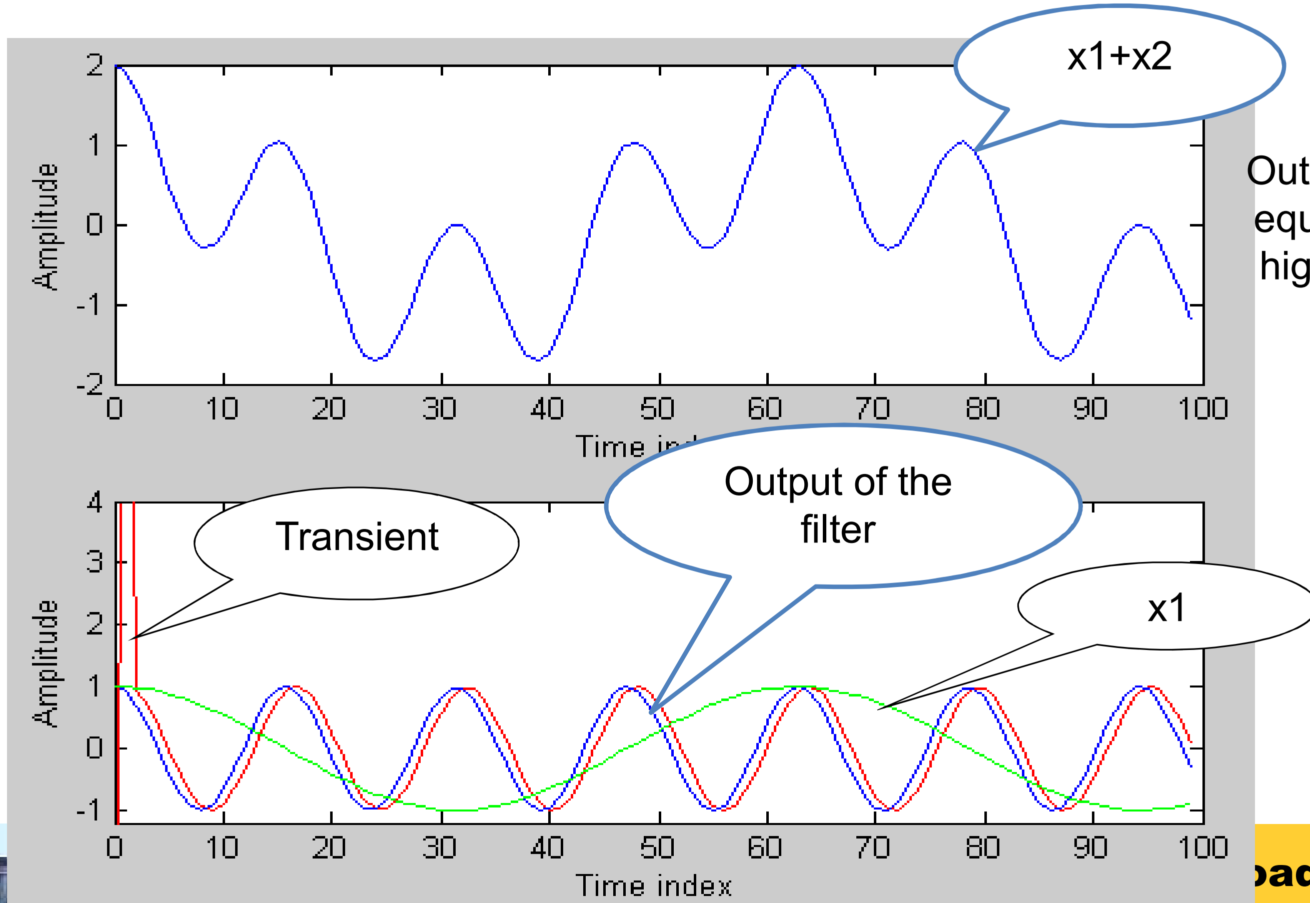
y[n]=h[n]*x[n]

y[n]=h[0]x[n]+h[1]x[n-1]+h[2]x[n-2]=ax[n]+bx[n-1]+ax[n-2]

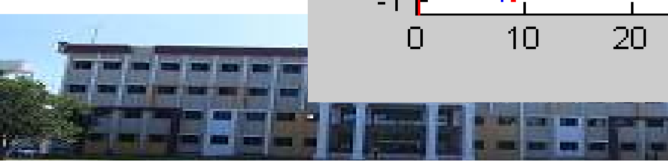


$$y[n] = -6.76195\,(\,x[n]+x[n-2]\,)+13.456335\,x[n-1]$$

If $x[n]=\{\cos(0.1n)+\cos(0.4n)\}u(n)$



Output is almost equal to x2, the high frequency



Classification of filters as analog or digital

Analog filters

An **analog** filter processes analog inputs and generates analog outputs.

Analog filters are constructed from passive or active electronic components such as resistors, capacitors and opamps to produce the required filtering effect.

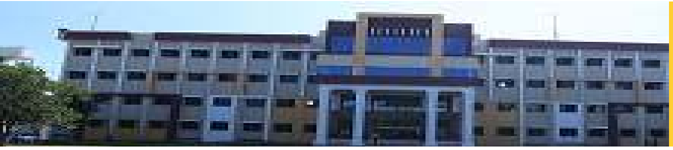
An Analog filter is described by a differential equation.

Digital filters

A **digital** filter processes and generates digital data.

A digital filter consists of elements like adder, multiplier and delay element

Digital filter is described by difference equation.



Classification of filters as analog or digital

Analog filters

The frequency response of an analog filter can be modified by changing the components.

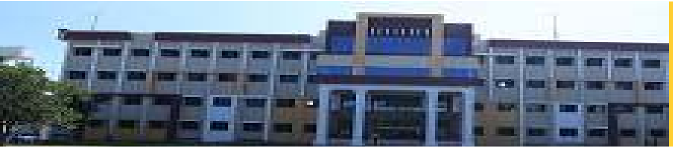
Such filter circuits are widely used in such applications as noise reduction, video signal enhancement, graphic equalizers in hi-fi systems, and many other areas.

Digital filters

The frequency response of digital filter can be changed by changing the filter coefficients

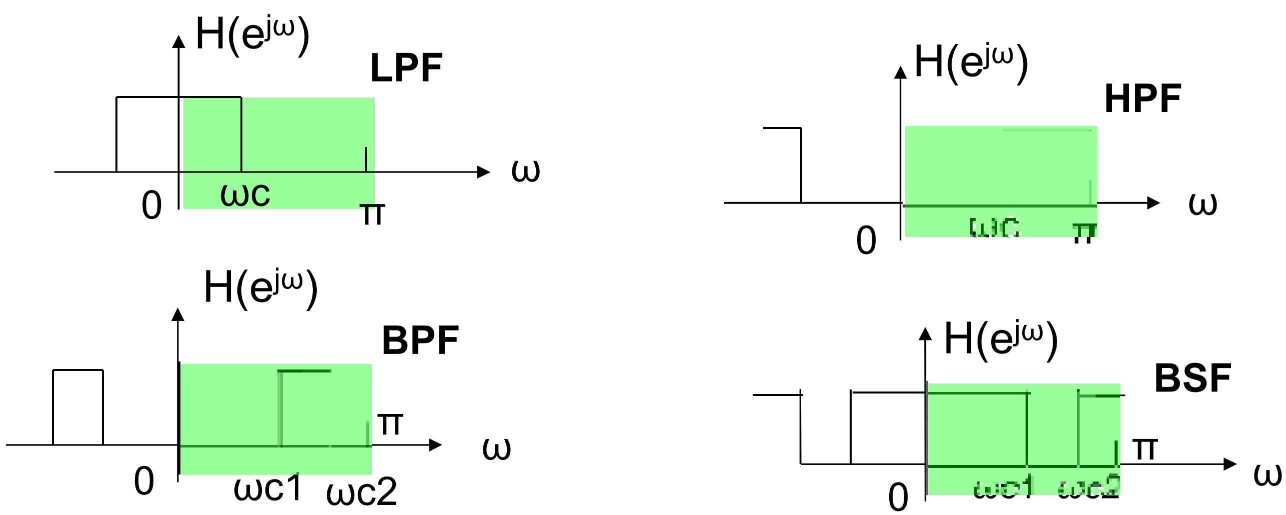
A **digital** filter uses a digital processor to perform numerical calculations on sampled values of the signal.

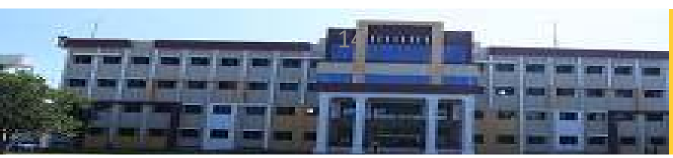
The processor may be a general-purpose computer such as a PC, or a specialized DSP (Digital Signal Processor) chip.



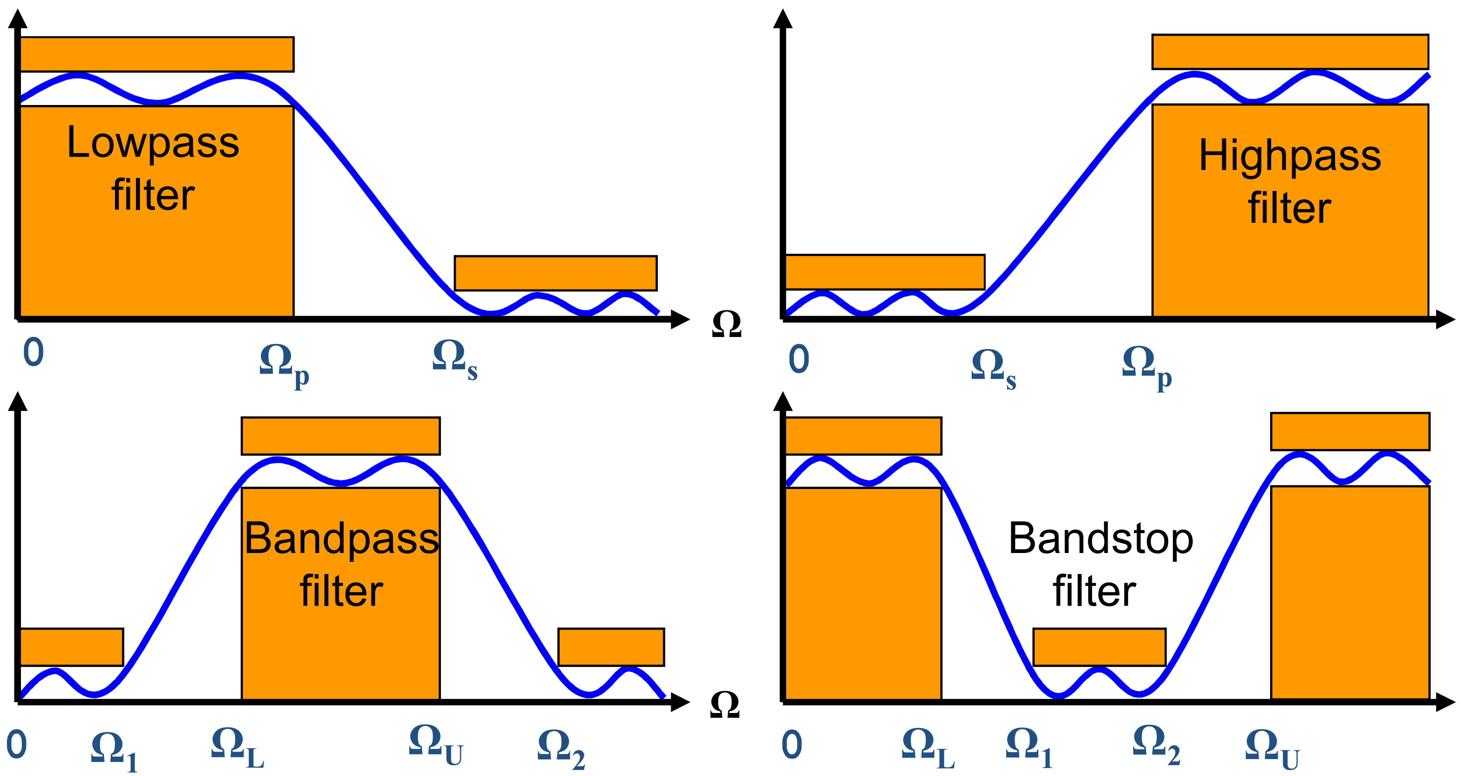
Classification of filters According to frequency response

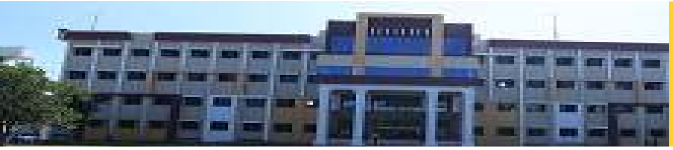
- A analog filter is a network used to shape the frequency spectrum of an electrical signal.
- These networks are essential parts of communication and control systems.
- Filters are classified as low pass, high pass, band pass and band reject, amplitude equalizers and delay equalizers.



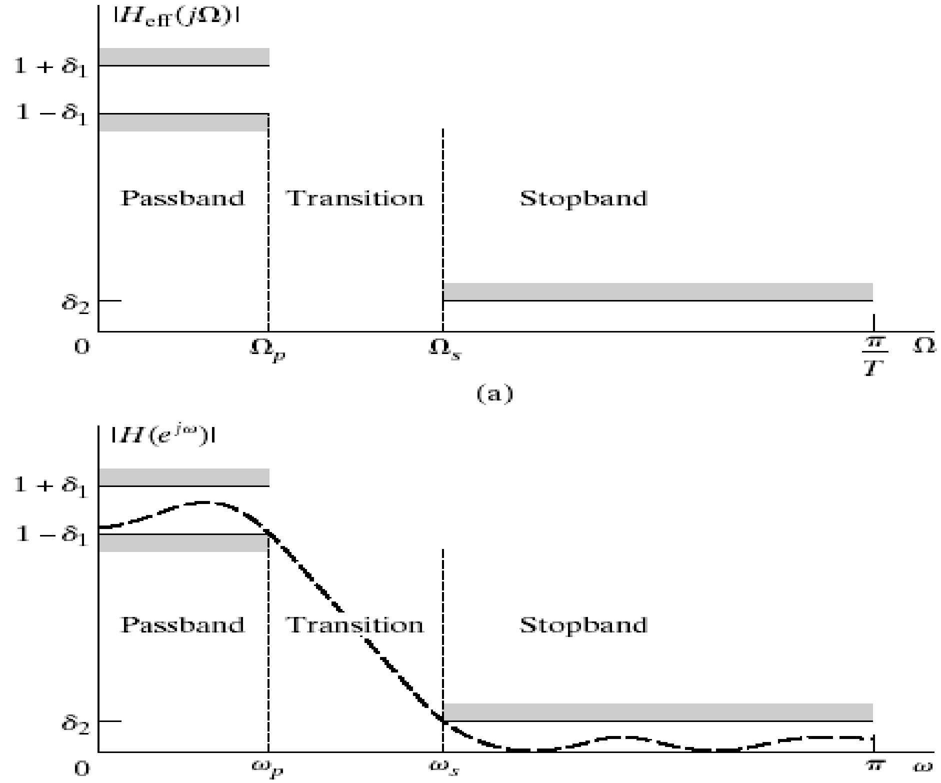


Practical Analog Filter specifications

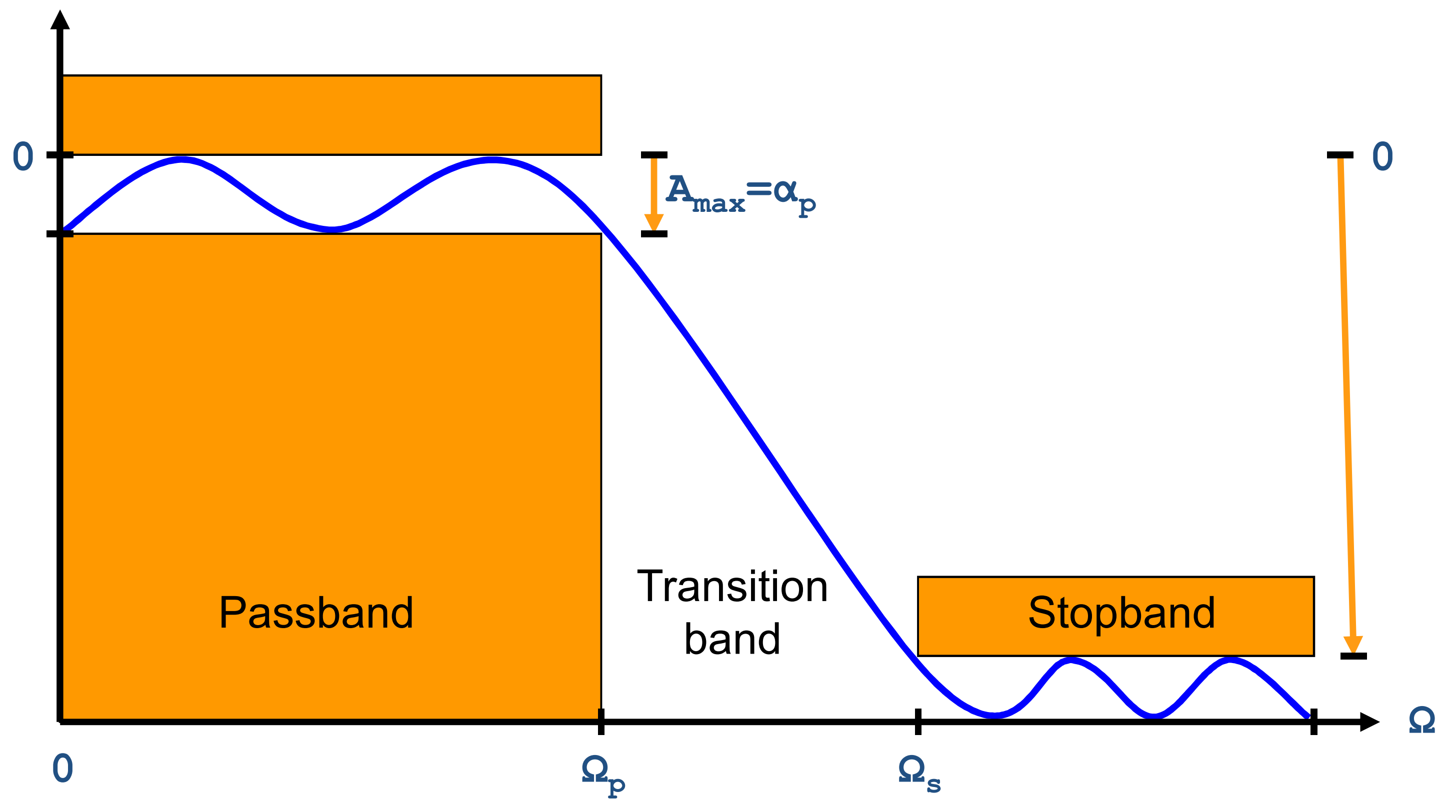




Practical analog Low pass filter specifications

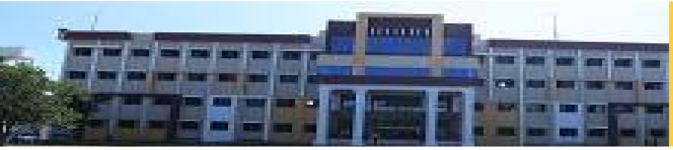


Practical analog low pass filter specifications



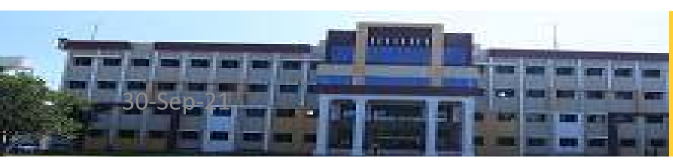
Practical analog Low pass filter specifications

- The basic function a of **LOW PASS filter** is to pass LOW frequencies with very little loss and to attenuate high frequencies.
- It is required to pass signals from DC up to pass band edge frequency Ω_p with at most $\mathbf{A_{max}(\alpha_p)dB}$ of attenuation.
- The frequencies above stop band edge frequency Ω_s are required to have atleast $\mathbf{A_{min}(\alpha_s)dB}$ of attenuation.
- The band of frequencies from 0 to Ω_p is called the **pass band**.
- The band of frequencies from Ω_s to infinity is called the **stop band**.
- The frequency band from Ω_p to Ω_s is referred to as **transition band.**

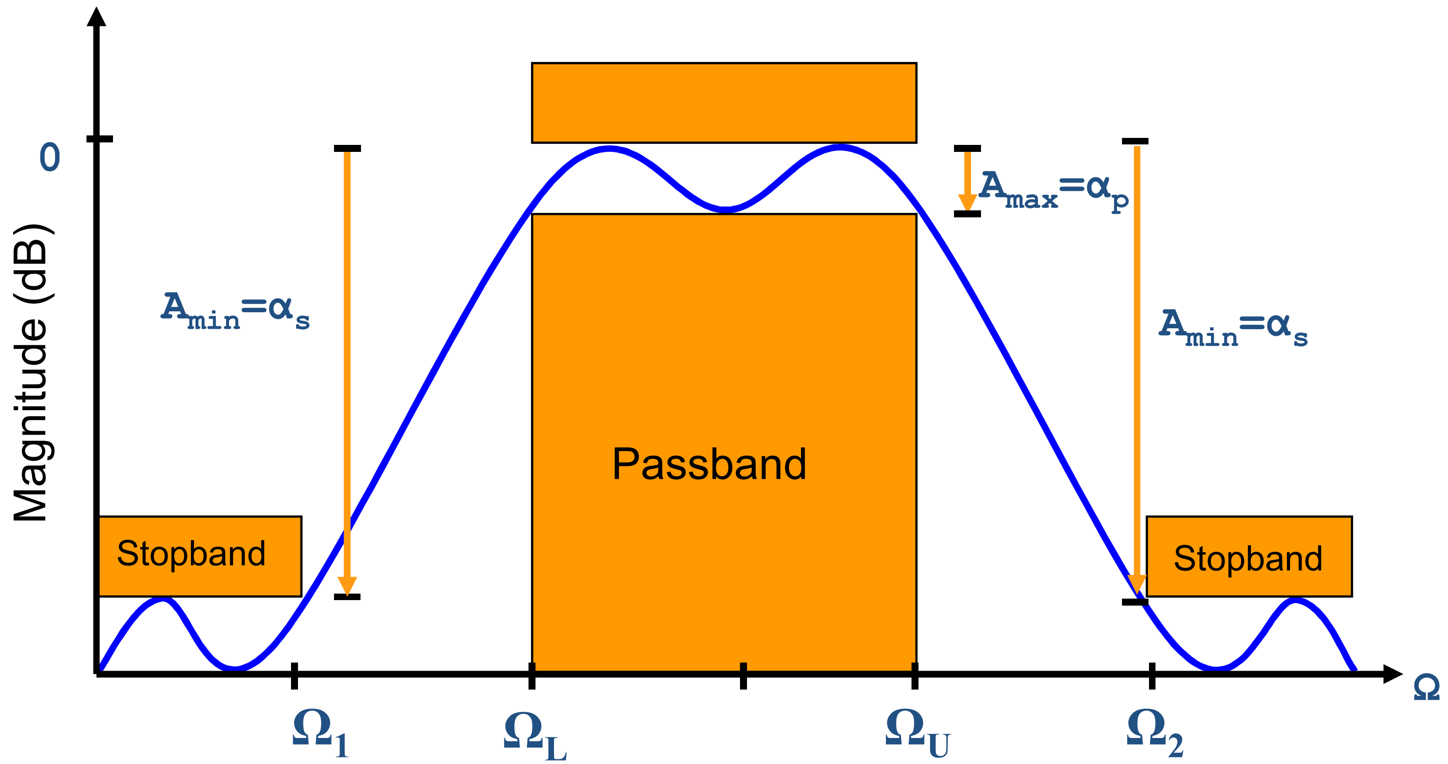


Practical analog high pass filter specifications

- The basic function a of HIGH PASS filter is to pass HIGH frequencies with very little loss and to attenuate low frequencies.
- It is required to pass signals from pass band edge frequency Ω_p up to infinity with at most $A_{max}(\alpha_p)$ dB of attenuation.
- The frequencies below stop band edge frequency Ω_s are required to have atleast $A_{min}(\alpha_s)$dB of attenuation.
- The band of frequencies from Ω_p to infinity is called the pass band.
- The band of frequencies from zero to Ω_s is called the stop band.
- The frequency band from Ω_s to Ω_p is referred to as transition band.

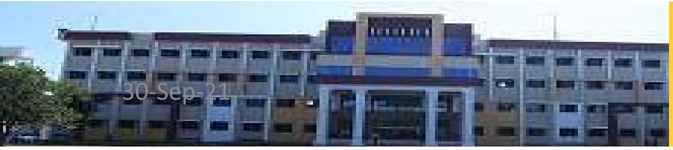


Practical analog band pass filter specifications



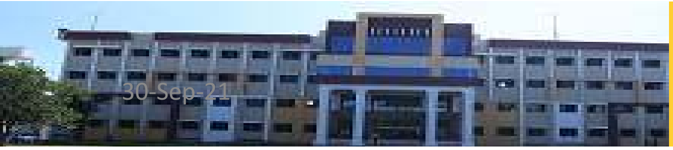
Practical analog Band pass filter specifications

- The basic function a of BAND PASS filter is to pass MIDDLE frequencies with very little loss and to attenuate low and high frequencies.
- It is required to pass signals from lower pass band edge frequency Ω_L to upper pass band edge frequency Ω_u with at most $A_{max}(\alpha_p)$ dB of attenuation.
- The frequencies below lower stop band edge frequency Ω_1 and above upper stop band edge frequency Ω_2 are required to have atleast $A_{min}(\alpha_s)$dB of attenuation.
- The frequency band from Ω_L to Ω_U is called the pass band.
- The band of frequencies from 0 to Ω_1 and Ω_2 to infinity are called the stop bands.
- The band of frequencies from Ω_1 to Ω_L and Ω_U to Ω_2 are referred to as transition bands.



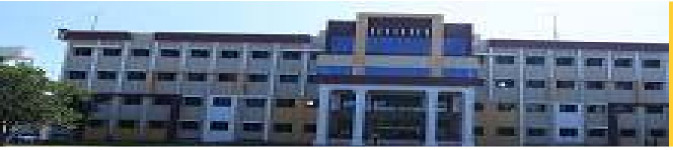
Practical analog Band stop filter specifications

- The basic function a of BAND STOP filter is to attenuate MIDDLE frequencies and to pass low and high frequencies with very little loss .
- It is required to attenuate signals from lower stop band edge frequency Ω_1 to upper stop band edge frequency Ω_2 with at least $A_{min}(\alpha_s)$dB of attenuation.
- The frequencies below lower pass band edge frequency Ω_L and above upper pass band edge frequency Ω_U are required to have at most $A_{max}(\alpha_p)$ dB of attenuation.
- The frequency band from Ω_1 to Ω_2 is called the stop band.
- The band of frequencies from 0 to Ω_L and Ω_U to infinity are called the pass bands.
- The band of frequencies from Ω_L to Ω_1 and Ω_2 to Ω_U are referred to as transition bands.



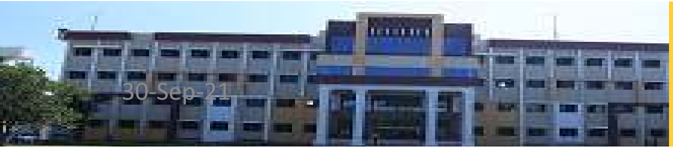
Design of digital filters from analog filters

- The most common techniques used for designing IIR digital filters known as indirect method, involves first designing an analog prototype filter and then transforming the prototype to a digital filter.
- For the given specifications of a digital filter, the derivation of the digital filter transfer function requires three steps
  - Map the desired digital filter transfer function into equivalent analog filter.
  - Derive the analog transfer function for the analog prototype.
  - Transform the transfer function from the analog prototype into an equivalent digital transfer function.



Advantages of digital filters

1. Unlike analog filters, the digital filters performance is not influenced by component aging, temperature and power supply variations.
2. A digital filter is highly immune to noise and posses considerable parameter stability.
3. Digital filters afford a wide variety of shapes for the amplitude and phase responses.
4. There are no problems of input or output impedance matching with digital filters.
5. Digital filters can be operated over a wide range of frequencies.



Advantages of digital filters

6. The coefficients of digital filter can be programmed and altered any time to obtain the desired characteristics.
7. Multiple filtering is possible only in digital filters.

Disadvantage of digital filters

1. The quantization error arises due to finite word length in the representation of signals and parameters.

Design Procedure

- Digital filter is LTI discrete Time System
- filters < IIR filters / FIR filters
- IIR digital filter have transfer function of the form

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = \frac{\sum_{k=0}^{M-1} b_k z^{-k}}{1+\sum_{k=1}^{N-1} a_k z^{-k}}$$

- FIR digital filter $H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = \sum_{k=0}^{M-1} b_k z^{-k}$
- Designing digital filter for given specifications is to find $H(z)$ or a_k, b_k Coefficients

** Analog frequency $(\Omega) \rightarrow$ Radians/sec
Digital frequency $(\omega) \rightarrow$ radians $\quad 0 \leq \omega \leq \pi$

Filter design: (Low pass filter is prototype)

1. Any Analog filter design

1. Design Low pass filter [design H(s)]
2. Analog frequency band transformations

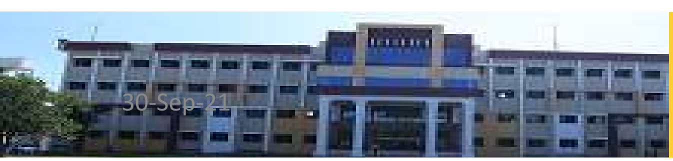
$H(s)_{LP} \rightarrow H(s)_{HP}$ High pass

$H(s)_{LP} \rightarrow H(s)_{BP}$ Band pass

$H(s)_{LP} \rightarrow H(s)_{BS}$ Band stop

2. Any digital filter design

1. Digital filter Specifications (ω) radians $0 \leq \omega \leq \pi$ $\xrightarrow{\text{Transformed to}}$ Analog filter Specifications (Ω) radians/sec
   - Bilinear Transformation (or)
   - Impulse Invarient Method



2. Design Analog low pass filter using analog filter specifications (design $H(s)$)

3. $H(s)_{LP}$ (low pass) $\xrightarrow{\text{Transformed to}}$ $H(z)_{LP}$ (low pass)

- Bilinear Transformation
- Impulse Invariant Method

4. $H(z)$ digital frequency band Transformation

$H(z)_{LP} \rightarrow H(z)_{HP}$ High pass

$H(z)_{LP} \rightarrow H(z)_{BP}$ Band pass

$H(z)_{LP} \rightarrow H(z)_{BS}$ Band stop

5. Realize $H(z)$ using any of the structures (form I, form II, Transpose, Cascade, parallel ...)

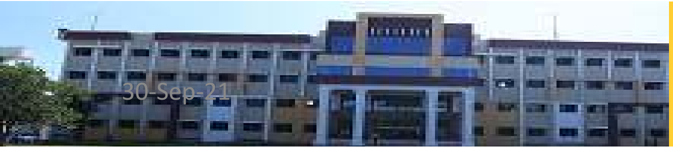
High pass design

$$H(s)_{LP} \xrightarrow[\text{Transformation}]{\text{filter}} H(z)_{LP} \xrightarrow[\text{frequency band Transformation}]{\text{Digital}} H(z)_{HP}$$

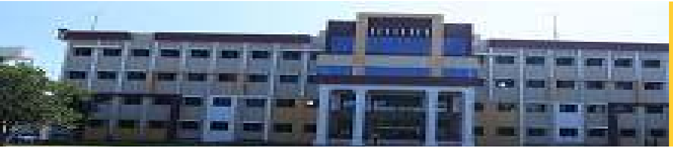
$$H(s)_{LP} \xrightarrow[\text{frequency band Transformation}]{\text{Analog}} H(s)_{HP} \xrightarrow[\text{Transformation}]{\text{filter}} H(z)_{HP}$$

In literature

- There are several methods to design IIR filters.
- All are based on converting stable analog filter into stable Infinite duration Impulse Response Digital filters.



Analog filter approximations



Analog filter approximations

The rational function low pass approximations which we describe in this have the general form.

Magnitude function

$$|H(S)|^2 = |H(j\Omega)|^2 = \left|\frac{V_{OUT}}{V_{IN}}\right|^2 = \frac{1}{1+|K(j\Omega)|^2} = \frac{1}{1+\left|\frac{N(j\Omega)}{D(j\Omega)}\right|^2} \rightarrow (1)$$

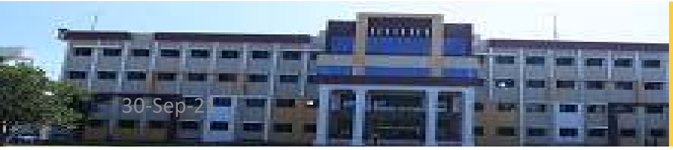
Where H(S) is the desired magnitude function and K(S) is the rational function in S.

- The function K(S) is chosen such that
  - its magnitude is small in pass band to make the magnitude of H(S) close to UNITY.
  - Its magnitude is large in the stop band to make the magnitude of H(S) close to ZERO.

In particular K(S) may be chosen to be a polynomial of the form

$$K(S) = P_N(S) = a_0 + a_1 S + a_2 S^2 + \ldots + a_N S^N \rightarrow (2)$$

Where the coefficients of the n^{th} order polynomial $P_n(S)$ are chosen so that the corresponding loss function satisfies the given filter requirements.



Low pass filter approximation

In particular K(S) may be chosen to be a polynomial of the form

In the pass band

$$i.e \quad as\ \Omega \to 0, \quad K(j\Omega) \to 0 \Rightarrow \left|\frac{V_{IN}}{V_{OUT}}\right|^2 = 1 + \left|K(j\Omega)^2\right| = 1 \Rightarrow V_{OUT} = V_{IN}$$

As expected in the pass band of Low pass filter (near to DC) no loss of signal the signal. But practically there will be some loss.

In the stop band

$$i.e\ as\ \Omega \to \infty, \quad K(j\Omega) \to \infty \Rightarrow \left|\frac{V_{IN}}{V_{OUT}}\right|^2 = 1 + \left|K(j\Omega)^2\right| = \infty \Rightarrow V_{OUT} = 0$$

As expected in the stop band of Low pass filter (high frequencies) no pass of signal the signal. But practically there will be some Pass of signal

Low pass filter approximation

There are four types of polynomials which satisfy these conditions.

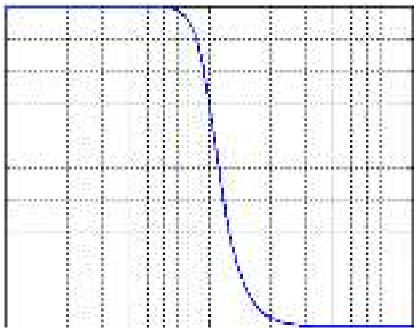
They are

1. Butterworth Polynomial
   (Maximally flat approximation)
1. Chebyshev polynomial
2. Elliptic polynomials
3. Bessel Polynomials

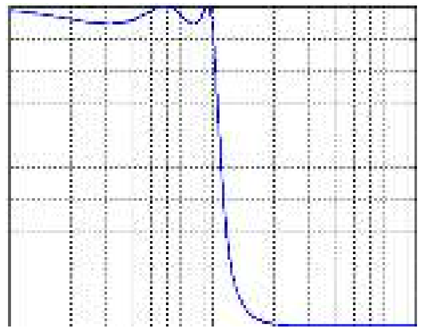
We are going to study

Butterworth filter approximation and Chebyshev filter approximation

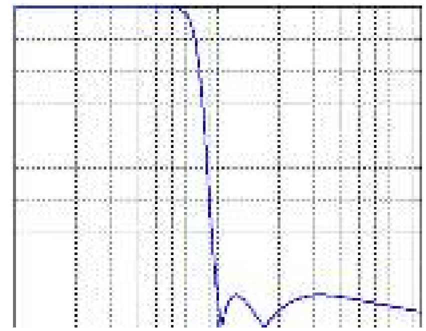
IIR Filter Types



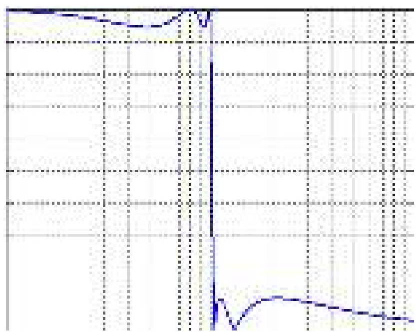
Butterworth



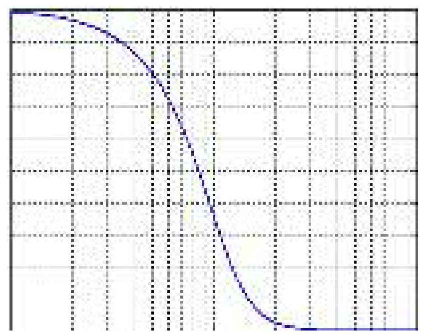
Chebyshev Type I



Chebyshev Type II



Elliptic



Bessel

Butterworth filter Approximation
(Maximally flat approximation)

$$K(S) = P_N(S) = \varepsilon\left(\frac{S}{S_P}\right)^N = \varepsilon\left(\frac{\Omega}{\Omega_P}\right)^N \quad or \quad \left(\frac{\Omega}{\Omega_C}\right)^N \rightarrow (4)$$

The corresponding magnitude function is

$$|H(j\Omega)| = \left|\frac{V_{OUT}}{V_{IN}}\right| = \frac{1}{\sqrt{1+\left(\frac{\Omega}{\Omega_C}\right)^{2N}}} \quad for \quad N = 1,2,3,\ldots \rightarrow (5)$$

Where

N is the order of the filter

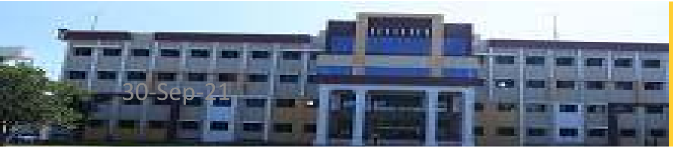
ε is a constant

Ω is the operating frequency and

Ω_C is the cutoff frequency

Ω_P is the pass band edge frequency

$\left(\frac{\Omega}{\Omega_C}\right)^N = \varepsilon\left(\frac{\Omega}{\Omega_P}\right)^N \Rightarrow \frac{\Omega}{\Omega_C} = \varepsilon^{1/N}\frac{\Omega}{\Omega_P}$ or $\Omega_P = \varepsilon^{1/N}\Omega_C \rightarrow (6)$



Butterworth filter Approximation
(Maximally flat approximation)

At DC means near Ω =0 $\varepsilon^2\left(\frac{\Omega}{\Omega_P}\right)^{2N} << 1$

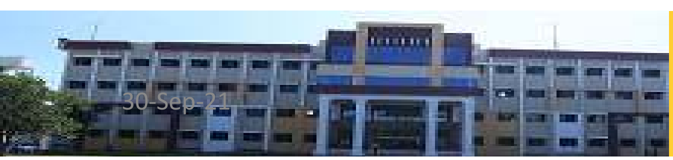
So

$$\left[1+\varepsilon^2\left(\frac{\Omega}{\Omega_P}\right)^{2N}\right]^{\frac{1}{2}} = 1+\frac{1}{2}\varepsilon^2\left(\frac{\Omega}{\Omega_P}\right)^{2N} - \frac{1}{8}\varepsilon^4\left(\frac{\Omega}{\Omega_P}\right)^{4N} + \frac{1}{16}\varepsilon^6\left(\frac{\Omega}{\Omega_P}\right)^{6N} + \ldots$$

This expression shows that the first 2N-1 derivatives are zero at Ω=0.

Since K(S) was chosen to be an nth order polynomial this is the maximum number of derivatives that can be made zero. Thus the slope is as flat as possible at DC.

For this reason the butter worth approximation is also known as the Maximally flat Approximation.



Butterworth filter Approximation
(Maximally flat approximation)

The loss in dB is given from equ i.e $|H(j\Omega)|^2 = \left|\frac{V_{OUT}}{V_{IN}}\right|^2 = \frac{1}{1+\varepsilon^2\left(\frac{\Omega}{\Omega_P}\right)^{2N}} \rightarrow (7)$

as $$A(\Omega) = 10\log_{10}\left[1+\varepsilon^2\left(\frac{\Omega}{\Omega_P}\right)^{2N}\right] dB \rightarrow (8)$$

At pass band edge frequency $i.e\ at\ \Omega = \Omega_P \quad the\ loss\ is\ A_{\max}(\alpha_P)$

➔ $\alpha_P = A(\Omega_P) = 10\log_{10}\left[1+\varepsilon^2\right] \Rightarrow \qquad \varepsilon = \sqrt{10^{0.1\alpha_P} - 1} \rightarrow (9)$

It is the parameter related to pass band

At high frequencies the loss asymptotically approaches $20\log_{10}\varepsilon\left(\frac{\Omega}{\Omega_P}\right)^N$

because

$$10\log_{10}\left[1+\varepsilon^2\left(\frac{\Omega}{\Omega_P}\right)^{2N}\right] = 10\log_{10}\left[\varepsilon\left(\frac{\Omega}{\Omega_P}\right)^N\right]^2 = 20\log_{10}\varepsilon\left(\frac{\Omega}{\Omega_P}\right)^N$$

These loss is seen to increase with the order N. At high frequencies the slope is 6N dB/Octave. Therefore the stop band loss increases with the order N.

Butterworth filter Approximation
(Maximally flat approximation)

At stop band edge frequency $i.e\ at\ \Omega = \Omega_S \quad the\ loss\ is\ A_{\min}(\alpha_S)$

$$\Rightarrow \alpha_S = A(\Omega_S) = 10\log_{10}\left[1 + \varepsilon^2\left(\frac{\Omega_S}{\Omega_P}\right)^{2N}\right]$$

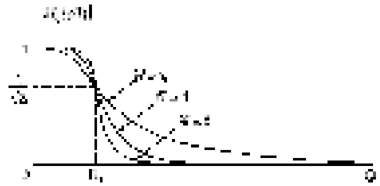
$$\Rightarrow 10^{0.1\alpha_S} - 1 = \varepsilon^2\left(\frac{\Omega_S}{\Omega_P}\right)^{2N} \Rightarrow \left(\frac{\Omega_S}{\Omega_P}\right)^{2N} = \frac{10^{0.1\alpha_S}-1}{10^{0.1\alpha_P}-1} \Rightarrow \left(\frac{\Omega_S}{\Omega_P}\right)^{N} = \sqrt{\frac{10^{0.1\alpha_S}-1}{10^{0.1\alpha_P}-1}}$$

$$\Rightarrow N\log_{10}\left(\frac{\Omega_S}{\Omega_P}\right) = \log_{10}\sqrt{\frac{10^{0.1\alpha_S}-1}{10^{0.1\alpha_P}-1}} \Rightarrow N = \frac{\log_{10}\sqrt{\frac{10^{0.1\alpha_S}-1}{10^{0.1\alpha_P}-1}}}{\log_{10}\left(\frac{\Omega_S}{\Omega_P}\right)} = \frac{\log_{10}\left(\frac{\lambda}{\varepsilon}\right)}{\log_{10}\left(\frac{\Omega_S}{\Omega_P}\right)}$$

Where $\lambda = \sqrt{10^{0.1\alpha_S} - 1} \rightarrow (10)$ is a Parameter related to stop band

Since this expression normally does not result in an integer value we therefore round off N to the next higher integer to satisfy the minimum required specifications.

$$N \geq \frac{\log_{10}\left(\frac{\lambda}{\varepsilon}\right)}{\log_{10}\left(\frac{\Omega_S}{\Omega_P}\right)} \rightarrow (11)$$



Stable normalized Butterworth filter design

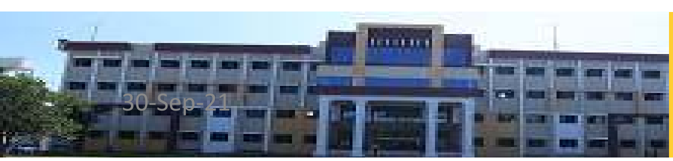
The magnitude function of the Butterworth low pass filter is given by

$$\left|H(j\Omega)\right| = \frac{1}{\left[1+\left(\frac{\Omega}{\Omega_C}\right)^{2N}\right]^{\frac{1}{2}}} \rightarrow (11)$$

The magnitude squared function of a normalized Butterworth low pass filter with Ω_c=1 rad/sec is given by

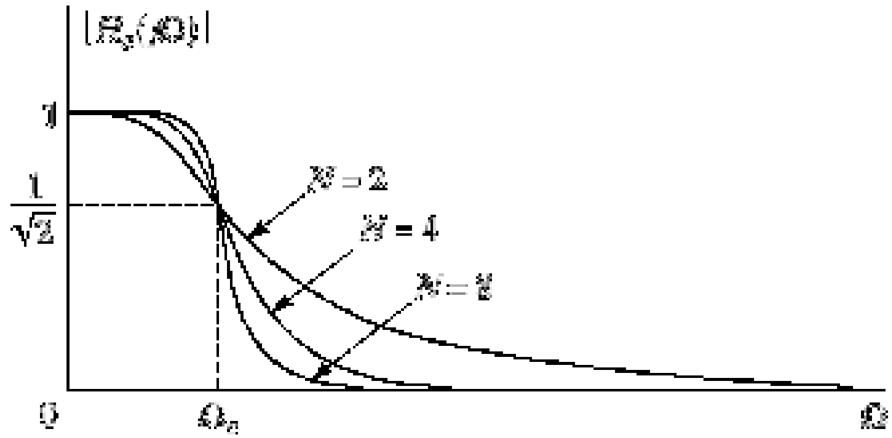
$$\left|H(j\Omega)\right|^2 = \frac{1}{1+\Omega^{2N}} \rightarrow (12)$$

The function is monotonically decreasing. The maximum response is zero at Ω=0. The response approaches ideal characteristics as the order N increases.



Stable normalized Butterworth filter design

At $\Omega=\Omega_c$ the curve passes through 0.707 which corresponds to 3dB point.



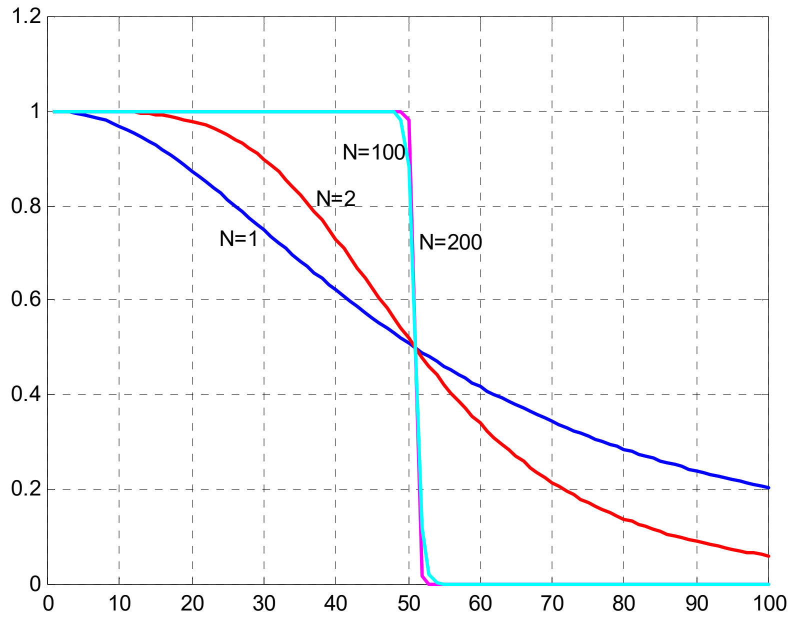
Now let us derive the transfer function of a stable filter.

For this purpose substitute $\Omega=S/j$ in equ.12 then we have

$$|H(j\Omega)|^2 = |H(j\Omega)||H(-j\Omega)| = \frac{1}{1+\left(\frac{S}{j}\right)^{2N}} \rightarrow (13)$$

$$\Rightarrow \quad |H(S)||H(-S)| = \frac{1}{1+(-1)^N S^{2N}} \rightarrow (14)$$

Stable normalized Butterworth filter design



Stable normalized Butterworth filter design

The above relation tells us that this function has poles in the LHP as well as in the RHP because of the presence of two factors H(S) and H(-S).

If H(S) has poles in the LHP then H(-S) has the corresponding poles in the RHP.

These roots we can get by equating the denominator to zero i.e

$$1+(-S^2)^N = 0 \rightarrow (15)$$

For N odd it reduces to $S^{2N} = 1 = e^{j2\pi k} \Rightarrow S_k = e^{j\frac{\pi k}{N}} \rightarrow (16)$

For N even it reduces to $S^{2N} = -1 = e^{j(2k-1)\pi} \Rightarrow S_k = e^{j\frac{(2k-1)\pi}{N}} \rightarrow (17)$

The solution of the above equation is

$$S_k = e^{j\frac{\pi}{2}\left[\frac{2k+N-1}{N}\right]} \quad for \quad k = 1,2,\ldots,2N \rightarrow (18)$$

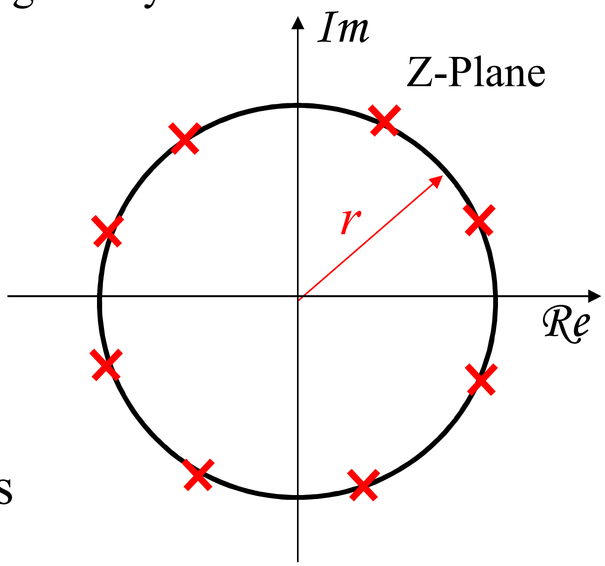
Stable normalized Butterworth filter design

These 2N roots are located on the unit circle and are equally spaced at л/N radians intervals

The S domain magnitude function is therefore given by

$$H(S) = \frac{1}{\prod_j (S - S_j)} \rightarrow (19)$$

Where S_j are the left half plane poles.



Poles of Butterworth filter are located on the circle in the S-plane and are equally spaced at л/N radians intervals

Stable normalized Butterworth filter design

Example.3 Find the Butterworth approximation function for the 3rd order Normalized low Pass Filter

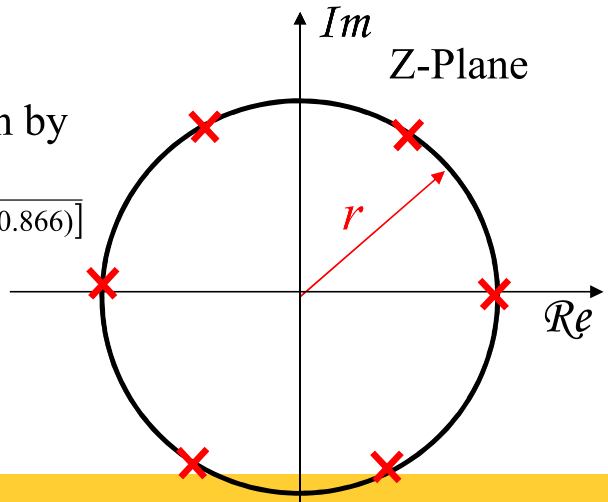
Solution: N=3 for third order and k=1 to 2N = 1 to 6.

$$S_k = e^{j\frac{\pi}{2}\left[\frac{2k+N-1}{N}\right]} = e^{j\frac{\pi}{2}\left[\frac{2k+2}{3}\right]} = e^{j\left[\frac{k+1}{3}\right]\pi}$$

$$S_1 = e^{j\left[\frac{2\pi}{3}\right]} = \cos\left(\tfrac{2\pi}{3}\right) + j\sin\left(\tfrac{2\pi}{3}\right) = -0.5 + j0.866$$

$$S_2 = e^{j\left[\frac{3\pi}{3}\right]} = e^{j\pi} = \cos(\pi) + j\sin(\pi) = -1$$

$$S_3 = e^{j\left[\frac{4\pi}{3}\right]} = \cos\left(\tfrac{4\pi}{3}\right) + j\sin\left(\tfrac{4\pi}{3}\right) = -0.5 - j0.866$$



The S domain magnitude function is therefore given by

$$H(S) = \frac{1}{(S-S_1)(S-S_2)(S-S_3)} = \frac{1}{[S-(-0.5+j0.866)][S-(-1)][S-(-0.5-j0.866)]}$$

$$H(S) = \frac{1}{[(S+0.5)^2-(j0.866)^2][S+1]} = \frac{1}{(S^2+S+0.25+0.75)(S+1)}$$

$$\boxed{H(S) = \frac{1}{(S+1)(S^2+S+1)}}$$

Stable normalized Butterworth filter design

Example.4 Find the Butterworth approximation function for the 4th order Normalized low Pass Filter

Solution: N=4 for fourth order and k=1 to 2N = 1 to 8.

$$S_k = e^{j\frac{\pi}{2}\left[\frac{2k+N-1}{N}\right]} = e^{j\frac{\pi}{2}\left[\frac{2k+3}{4}\right]}$$

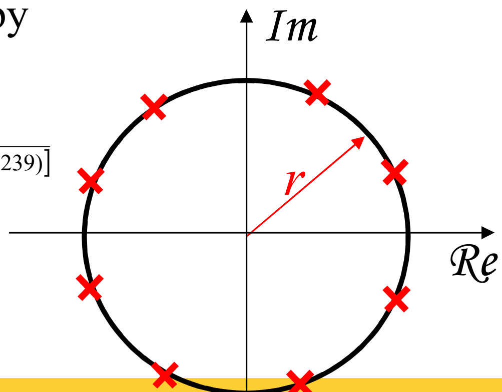
$$S_1 = e^{j\left[\frac{5\pi}{8}\right]} = \cos\left(\tfrac{5\pi}{8}\right) + j\sin\left(\tfrac{5\pi}{8}\right) = -0.3827 + j0.9239$$

$$S_2 = e^{j\left[\frac{7\pi}{8}\right]} = \cos\left(\tfrac{7\pi}{8}\right) + j\sin\left(\tfrac{7\pi}{8}\right) = -0.9239 + j0.3827$$

$$S_3 = e^{j\left[\frac{9\pi}{8}\right]} = \cos\left(\tfrac{9\pi}{8}\right) + j\sin\left(\tfrac{9\pi}{8}\right) = -0.9239 - j0.3827$$

$$S_4 = e^{j\left[\frac{11\pi}{8}\right]} = \cos\left(\tfrac{11\pi}{8}\right) + j\sin\left(\tfrac{11\pi}{8}\right) = -0.3827 - j0.9239$$

Z-Plane



The S domain magnitude function is therefore given by

$$H(S) = \frac{1}{(S-S_1)(S-S_2)(S-S_3)(S-S_4)}$$

$$H(S) = \frac{1}{[S-(-0.3827+j0.9239)][S-(-0.9239+j0.3827)][S-(-0.9239-j0.3827)][S-(-0.3827-j0.9239)]}$$

$$H(S) = \frac{1}{\left[(S+0.3827)^2-(j0.9239)^2\right]\left[(S+0.9239)^2-(j0.3827)^2\right]}$$

$$H(S) = \frac{1}{(S^2+0.76536S+1)(S^2+1.84776S+1)}$$

List of Butterworth polynomials

N	Denominator of H(S)
1	$(S+1)$
2	$(S^2+\sqrt{2}S+1)$
3	$(S+1)\ (S^2+S+1)$
4	$(S^2+0.76537S+1)\ (S^2+1.84776S+1)$
5	$(S+1)\ (S^2+0.61803S+1)\ (S^2+1.61803S+1)$
6	$(S^2+1.931855S+1)(S^2+\sqrt{2}S+1)$ $(S^2+0.51764S+1)$
7	$(S+1)\ (S^2+1.80194S+1)(S^2+1.247S+1)$ $(S^2+0.445S+1)$

Chebyshev filter Approximation

The main feature of the Butterworth approximation is that the loss is maximally flat at the origin.

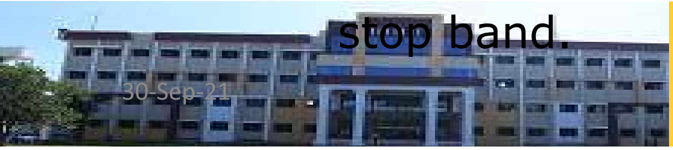
Thus the approximation to a flat pass band is very good at the origin but it gets progressively poorer as frequency approaches pass band edge.

Moreover the attenuation provided in the stop band is less than that attainable using some other polynomial types, such as Chebyshev polynomial.

There are two types of Chebyshev filters .i.e. Type-I and Type-II

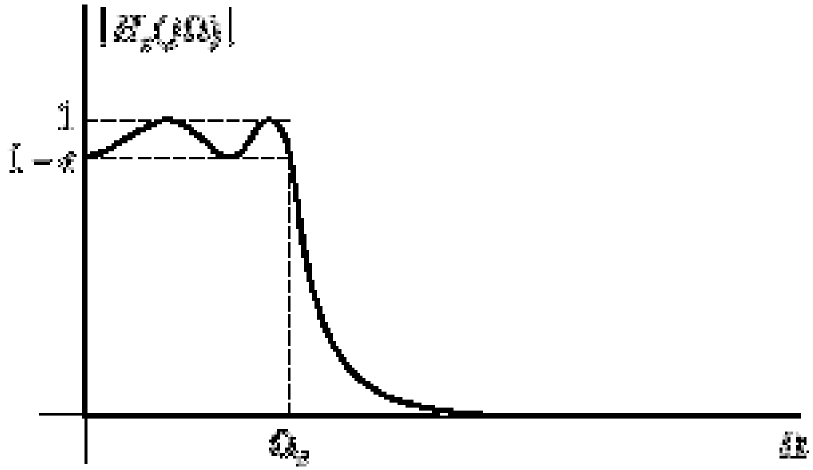
Type-I are all-pole filters that exhibits equiripple behavior in the Pass band and a monotonic characteristics in the stop band.

Type-II contains both poles and zeros and exhibits a monotonic behavior in the pass band and an equiripple behavior in the stop band.



Chebyshev filter Approximation

TYPE-I



Chebyshev filter Approximation

The magnitude square response of Nth order Type-I filter can be expressed as

$$|H(j\Omega)|^2 = \left|\frac{V_{OUT}}{V_{IN}}\right|^2 = \frac{1}{1+\varepsilon^2 C_N^2\left(\frac{\Omega}{\Omega_P}\right)} \quad for \quad N = 1,2,3,\ldots \rightarrow (1)$$

Where

ε is a filter parameter related to the ripple in the pass band.

and $C_N(\Omega)$ is the Nth order Chebyshev polynomial defined as

$$C_N(\Omega) = \begin{cases} \cos(N\cos^{-1}\Omega) & |\Omega| \le 1 \quad passband \\ \cosh(N\cosh^{-1}\Omega) & |\Omega| > 1 \quad stopband \end{cases} \rightarrow (2)$$

It can be expressed by recursive formula from

$$\begin{aligned} C_{N+1}(\Omega) + C_{N-1}(\Omega) &= \cos\left[(N+1)\cos^{-1}\Omega\right] + \cos\left[(N-1)\cos^{-1}\Omega\right] \\ &= 2\cos(N\cos^{-1}\Omega)\cos(\cos^{-1}\Omega) \\ &= 2\cos(N\cos^{-1}\Omega)\Omega = 2\Omega C_N(\Omega) \end{aligned}$$

as $$C_{N+1}(\Omega) = 2\Omega C_N(\Omega) - C_{N-1}(\Omega) \rightarrow (3)$$

Chebyshev filter Approximation

We know that $C_0(\Omega) = 1$ and $C_1(\Omega) = \Omega$

Then from

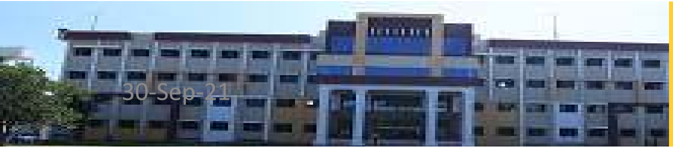
$$C_{N+1}(\Omega) = 2\Omega C_N(\Omega) - C_{N-1}(\Omega)$$

$$C_2(\Omega) = 2\Omega C_1(\Omega) - C_0(\Omega) = 2\Omega^2 - 1$$

$$C_3(\Omega) = 2\Omega C_2(\Omega) - C_1(\Omega) = 2\Omega\left(2\Omega^2 - 1\right) - \Omega = 4\Omega^3 - 3\Omega$$

$$C_4(\Omega) = 2\Omega C_3(\Omega) - C_2(\Omega) = 2\Omega\left(4\Omega^3 - 3\Omega\right) - \left(2\Omega^2 - 1\right) = 8\Omega^4 - 8\Omega^2 + 1$$

$$C_5(\Omega) = 2\Omega C_4(\Omega) - C_3(\Omega) = 2\Omega\left(8\Omega^4 - 8\Omega^2 + 1\right) - \left(4\Omega^3 - 3\Omega\right) = 16\Omega^5 - 20\Omega^4 + 5\Omega$$



Chebyshev filter Approximation

$$C_0(\Omega) = 1$$

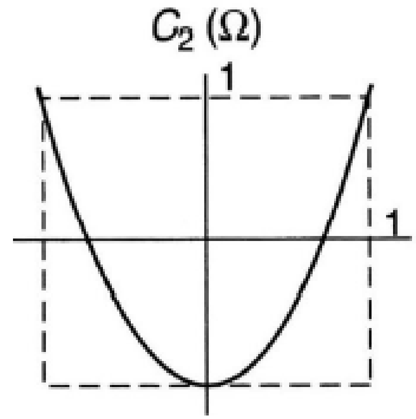
$$C_1(\Omega) = \Omega$$

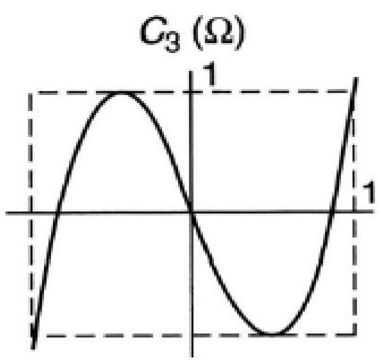
$$C_2(\Omega) = 2\Omega^2 - 1$$

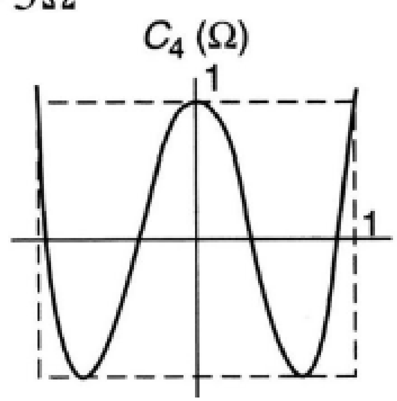
$$C_3(\Omega) = 4\Omega^3 - 3\Omega$$

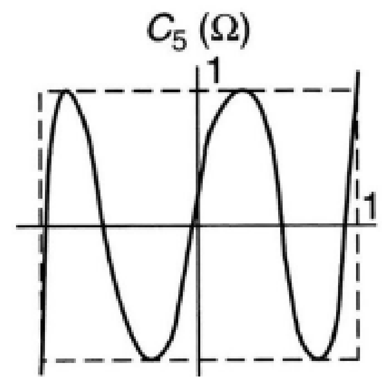
$$C_4(\Omega) = 8\Omega^4 - 8\Omega^2 + 1$$

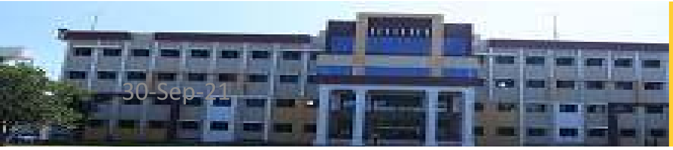
$$C_5(\Omega) = 16\Omega^5 - 20\Omega^3 + 5\Omega$$











Chebyshev filter Approximation

Chebyshev Polynomial has the following properties

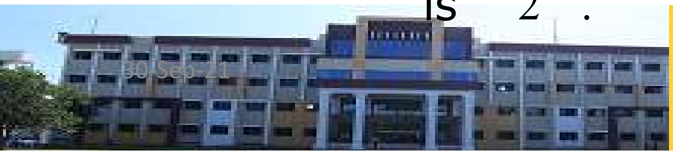
1. $$C_N(\Omega) = \begin{cases} -C_N(-\Omega) \; for \quad N\,odd \\ C_N(-\Omega) \; for \quad N\,even \end{cases}$$

2. $$C_N(0) = \begin{cases} 0 \; for \quad N\,odd \\ (-1)^{N/2} \; for \quad N\,even \end{cases}$$

3. $C_N(1) = 1 \; for \quad all\ N$

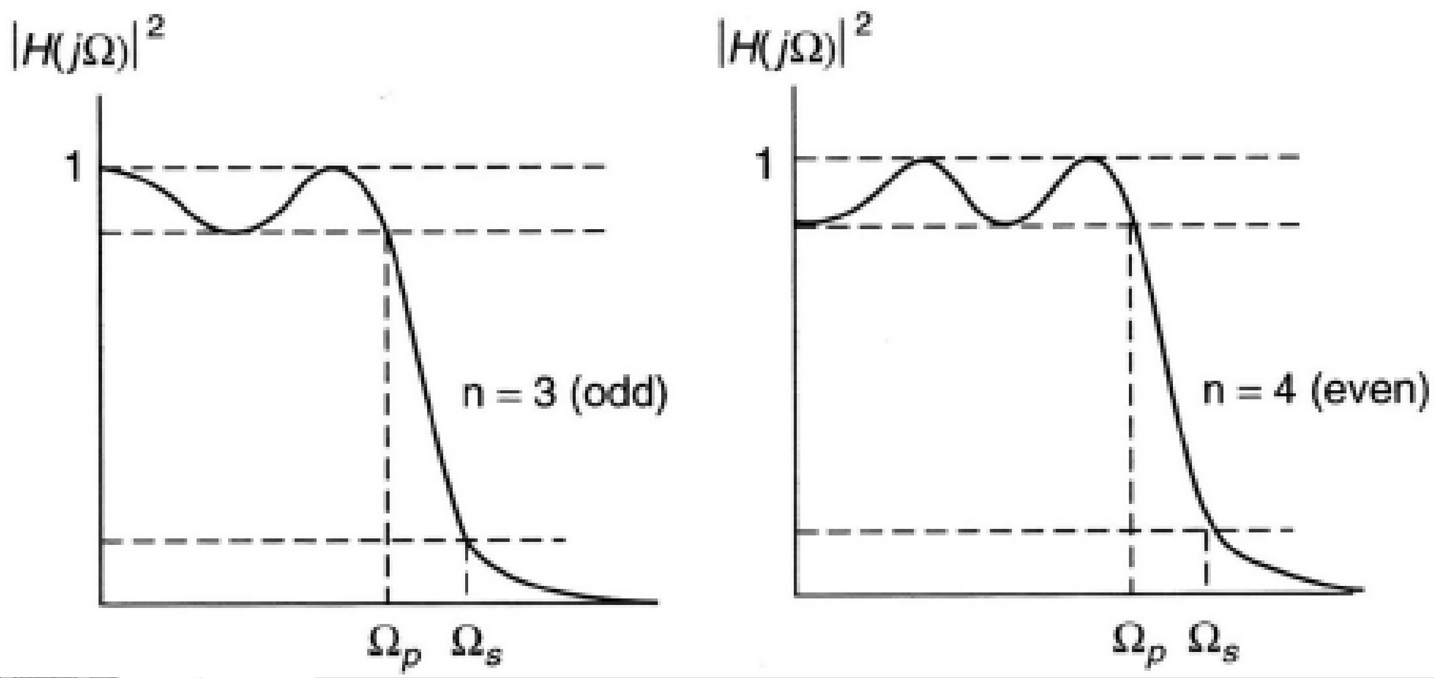
4. $$C_N(-1) = \begin{cases} -1 \; for \quad N\,odd \\ 1 \; for \quad N\,even \end{cases}$$

5. $C_N(\Omega)$ oscillates with equal ripple between ± 1 for $|\Omega| \le 1$.
6. For all N $0 \le |C_N(\Omega)| \le 1 \; for \; 0 \le |\Omega| \le 1$ and $|C_N(\Omega)| > 1 \; for \; |\Omega| > 1$
7. $C_N(\Omega)$ is monotonically increasing for $|\Omega| > 1 \; for\ all\ N.$
8. Every coefficient is an integer and the one associated with Ω^N is 2^{N-1}.



Chebyshev filter Approximation

Chebyshev Type-I



Chebyshev filter Approximation

The loss in dB is given from equ.1 i.e $|H(j\Omega)|^2 = \left|\frac{V_{OUT}}{V_{IN}}\right|^2 = \frac{1}{1+\varepsilon^2 C_N^2\left(\frac{\Omega}{\Omega_P}\right)}$

as $A(\Omega) = 10\log_{10}\left[1+\varepsilon^2 C_N^2\left(\frac{\Omega}{\Omega_P}\right)\right] dB \rightarrow (4)$

At pass band edge frequency $i.e\ at\ \Omega = \Omega_P \quad the\ loss\ is\ A_{\max}(\alpha_P)$

$\Rightarrow \alpha_P = A(\Omega_P) = 10\log_{10}\left[1+\varepsilon^2 C_N^2(1)\right] = 10\log_{10}\left[1+\varepsilon^2\right] \Rightarrow \quad \varepsilon = \sqrt{10^{0.1\alpha_P}-1} \rightarrow (5)$

At stop band edge frequency $i.e\ at\ \Omega = \Omega_S \quad the\ loss\ is\ A_{\min}(\alpha_S)$

$\Rightarrow \alpha_S = A(\Omega_S) = 10\log_{10}\left[1+\varepsilon^2 C_N^2\left(\frac{\Omega_S}{\Omega_P}\right)\right] = 10\log_{10}\left[1+\varepsilon^2 \cosh^2\left(N\cosh^{-1}\left(\frac{\Omega_S}{\Omega_P}\right)\right)\right]$

$$\frac{10^{0.1\alpha_S}-1}{\varepsilon^2} = \cosh^2\left(N\cosh^{-1}\left(\frac{\Omega_S}{\Omega_P}\right)\right) \Rightarrow N\cosh^{-1}\left(\frac{\Omega_S}{\Omega_P}\right) = \cosh^{-1}\sqrt{\frac{10^{0.1\alpha_S}-1}{10^{0.1\alpha_P}-1}}$$

$$N \geq \frac{\cosh^{-1}\sqrt{\frac{10^{0.1\alpha_S}-1}{10^{0.1\alpha_P}-1}}}{\cosh^{-1}\left(\frac{\Omega_S}{\Omega_P}\right)} = \frac{\cosh^{-1}\left(\frac{\lambda}{\varepsilon}\right)}{\cosh^{-1}\left(\frac{\Omega_S}{\Omega_P}\right)} \rightarrow (6)$$

Where $\lambda = \sqrt{10^{0.1\alpha_S}-1} \rightarrow (7)$

Chebyshev Approximation provides 6(N-1)dB more attenuation than Butterworth for the same order

The attenuation in STOP BAND of Butterworth filter for in dB $\Omega >> \Omega_P$ is given by

$$A(\Omega) = 10\log_{10}\left[1 + \varepsilon^2\left(\frac{\Omega_S}{\Omega_P}\right)^{2N}\right] \approx 10\log_{10}\left[\varepsilon^2\left(\frac{\Omega_S}{\Omega_P}\right)^{2N}\right] = 20\log_{10}\left[\varepsilon\left(\frac{\Omega_S}{\Omega_P}\right)^{N}\right] \rightarrow I$$

The attenuation in STOP BAND of Chebyshev filter for $\Omega >> \Omega_P$ in dB is given by

$$A(\Omega) = 10\log_{10}\left[1 + \varepsilon^2 C_N^2\left(\frac{\Omega_S}{\Omega_P}\right)\right] \approx 10\log_{10}\left[\varepsilon^2 C_N^2\left(\frac{\Omega_S}{\Omega_P}\right)\right] = 20\log_{10}\left[\varepsilon C_N\left(\frac{\Omega_S}{\Omega_P}\right)\right] \rightarrow II$$

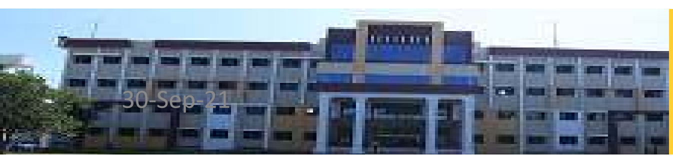
But for $\Omega >> \Omega_P$ $\quad C_N\left(\frac{\Omega_S}{\Omega_P}\right) \cong 2^{N-1}\left(\frac{\Omega_S}{\Omega_P}\right)^N \rightarrow III$

So above equation-II reduces to

$$A(\Omega) = 20\log_{10}\left[\varepsilon C_N\left(\frac{\Omega_S}{\Omega_P}\right)\right] = 20\log_{10}\left[\varepsilon 2^{N-1}\left(\frac{\Omega_S}{\Omega_P}\right)^N\right] = 20\log_{10}\varepsilon\left(\frac{\Omega_S}{\Omega_P}\right)^N + 20\log_{10}2^{N-1}$$

$$A(\Omega) = 20\log_{10}\varepsilon\left(\frac{\Omega_S}{\Omega_P}\right)^N + 20(N-1)\log_{10}2 = 20\log_{10}\varepsilon\left(\frac{\Omega_S}{\Omega_P}\right)^N + 6(N-1) \rightarrow IV$$

Comparing above equ.I and IV it is seen that the Chebyshev approximation provides more attenuation than a Butterworth of the same order. $20\log_{10}2^{N-1} = 6(N-1)dB$



Stable normalized Chebyshev filter design

To find the poles of the Chebyshev approximation transfer function Take the denominator of the equ(1) substitute $\Omega_P = 1$ to get normalized function and equate it to zero. .i.e.

$$|H(j\Omega)|^2 = |H(j\Omega)||H(-j\Omega)| = \frac{1}{1+\varepsilon^2 C_N^2(\Omega)} \quad \rightarrow 1+\varepsilon^2 C_N^2(\Omega) = 0 \Rightarrow C_N(\Omega) = \pm\frac{j}{\varepsilon}$$

It can be proved that the roots of above equation are

$$S_k = \sigma_k \pm j\Omega_k \;\; for\ k = 1,2,\ldots,2N$$

Where

$$\sigma_k = \pm\sin\left[\tfrac{\pi}{2}\left(\tfrac{2k-1}{N}\right)\right]\sinh\left(\tfrac{1}{N}\sinh^{-1}\left(\tfrac{1}{\varepsilon}\right)\right)$$

$$\Omega_k = \cos\left[\tfrac{\pi}{2}\left(\tfrac{2k-1}{N}\right)\right]\cosh\left(\tfrac{1}{N}\sinh^{-1}\left(\tfrac{1}{\varepsilon}\right)\right)$$

further

$$\left[\frac{\sigma_k}{\sinh\left(\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right)}\right]^2 + \left[\frac{\Omega_k}{\cosh\left(\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right)}\right]^2 = 1$$

Stable normalized Chebyshev filter design

These 2N roots are located on the ellipse in the s-plane spaced at л/N radians intervals

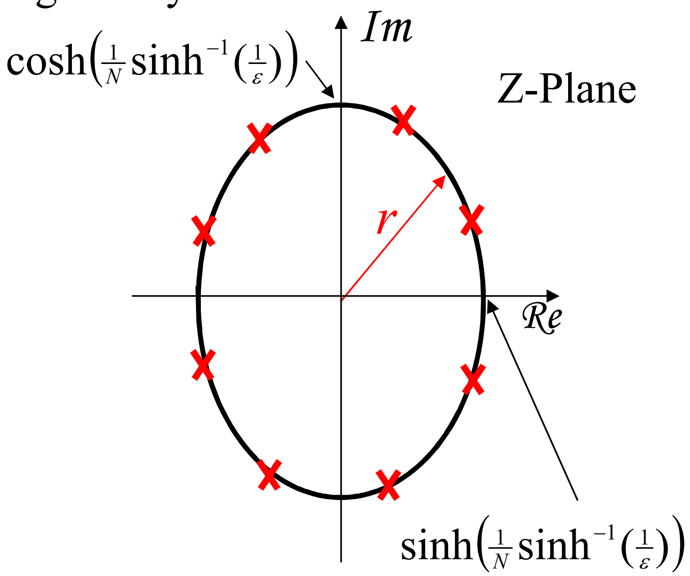
The S domain magnitude function is therefore given by

$$H(S) = \frac{H_0}{\prod_j (S - S_j)}$$

Where S_j are the left half plane poles and H_0 is the order dependent constant

It can be found from

$$H(S)\big|_{S=0} = \begin{cases} 1 & \text{for } N \text{ odd} \\ \dfrac{1}{\sqrt{1+\varepsilon^2}} & \text{for } N \text{ even} \end{cases}$$



Stable normalized Chebyshev filter design

Example.5 Find the Chebyshev approximation function order for the filter requirements
Ωp=200 rad/s, Ωs=600 rad/s, αp=0.5dB, αs=20dB.

Solution: Given Ωp=200 rad/s, Ωs=600 rad/s, αp=0.5dB, αs=20dB.

$$N \geq \frac{\cosh^{-1}\sqrt{\frac{10^{0.1\alpha_S}-1}{10^{0.1\alpha_P}-1}}}{\cosh^{-1}\left(\frac{\Omega_S}{\Omega_P}\right)} = \frac{\cosh^{-1}\sqrt{\frac{10^{2}-1}{10^{0.05}-1}}}{\cosh^{-1}\left(\frac{600}{200}\right)} = \frac{\cosh^{-1}(28.484)}{\cosh^{-1}(3)} = 2.293$$

So the required order is N=3, for third order k=1 to 2N = 1 to 6.

$$S_k = \sigma_k \pm j\Omega_k \quad for\ k = 1,2,\ldots,2N$$

Where

$$\sigma_k = \pm\sin\left[\tfrac{\pi}{2}\left(\tfrac{2k-1}{N}\right)\right]\sinh\left(\tfrac{1}{N}\sinh^{-1}\left(\tfrac{1}{\varepsilon}\right)\right)$$

$$\Omega_k = \cos\left[\tfrac{\pi}{2}\left(\tfrac{2k-1}{N}\right)\right]\cosh\left(\tfrac{1}{N}\sinh^{-1}\left(\tfrac{1}{\varepsilon}\right)\right)$$

Example.6 **obtain an analog Chebyshev filter transfer function that satisfies the constraints**

$$\frac{1}{\sqrt{2}} \le |H(j\Omega)| \le 1 \; for \; 0 \le \Omega \le 2 \qquad |H(j\Omega)| \le 0.1 \; for \; \Omega \ge 4$$

Solution: In general specifications are given as

$$\frac{1}{\sqrt{1+\varepsilon^2}} \le |H(j\Omega)| \le 1 \; for \; 0 \le \Omega \le \Omega_p \qquad |H(j\Omega)| \le \frac{1}{\sqrt{1+\lambda^2}} \; for \; \Omega \ge \Omega_s$$

$$\Omega_P = 2 \qquad \Omega_S = 4$$

$$\frac{1}{\sqrt{1+\varepsilon^2}} = \frac{1}{\sqrt{2}} \Rightarrow 1+\varepsilon^2 = 2 \Rightarrow \varepsilon = 1$$

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.1 \Rightarrow 1+\lambda^2 = 100 \Rightarrow \lambda = \sqrt{99} = 9.95$$

$$N \ge \frac{\cosh^{-1}\sqrt{\frac{10^{0.1\alpha_S}-1}{10^{0.1\alpha_P}-1}}}{\cosh^{-1}\left(\frac{\Omega_S}{\Omega_P}\right)} = \frac{\cosh^{-1}\left(\frac{\lambda}{\varepsilon}\right)}{\cosh^{-1}\left(\frac{\Omega_S}{\Omega_P}\right)} = \frac{\cosh^{-1}\left(\frac{9.95}{1}\right)}{\cosh^{-1}\left(\frac{4}{2}\right)} = 2.269$$

Order to be selected is N=3 and k=1 to 2N = 1 to 6.

Example.6 obtain an analog Chebyshev filter transfer function that satisfies the constraints

$\frac{1}{\sqrt{2}} \le |H(j\Omega)| \le 1 \; for \; 0 \le \Omega \le 2$ $\qquad |H(j\Omega)| \le 0.1 \; for \; \Omega \ge 4$

Solution cont.d: $\qquad S_k = \sigma_k \pm j\Omega_k \;\; for \; k = 1,2,\ldots,2N$

Where $\quad \sigma_k = \pm\sin\left[\frac{\pi}{2}\left(\frac{2k-1}{N}\right)\right]\sinh\left(\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right)$

$$\Omega_k = \cos\left[\frac{\pi}{2}\left(\frac{2k-1}{N}\right)\right]\cosh\left(\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right)$$

$$\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right) = \frac{1}{3}\sinh^{-1}\left(\frac{1}{1}\right) = 0.29379$$

$$A = \sinh\left(\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right) = \sinh(0.29379) = 0.298$$

$$B = \cosh\left(\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right) = \cosh(0.29379) = 1.043$$

Example.6 cont.d

$$\sigma_k = \pm\sin\left[\tfrac{\pi}{2}\left(\tfrac{2k-1}{3}\right)\right]\ A \qquad \Omega_k = \cos\left[\tfrac{\pi}{2}\left(\tfrac{2k-1}{3}\right)\right]B$$

$$\sigma_1 = \pm\sin\left[\tfrac{\pi}{2}\left(\tfrac{1}{3}\right)\right](0.298) = \pm 0.149 \qquad \Omega_1 = \cos\left[\tfrac{\pi}{2}\left(\tfrac{1}{3}\right)\right](1.043) = 0.903$$

$$\sigma_2 = \pm\sin\left[\tfrac{\pi}{2}\left(\tfrac{3}{3}\right)\right](0.298) = \pm 0.298 \qquad \Omega_2 = \cos\left[\tfrac{\pi}{2}\left(\tfrac{3}{3}\right)\right](1.043) = 0$$

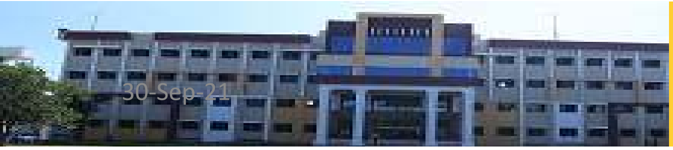
$$\sigma_3 = \pm\sin\left[\tfrac{\pi}{2}\left(\tfrac{5}{3}\right)\right](0.298) = \pm 0.149 \qquad \Omega_3 = \cos\left[\tfrac{\pi}{2}\left(\tfrac{5}{3}\right)\right](1.043) = -0.903$$

Left half plane Poles are given by

$$S_1 = \sigma_1 + j\Omega_1 = \pm 0.149 + j0.903$$

$$S_2 = \sigma_2 + j\Omega_2 = \pm 0.298$$

$$S_3 = \sigma_3 + j\Omega_3 = \pm 0.149 - j0.903$$



Example.6 cont.d

Normalized Transfer function is

$$H(S)=\frac{H_0}{(S-S_1)(S-S_2)(S-S_3)}=\frac{H_0}{\left[S-(-0.149+j0.903)\right]\left[S-(-0.298)\right]\left[S-(-0.149-j0.903)\right]}$$

$$H(S)=\frac{H_0}{\left[(S+0.149)^2-(j0.903)^2\right]\left[S+0.298\right]}=\frac{H_0}{(S+0.298)(S^2+0.298S+0.8388)}$$

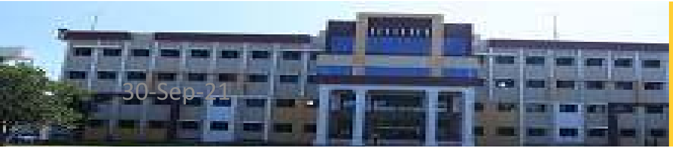
Using $\quad H(S)\big|_{S=0}=1 \; for\ N\ Odd$

$$H_0=(0.298)(0.8388)=0.25 \Rightarrow H(S)=\frac{0.25}{(S+0.298)(S^2+0.298S+0.8388)}$$

Denormalized Transfer function with $\Omega_C=\Omega_P/\varepsilon^{1/N}=2/1=2$ is

$$H(s)=H(S)\big|_{S=\frac{s}{\Omega_C}}=\frac{0.25}{(S+0.298)(S^2+0.298S+0.8388)}\bigg|_{S=\frac{s}{2}}=\frac{0.25}{(\frac{s}{2}+0.298)[(\frac{s}{2})^2+0.298(\frac{s}{2})+0.8388]}$$

$$H(s)=\frac{2}{(s+0.596)(s^2+0.596s+3.354)}$$



Design of Analog Butterworth LOW PASS filter

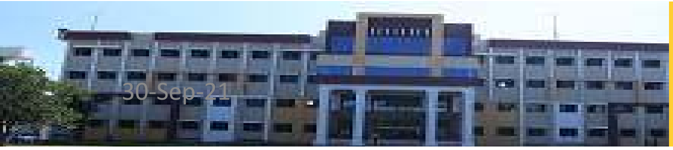
1. From the given specifications find the order of the filter N.

2. Round off Order N to the next higher integer.

3. Find the Normalized Transfer function H(S).

4. Calculate the value of cut off frequency Ω_c.

5. Find the De-normalized transfer function H(s) by replacing S with s/Ω_c.



Design of Analog Chebyshev LOWPASS filter

1. From the given specifications find the order of the filter N.

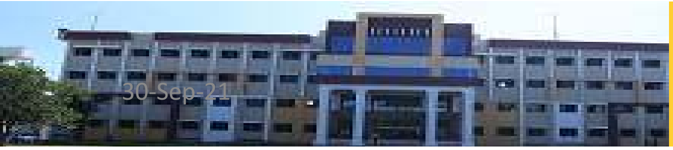
2. Round off Order N to the next higher integer.

3. Find the denominator of the Normalized Transfer function H(S).

4. Calculate the value of cut off frequency Ω_c and find numerator constant H_0 depending on the value of N.

$$H(S)\big|_{S=0} = \begin{cases} 1 & \textit{for } N \textit{ odd} \\ \dfrac{1}{\sqrt{1+\varepsilon^2}} & \textit{for } N \textit{ even} \end{cases}$$

5. Find the De-normalized transfer function H(s) by replacing S with s/Ω_c.



Frequency transformations of analog filters

- The approximations described so far were directly applicable to low-pass filters.
- These approximations can be adapted to high pass, symmetrical band pass and symmetrical band reject filters from a normalized low pass filter(Ω_c=1 rad/sec)

HP, BP or BR requirements → Equivalent LP Requirements → $T_{LP}(S)$ → $T_{HP}(s)$, $T_{BP}(s)$, $T_{BR}(s)$

Take the given filter requirements

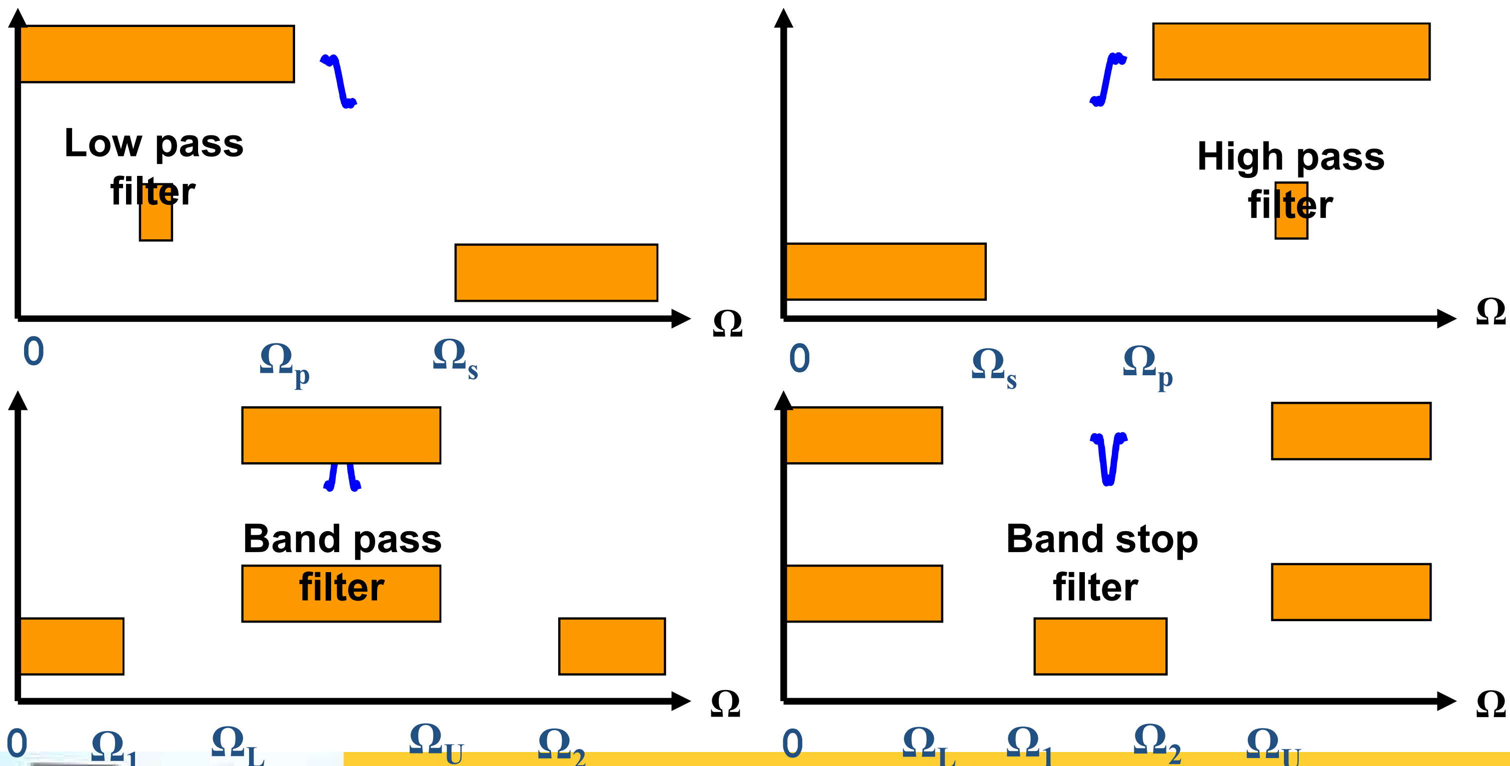
Translate the given requirements to EQUIVALENT low pass requirements.

Approximate the resulting low pass requirement using the specified approximation method

Finally translate the low pass approximation function to the desired HP, BP or BR approximation function

Frequency transformations of analog filters

Practical Analog Filter specifications



Design of HP/BP/BR filters of Butterworth / Chebyshev type Analog filter

1. Find the equivalent low pass requirements.
   .i.e. $\alpha_p \ \alpha_s \ \Omega_p = 1 \ and$

$$for \quad HPF \quad \Omega_r = \frac{\Omega_p}{\Omega_s}$$

$$for \quad BPF \quad \Omega_r = \frac{\Omega_2 - \Omega_1}{\Omega_u - \Omega_l}$$

$$for \quad BRF \quad \Omega_r = \frac{\Omega_u - \Omega_l}{\Omega_2 - \Omega_1}$$

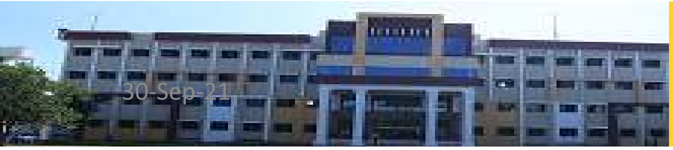
2. Find the normalized Low pass filter order and Transfer function for the given approximation Type.

3. Find the required De-normalized Transfer function by replacing S in H(S) with below transformations

$$for \qquad HPF \qquad H(s) = H(S)\Big|_{S = \frac{\Omega_C}{s}}$$

$$for \qquad BPF \qquad H(s) = H(S)\Big|_{S = \frac{s^2 + \Omega_u \Omega_l}{s(\Omega_u - \Omega_l)}}$$

$$for \quad BRF \ \ or \ \ BRF \quad H(s) = H(S)\Big|_{S = \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u \Omega_l}}$$



Stable Analog filter design

For the given specifications αp=3dB, αs=15dB, Ωp=1000 rad/s and Ωs=500 rad/s design a butter worth approximated High pass filter.

Solution: $\alpha_p = 3dB \quad \Omega_P = 1000\ rad/s \qquad \alpha_s = 15dB \quad \Omega_S = 500\ rad/s$

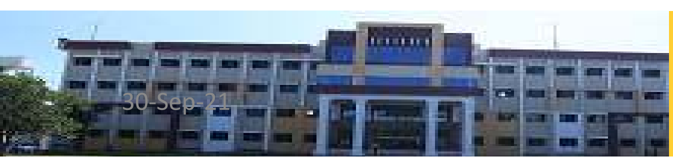
$$\varepsilon = \sqrt{10^{0.1\alpha_P} - 1} = \sqrt{10^{0.3} - 1} = 1 \qquad \lambda = \sqrt{10^{0.1\alpha_S} - 1} = \sqrt{10^{1.5} - 1} = 5.533$$

$$\lambda / \varepsilon = 5.533$$

Equivalent low pass requirements are

$$\alpha_p = 3dB \quad \alpha_s = 15dB \quad \Omega_p = 1\ rad/s \quad \Omega_r = \frac{\Omega_P}{\Omega_S} = \frac{1000}{500} = 2\ rad/s$$

$$N \geq \frac{\log_{10}\left(\frac{\lambda}{\varepsilon}\right)}{\log_{10}\left(\frac{\Omega_r}{\Omega_p}\right)} = \frac{\log_{10}(5.533)}{\log_{10}(2)} = 2.468$$



Solution: select N=3 and k=1 to 2N = 1 to 6.

$$S_k = e^{j\frac{\pi}{2}\left[\frac{2k+N-1}{N}\right]} = e^{j\frac{\pi}{2}\left[\frac{2k+2}{3}\right]} = e^{j\left[\frac{k+1}{3}\right]\pi}$$

$$S_1 = e^{j\left[\frac{2\pi}{3}\right]} = \cos\left(\frac{2\pi}{3}\right) + j\sin\left(\frac{2\pi}{3}\right) = -0.5 + j0.866$$

$$S_2 = e^{j\left[\frac{3\pi}{3}\right]} = e^{j\pi} = \cos(\pi) + j\sin(\pi) = -1$$

$$S_3 = e^{j\left[\frac{4\pi}{3}\right]} = \cos\left(\frac{4\pi}{3}\right) + j\sin\left(\frac{4\pi}{3}\right) = -0.5 - j0.866$$

Normalized Transfer function of equivalent Low-pass filter is

$$H(S) = \frac{1}{(S-S_1)(S-S_2)(S-S_3)} = \frac{1}{[S-(-0.5+j0.866)][S-(-1)][S-(-0.5-j0.866)]}$$

$$H(S) = \frac{1}{[(S+0.5)^2-(j0.866)^2][S+1]} = \frac{1}{(S^2+S+0.25+0.75)(S+1)}$$

$$H(S) = \frac{1}{(S+1)(S^2+S+1)}$$

Stable analog filter design

Solution: $\Omega_P = \varepsilon^{1/N}\Omega_C \Rightarrow \Omega_C = \Omega_P / \varepsilon^{1/N} = 1000/1 = 1000$

De-normalized Transfer function of required High-pass filter is
Given by

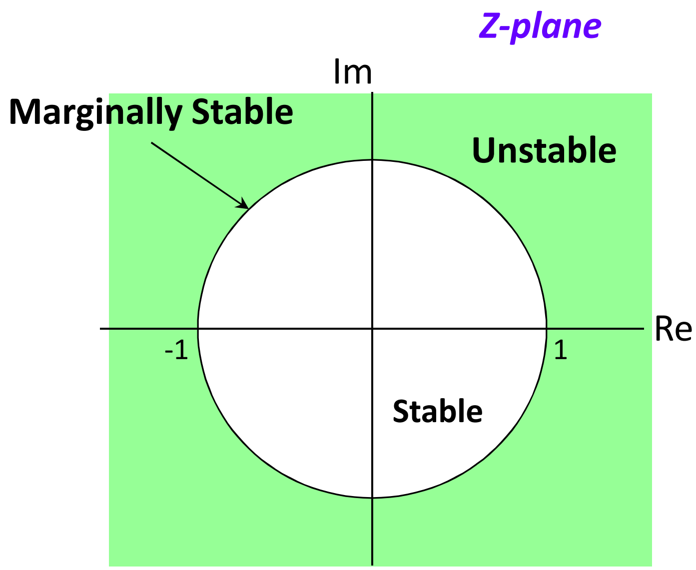
$$H(s) = H(S)\Big|_{S=\frac{\Omega_c}{s}} = \left.\frac{1}{(S+1)(S^2+S+1)}\right|_{S=\frac{1000}{s}}$$

$$H(s) = \frac{1}{\left(\frac{1000}{s}+1\right)\left(\left(\frac{1000}{s}\right)^2 + \frac{1000}{s}+1\right)}$$

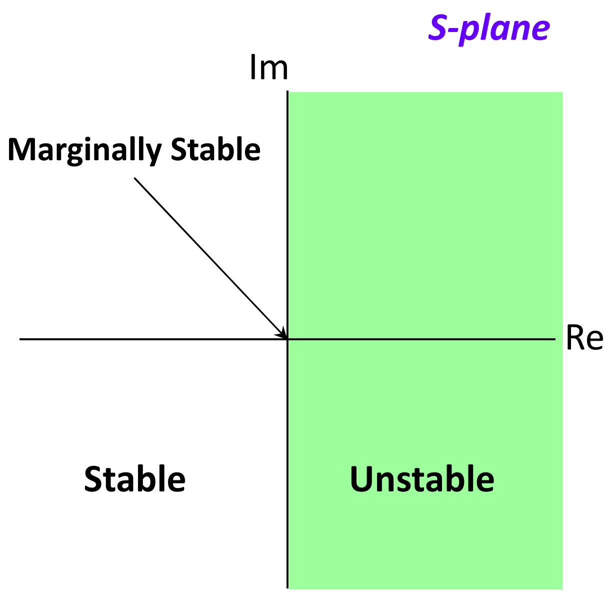
De-normalized Transfer function of required High-pass filter is

$$H(s) = \frac{s^3}{\left(s+10^3\right)\left(s^2+10^3 s+10^6\right)}$$

Stability in Both Domains



Discrete-Time Systems



Continuous-Time Systems

Infinite Impulse Response (IIR) Filters

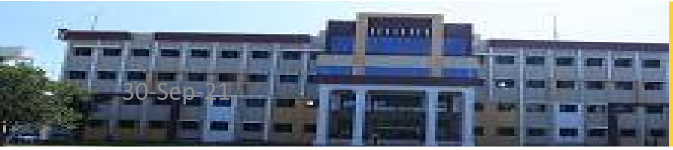
- If the conversion technique is to be effective it should posses the following desirable properties

1. The jΩ-axis in the S-plane should map into the unit circle in the Z-plane.
2. The left half of the S-plane should map in to the inside of the unit circle in the Z-plane.

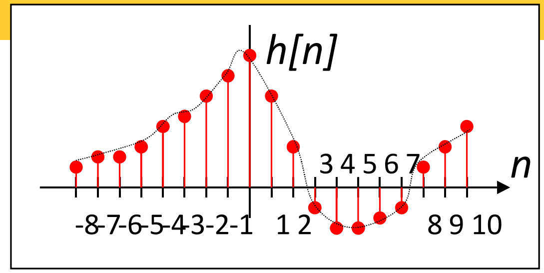
Thus a stable Analog filter must be converted to Stable Digital Filter.

some of the methods are

1. Approximate difference method
2. Impulse Invariance Transformation
3. Bilinear Transformation



Impulse invariant method

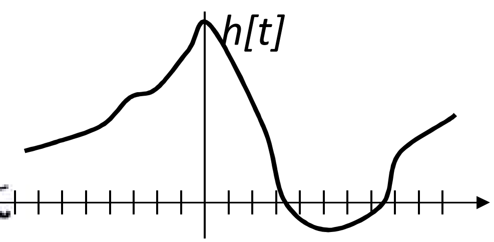


- Basic principle**: Sampling of impulse response of an analog filter.**

$h_a(t)$ → Impulse response of analog filter

T → Sampling period

h(n) → Impulse response of digital filter



$$H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s + p_k}$$

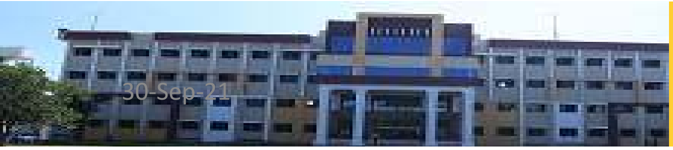
Impulse-Invariant Method
(Impulse Invariant Transformation)

Objective: To design an IIR filter having an impulse response ***h(n)*** as the sampled version of the impulse response of the analog filter $h(t)$:

$$h(t) = h(nT) = h(n) \;\; for \; n = 0,1,2,...$$

where *T* is the sampling interval.

In consequence of this result, the frequency response of the digital filter is an aliased version of the frequency response of the corresponding analog filter.



$$h_a(t) = \sum_{k=1}^{N} A_k e^{-p_k t} u(t)$$

If we sample $h_a(t)$ at $t = nT$, we get

$$h(n) = h_a(nT) = \sum_{k=1}^{N} A_k e^{-p_k nT} u(n) = \sum_{k=1}^{N} A_k \left(e^{-p_k T}\right)^n u(n)$$

$$H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - e^{-p_k T} z^{-1}}$$

The poles of the digital IIR filter are located at $z = e^{-p_k T}$.
(12.11), we get the transformation

$$\frac{1}{s + p_k} \longleftrightarrow \frac{1}{1 - e^{-p_k T} z^{-1}}$$

Substituting $s = j\Omega$ and $z = e^{j\omega}$ to get the frequency response of the analog filter and digital filter, we get.

$$H_a(j\Omega) = \frac{1}{p_k + j\Omega} \quad \text{and} \quad H(e^{j\omega}) = \frac{1}{1 - e^{-p_k T} e^{-j\omega}}$$

Comparing the values of the analog and digital filter frequency responses at zero frequency, we get

$$H_a(j\Omega)\Big|_{\Omega=0} = H_a(j0) = H_a(0) = \frac{1}{p_k} \quad \text{and}$$

$$H(e^{j\omega})\Big|_{\omega=0} = H(e^{j0}) = H(1) = \frac{1}{1 - e^{-p_k T}}$$

For small values of the sampling interval T, we can use the approximation $e^{-p_k T} \approx 1 - p_k T$, which gives $H(e^{j0}) = \frac{1}{p_k T}$. This suggests that the transfer function $H(z)$ must be multiplied by T so that the gain of the analog and digital filters matches closely at zero frequency. This modification results if we use the transformation

$$h(n) = Th_a(nT) = T \sum_{k=1}^{N} A_k \left(e^{-p_k T}\right)^n u(n)$$

Then, $H(z)$ is given by

$$H(z) = \sum_{k=1}^{N} \frac{TA_k}{1 - e^{-p_k T} z^{-1}}$$

Relation between analog and digital filter poles

$$H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s + p_k}$$

$$s = -p_k$$

$$z = e^{sT}$$

If $s = \sigma + j\Omega$ and $z = re^{j\omega}$

$$r\, e^{j\omega} = e^{\sigma T}\, e^{j\Omega T}$$

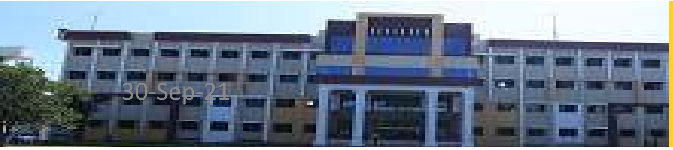
$$H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - e^{-p_k T} z^{-1}}$$

$$z = e^{-p_k T}$$

Therefore, we have

$$\omega = \Omega\, T$$

$$|z| = r = e^{\sigma T}$$



Impulse-Invariant Method

- Consider any pole

On jΩ-axis

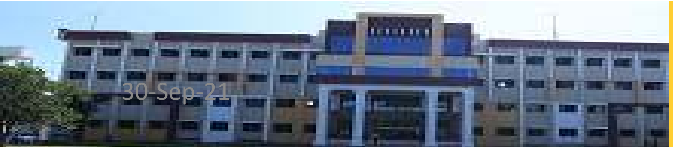
$$\sigma = 0 \Rightarrow r = e^{0.T} = 1 \quad \rightarrow \quad \text{UNIT CIRCLE}$$

Left-half S-plane

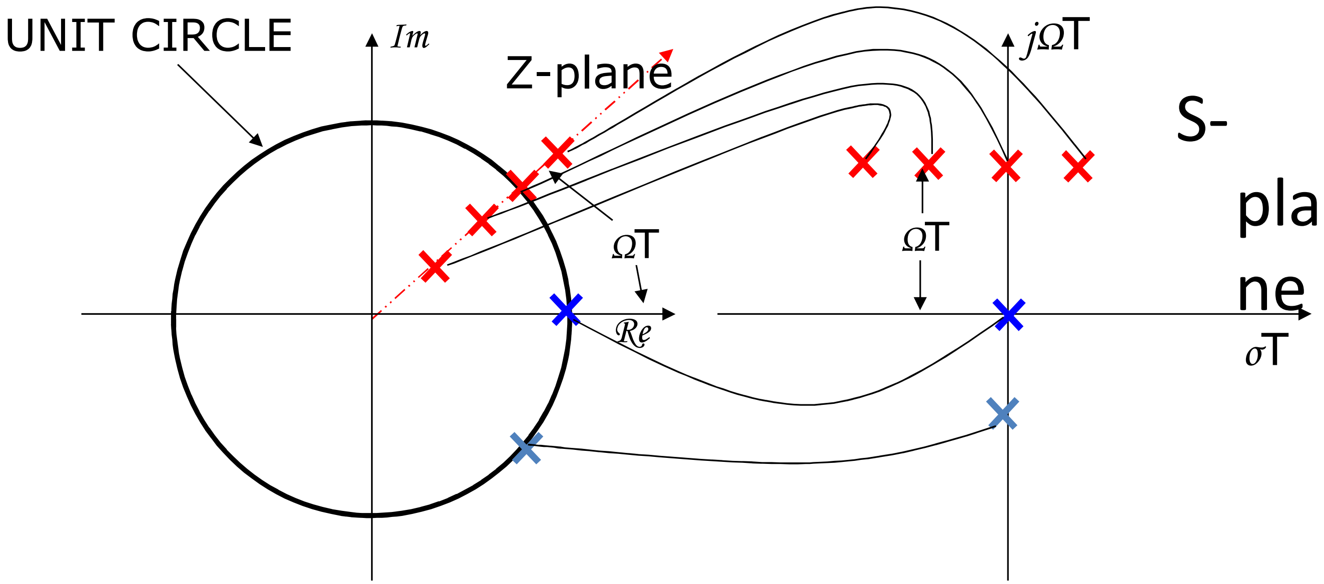
$$\sigma < 0 \Rightarrow r = e^{\sigma T} < 1 \quad \rightarrow \quad \text{INSIDE UNIT CIRCLE}$$

Right-half S-plane

$$\sigma > 0 \Rightarrow r = e^{\sigma T} > 1 \quad \rightarrow \quad \text{OUTSIDE UNIT CIRCLE}$$



Impulse-Invariant Method



Mapping of jΩ-axis, Stable and Unstable poles from S-plane to Z-plane

Impulse-Invariant Method

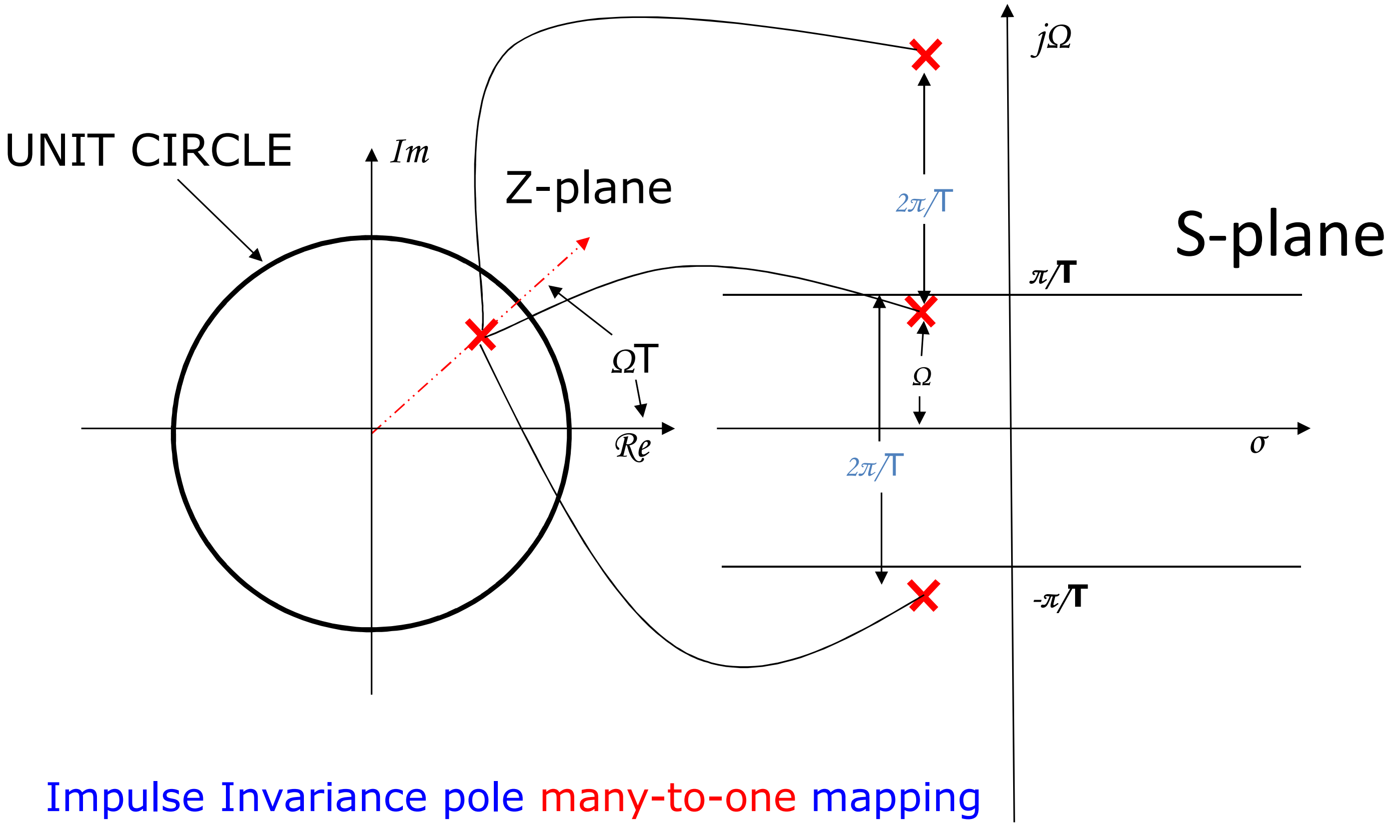
- Therefore the impulse invariance method maps

1. Poles from the jΩ-axis of S-plane to Unit circle of Z-plane.
2. Poles from Left-half (**Negative real part**) of S-plane to inside unit circle of Z-plane.
3. Poles from Right-half (**Positive real part**) S-plane to outside unit circle of Z-plane.

- Disadvantage:

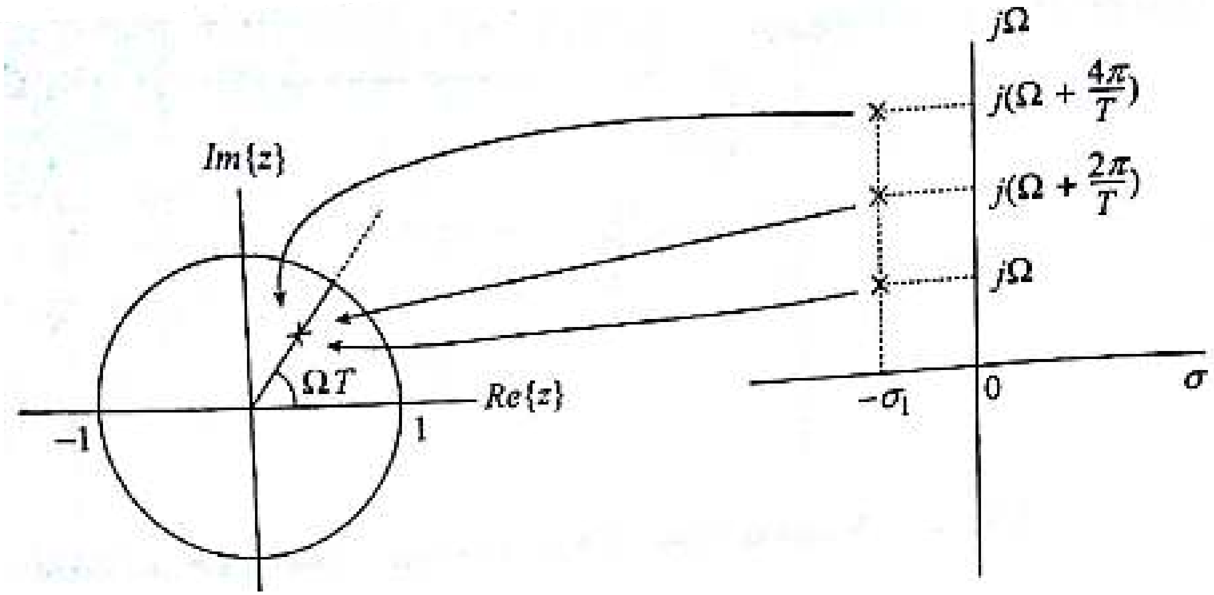
It is not one-to-one mapping it is many-to-one mapping.

Impulse-Invariant Method



Impulse Invariance pole many-to-one mapping

$$s_1 = -\sigma_1 + j\Omega, \quad s_2 = -\sigma_1 + j\left(\Omega + \frac{2\pi}{T}\right), \quad \text{and} \quad s_3 = -\sigma_1 + j\left(\Omega + \frac{4\pi}{T}\right)$$



Impulse-Invariant Method

- Impulse invariance pole mapping

$$s_1 = \sigma + j\Omega \Rightarrow z_1 = e^{(\sigma + j\Omega)T} = e^{\sigma T} e^{j\Omega T}$$

$$s_2 = \sigma + j\left(\Omega + \tfrac{2\pi}{T}\right) \Rightarrow z_2 = e^{\left[\sigma + j\left(\Omega + \frac{2\pi}{T}\right)\right]T} = e^{\sigma T} e^{j\Omega T} e^{j2\pi}$$

$$s_2 = \sigma + j\left(\Omega + \tfrac{2\pi}{T}\right) \Rightarrow z_2 = e^{\sigma T} e^{j\Omega T} \quad (because \quad e^{j2\pi} = 1)$$

•Here **z1=z2**, there are infinite number of S-plane poles that map to the same location in the Z-plane.

– They have same real part, but imaginary parts differ by 2π/T.

The S-plane poles having imaginary parts between −π/T to π/T causes aliasing, when sampling analog signals.

Impulse-Invariant Method

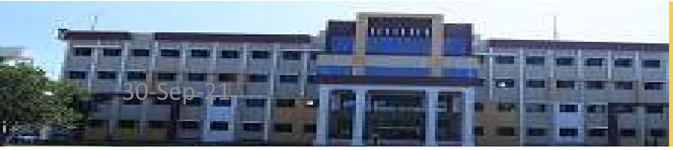
$$z_1 = e^{s_1 T} = e^{(-\sigma_1 + j\Omega)T} = e^{-\sigma_1 T} e^{j\Omega T},$$

$$z_2 = e^{s_2 T} = e^{\left[-\sigma_1 + j\left(\Omega + \frac{2\pi}{T}\right)T\right]} = e^{-\sigma_1 T} e^{j(\Omega T + 2\pi)}$$

$$= e^{-\sigma_1 T} e^{j\Omega T} \underbrace{e^{j2\pi}}_{=1} = e^{-\sigma_1 T} e^{j\Omega T} = z_1,$$

$$z_3 = e^{s_3 T} = e^{\left[-\sigma_1 + j\left(\Omega + \frac{4\pi}{T}\right)T\right]} = e^{-\sigma_1 T} e^{j(\Omega T + 4\pi)}$$

$$= e^{-\sigma_1 T} e^{j\Omega T} \underbrace{e^{j4\pi}}_{=1} = e^{-\sigma_1 T} e^{j\Omega T} = z_1$$



$$H_a(s) = \frac{b}{(s+a)^2 + b^2}$$

into a digital filter $H(z)$ using the impulse-invariant method.

$$\sin(bt)u(t) \longleftrightarrow \frac{b}{s^2 + b^2} \quad \text{and}$$

$$e^{-at}\sin(bt)u(t) \longleftrightarrow \frac{b}{(s+a)^2 + b^2}$$

Therefore, the inverse Laplace transform of $H_a(s)$ is given by

$$h_a(t) = e^{-at}\sin(bt)u(t)$$

If we sample $h_a(t)$ at $t = nT$, we get

$$h(n) = h_a(nT) = e^{-anT}\sin(bnT)u(n)$$

$$= e^{-anT}\left(\frac{e^{jbnT} - e^{-jbnT}}{2j}\right)u(n)$$

$$h(n) = \frac{1}{2j}\left[\left(e^{-(a-jb)T}\right)^n u(n) - \left(e^{-(a+jb)T}\right)^n u(n)\right]$$

The z-transform of this equation yields

$$H(z)=\frac{1}{2j}\left[\frac{1}{1-e^{-(a-jb)T}z^{-1}} - \frac{1}{1-e^{-(a+jb)T}z^{-1}}\right]$$

$$= \frac{e^{-aT}\sin(bT)z^{-1}}{1-2e^{-aT}\cos(bT)z^{-1}+e^{-2aT}z^{-2}}$$

$$\frac{b}{(s+a)^2+b^2} \longleftrightarrow \frac{e^{-aT}\sin(bT)z^{-1}}{1-2e^{-aT}\cos(bT)z^{-1}+e^{-2aT}z^{-2}}$$

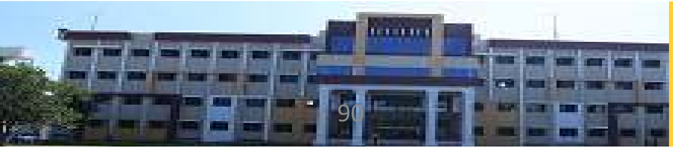
Form of $H(s)$	$H(z)$
$\frac{A}{s+p}$	$\frac{A}{1-e^{-pT}z^{-1}}$
$\frac{A}{(s+p)^2}$	$\frac{ATe^{-pT}z^{-1}}{(1-e^{-pT}z^{-1})^2}$
$\frac{A}{(s+p)^m}$	$\frac{(-1)^{m-1}}{(m-1)!}\frac{d^{m-1}}{dp^{m-1}}\frac{1}{1-e^{-pT}z^{-1}}$
$\frac{s+a}{(s+a)^2+b^2}$	$\frac{1-e^{-aT}\cos(bT)z^{-1}}{1-2e^{-aT}\cos(bT)z^{-1}+e^{-2aT}z^{-2}}$
$\frac{b}{(s+a)^2+b^2}$	$\frac{e^{-aT}\sin(bT)z^{-1}}{1-2e^{-aT}\cos(bT)z^{-1}+e^{-2aT}z^{-2}}$

Bilinear Transformation Method

The IIR filter design techniques described in the previous sections have severe **limitations in that they are appropriate only for low-pass filter design and limited class of band-pass filter design.**

In this section we describe a mapping from the s-plane to the *z*-plane, called **the bilinear transformation**, **that overcomes the limitation of the other design methods described previously.**

Basic principle: application of the trapezoidal formula for numerical integration of differential equation.



Bilinear Transformation method

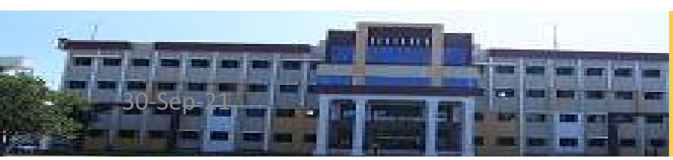
- **The Bilinear transformation is a conformal mapping that transforms the**

1. That transforms the jΩ-axis into the Unit circle in the Z-plane only once.

2. Thus avoiding aliasing of frequency components

3. All points in the L.H.P of ‘s’ are mapped inside the unit circle in the Z-plane and

4. All points in the R.H.P of ‘s’ are mapped into corresponding points outside the unit circle in the Z-plane.



Bilinear Transformation method

Bilinear transformation (or Tustin transformation) is based on the trapezoidal rule for integration. The trapezoidal rule is based on a piecewise linear approximation of the signal and sums the area of the trapezoidal strips, as shown in Fig. 12.9. Suppose that $x(t)$ is the input and $y(t)$ is the output of an integrator with the following transfer function:

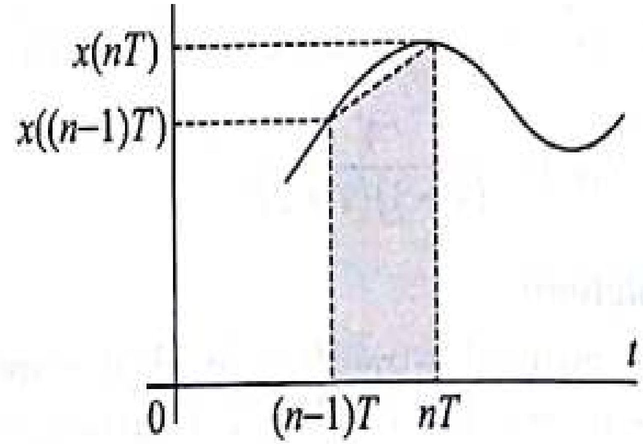


Fig. 12.9 Trapezoidal rule of integration

Bilinear Transformation method

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s}$$

$$sY(s) = X(s) \quad \Longrightarrow \quad \frac{dy(t)}{dt} = x(t)$$

Integrating both sides within the limits $(n-1)T$ and nT, we get

$$\int_{(n-1)T}^{nT} \frac{dy(t)}{dt} dt = \int_{(n-1)T}^{nT} x(t)\, dt$$

$$y(nT) - y((n-1)T) = \int_{(n-1)T}^{nT} x(t)\, dt$$

Instead of substituting a finite difference for the derivative, suppose that we integrate the derivative and approximate the integral by **the trapezoidal formula**:

$$\int_{x_1}^{x_2} f(x)dx \approx \frac{1}{2}(x_2 - x_1)[f(x_2) + f(x_1)]$$

Bilinear Transformation method

$$y(nT) - y((n-1)T) = \frac{T}{2}[x(nT) + x((n-1)T)]$$

$$y(n) - y(n-1) = \frac{T}{2}[x(n) + x(n-1)]$$

Taking the z-transform of this equation gives

$$Y(z) - z^{-1}Y(z) = \frac{T}{2}[X(z) + z^{-1}X(z)]$$

$$Y(z)[1 - z^{-1}] = \frac{T}{2}[1 + z^{-1}]X(z)$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{T}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right)$$

Thus, the transfer function of a trapezoidal rule integrator is

$$H(z) = \frac{T}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right)$$

Bilinear Transformation method-Derivation

Let us consider an analog linear filter of system function

$$H(s)=\frac{b}{s+a}\rightarrow(1)\Rightarrow \frac{Y(s)}{X(s)}=\frac{b}{s+a}\rightarrow(2)$$

So

$$sY(s)+aY(s)=bX(s)\rightarrow(3)$$

It's inverse Laplace transform (differential equation) is equal to

$$\frac{dy(t)}{dt}+ay(t)=bx(t)\rightarrow(4)$$

Integrating both sides within the limits $(n-1)T$ and nT, we get

$$y(nT)=\tfrac{T}{2}\left[-ay(nT)+bx(nT)-ay(nT-T)+bx(nT-T)\right]+y(nT-T)$$

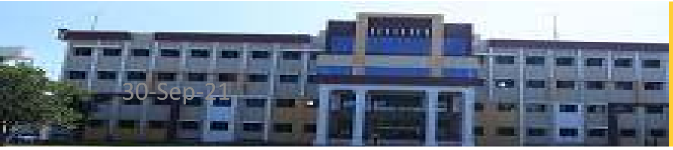
Bilinear Transformation method-Derivation

$$y(nT)=\tfrac{T}{2}\left[-ay(nT)+bx(nT)-ay(nT-T)+bx(nT-T)\right]+y(nT-T)$$

$$\left(1+\tfrac{aT}{2}\right)y(nT)-\left(1-\tfrac{aT}{2}\right)y(nT-T)=\tfrac{bT}{2}\left[x(nT)+x(nT-T)\right]\rightarrow(8)$$

The Z-transform of this differential equation is

$$\left(1+\tfrac{aT}{2}\right)Y(z)-\left(1-\tfrac{aT}{2}\right)z^{-1}Y(z)=\tfrac{bT}{2}\left[1+z^{-1}\right]X(z)$$



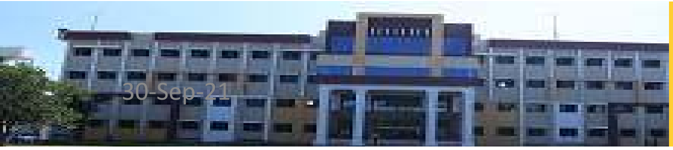
Bilinear Transformation method-Derivation

Transfer (System) function H(z) of the equivalent digital filter is is

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\frac{bT}{2}\left[1+z^{-1}\right]}{\left(1+\frac{aT}{2}\right)-\left(1-\frac{aT}{2}\right)z^{-1}}$$

$$=\frac{\frac{bT}{2}\left[1+z^{-1}\right]}{\left(1-z^{-1}\right)+\frac{aT}{2}\left(1+z^{-1}\right)}=\frac{\frac{bT}{2}}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+\frac{aT}{2}}$$

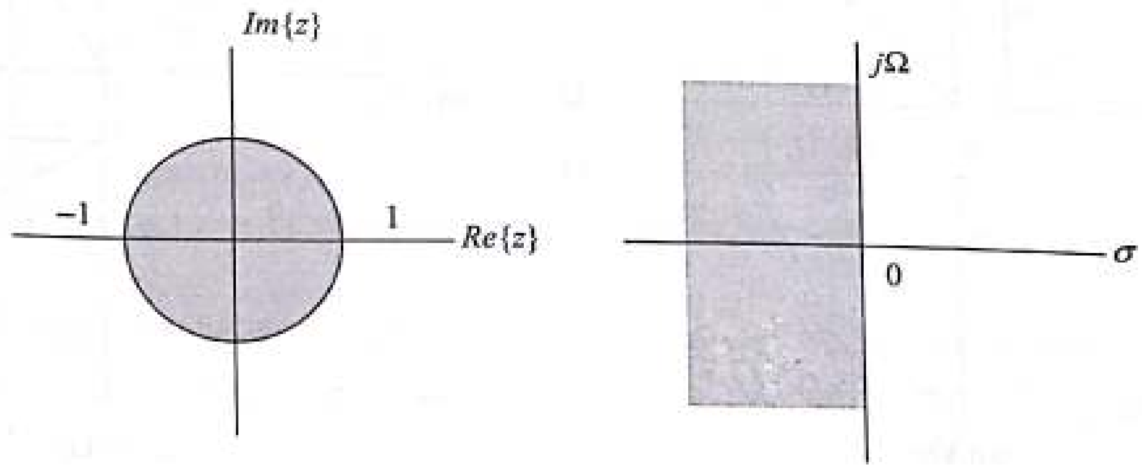
$$\boxed{H(z)=\frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+a}\rightarrow(9)}$$



12.7.1 Relationship Between Analog and Digital Frequencies

By substituting $s = j\Omega$ and $z = e^{j\omega}$ in Eq. (12.25), we get

$$j\Omega = \frac{2}{T}\left[\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right] = \frac{2}{T}\left[\frac{e^{-j\frac{\omega}{2}}(e^{j\frac{\omega}{2}}-e^{-j\frac{\omega}{2}})}{e^{-j\frac{\omega}{2}}(e^{j\frac{\omega}{2}}+e^{-j\frac{\omega}{2}})}\right]$$



$$= \frac{2}{T}\left[\frac{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}}{e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}}\right] = j\,\frac{2}{T}\left[\frac{\sin\frac{\omega}{2}}{\cos\frac{\omega}{2}}\right] = j\,\frac{2}{T}\tan\frac{\omega}{2}$$

Therefore, $\Omega = \frac{2}{T}\tan\frac{\omega}{2}$

or $\omega = 2\tan^{-1}\frac{\Omega T}{2}$

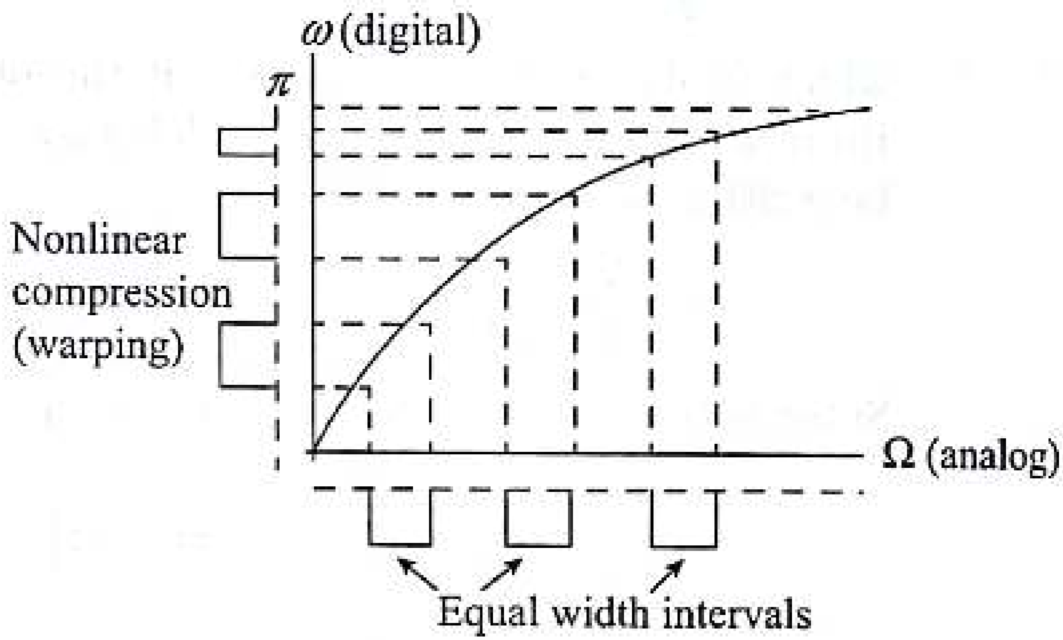


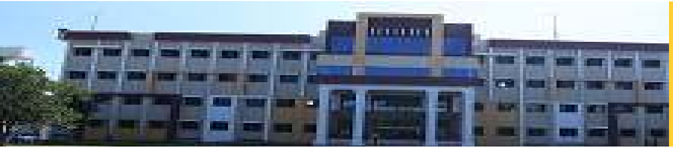
Fig. 12.11 Frequency mapping in bilinear transformation

Selection of Filter Type

- The transfer function $H(z)$ meeting the specifications must be a causal transfer function
- For IIR real digital filter the transfer function is a real rational function of z^{-1}

$$H(z) = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} + \cdots + p_M z^{-M}}{d_0 + d_1 z^{-1} + d_2 z^{-2} + \cdots + d_N z^{-N}}$$

- $H(z)$ must be stable and of lowest order N or M for reduced computational complexity



Selection of Filter Type

- FIR real digital filter transfer function is a polynomial in z^{-1} (order N) with real coefficients

$$H(z) = \sum_{n=0}^{N} h[n] z^{-n}$$

- For reduced computational complexity, degree N of $H(z)$ must be as small as possible
- If a linear phase is desired then we must have:

$$h[n] = \pm h[N-n]$$

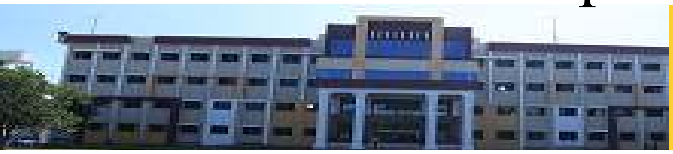
Selection of Filter Type

- Advantages in using an FIR filter -

  (1) Can be designed with exact linear phase

  (2) Filter structure always stable with quantised coefficients

- Disadvantages in using an FIR filter - Order of an FIR filter is considerably higher than that of an equivalent IIR filter meeting the same specifications; this leads to higher computational complexity for FIR



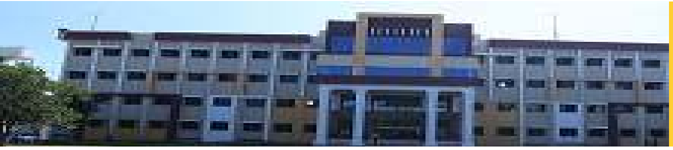
FIR Filter Design

Digital filters with finite-duration impulse response (all-zero, or FIR filters) have both advantages and disadvantages compared to infinite-duration impulse response (IIR) filters.

FIR filters have the following primary advantages:

- They can have exactly linear phase.
- They are always stable.
- The design methods are generally linear.
- They can be realized efficiently in hardware.
- The filter startup transients have finite duration.

The primary disadvantage of FIR filters is that they often require a much higher filter order than IIR filters to achieve a given level of performance. Correspondingly, the delay of these filters is often much greater than for an equal performance IIR filter.



FIR Design

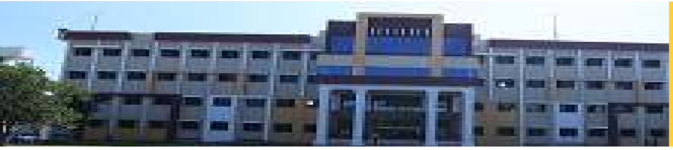
FIR Digital Filter Design

Three commonly used approaches to FIR filter design -

(1) Windowed Fourier series approach

(2) Frequency sampling approach

(3) Computer-based optimization methods

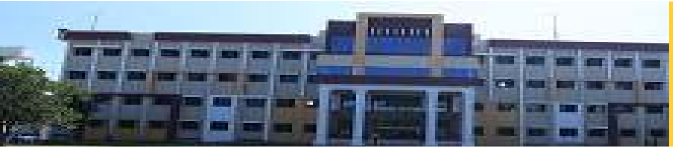


Finite Impulse Response Filters

- The transfer function is given by

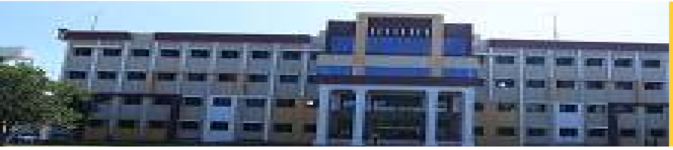
$$H(z) = \sum_{n=0}^{N-1} h(n).z^{-n}$$

- The length of Impulse Response is N
- All poles are at $z = 0$
- Zeros can be placed anywhere on the z-plane



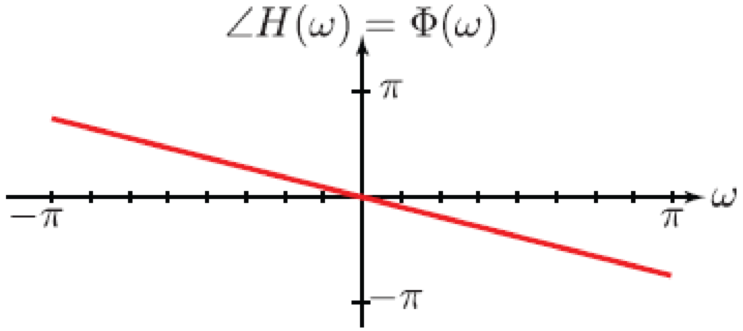
FIR: Linear phase

For phase linearity the FIR transfer function **must** have zeros outside the unit circle



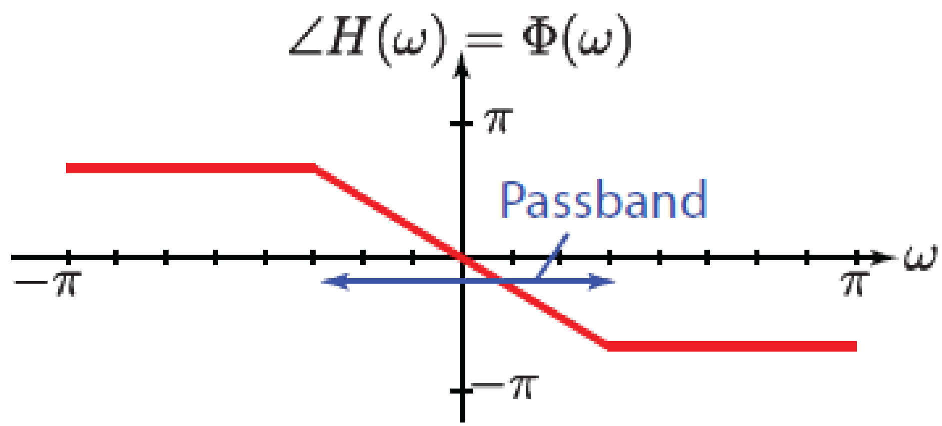
Linear Phase

- What is linear phase?
- Ans: The phase is a straight line in the passband of the system.
- Example: linear phase (all pass system)
- I Group delay is given by the negative of the slope of the line



Linear phase

- linear phase (low pass system)
- Linear characteristics only need to pertain to the passband frequencies only.



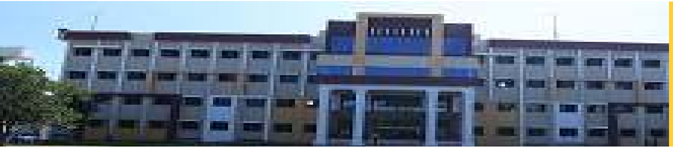
FIR: Linear phase

- For Linear Phase t.f. (order N-1)
- $$h(n) = \pm h(N-1-n)$$
- so that for N even:

$$H(z) = \sum_{n=0}^{N/2-1} h(n).z^{-n} \pm \sum_{n=N/2}^{N-1} h(n).z^{-n}$$

$$= \sum_{n=0}^{N/2-1} h(n).z^{-n} \pm \sum_{n=0}^{N/2-1} h(N-1-n).z^{-(N-1-n)}$$

$$= \sum_{n=0}^{N/2-1} h(n)\left[z^{-n} \pm z^{-m}\right] \quad m = N-1-n$$



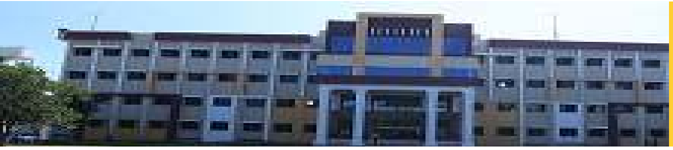
FIR: Linear phase

- for N odd:

$$H(z) = \sum_{n=0}^{\frac{N-1}{2}-1} h(n).\left[z^{-n} \pm z^{-m}\right] + h\left(\frac{N-1}{2}\right) z^{-\left(\frac{N-1}{2}\right)}$$

- I) On $C : |z| = 1$ we have for N even, and +ve sign

$$H(e^{j\omega T}) = e^{-j\omega T\left(\frac{N-1}{2}\right)} . \sum_{n=0}^{N/2-1} 2h(n).\cos\left(\omega T\left(n - \frac{N-1}{2}\right)\right)$$



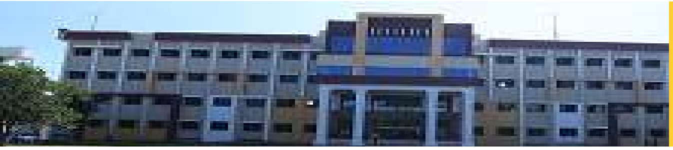
FIR: Linear phase

- II) While for –ve sign

$$H(e^{j\omega T}) = e^{-j\omega T\left(\frac{N-1}{2}\right)} . \sum_{n=0}^{N/2-1} j2h(n).\sin\left(\omega T\left(n-\frac{N-1}{2}\right)\right)$$

- [Note: antisymmetric case adds $\pi/2$ rads to phase, with discontinuity at $\omega = 0$]
- III) For N odd with +ve sign

$$H(e^{j\omega T}) = e^{-j\omega T\left[\frac{N-1}{2}\right]} \left\{ h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} 2h(n).\cos\left[\omega T\left(n-\frac{N-1}{2}\right)\right]\right\}$$



FIR: Linear phase

- IV) While with a –ve sign

$$H(e^{j\omega T}) = e^{-j\omega T\left[\frac{N-1}{2}\right]}\left\{\sum_{n=0}^{\frac{N-3}{2}} 2j.h(n).\sin\left[\omega T\left(n-\frac{N-1}{2}\right)\right]\right\}$$

- [Notice that for the antisymmetric case to have linear phase we require

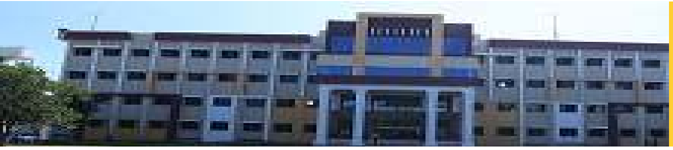
$$h\left(\frac{N-1}{2}\right) = 0.$$

The phase discontinuity is as for N even]

FIR: Linear phase

- The cases most commonly used in filter design are (I) and (III), for which the amplitude characteristic can be written as a polynomial in

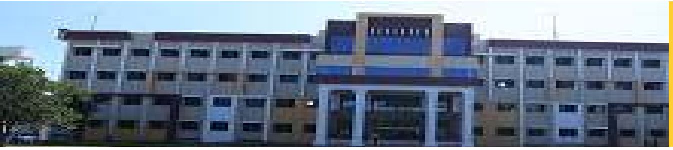
$$\cos\frac{\omega T}{2}$$



Summary of Properties

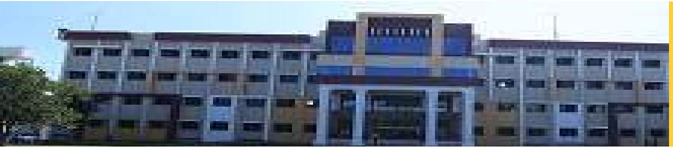
$$H(\omega) = e^{j\omega_0} e^{-j\omega N/2} F(\omega) \sum_{k=0}^{K} a_k \cos(k\omega)$$

Type	**I**	**II**	**III**	**IV**
Order *N*	even	odd	even	odd
Symmetry	symmetric	symmetric	anti-symmetric	anti-symmetric
Period	2π	4π	2π	4π
ω_0	0	0	$\pi/2$	$\pi/2$
$F(\omega)$	1	$\cos(\omega/2)$	$\sin(\omega)$	$\sin(\omega/2)$
K	$N/2$	$(N-1)/2$	$(N-2)/2$	$(N-1)/2$
$H(0)$	arbitrary	arbitrary	0	0
$H(\pi)$	arbitrary	0	0	arbitrary



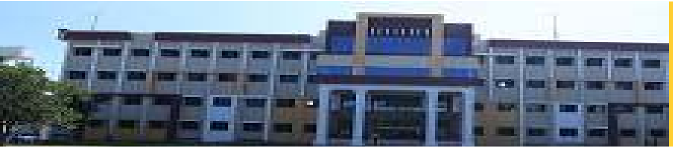
Summary of the characteristics of Linear phase FIR filters

Type	Frequency response $H(e^{j\omega})$	Magnitude response $\lvert\bar{H}(e^{j\omega})\rvert$	Phase response $\angle H(e^{j\omega})$	Applications.
Symmetrical Impulse response N=odd.	$e^{-j\omega\left(\frac{N-1}{2}\right)}\left[\sum_{n=0}^{\frac{N-1}{2}} a(n)\cos\omega n\right]$ $a(0)=h\left(\frac{N-1}{2}\right)$ $a(n)=2h\left[\left(\frac{N-1}{2}\right)-n\right]$	$\left\lvert\sum_{n=0}^{\frac{N-1}{2}} a(n)\cos\omega n\right\rvert$	$-\alpha\omega+\theta$ wher $\theta=0$ for $\bar{H}(e^{j\omega})>0$ $\theta=\pi$ for $\bar{H}(e^{j\omega})<0$	Low pass, High pass, Band pass, Band stop
Symmetrical Impulse response N=even.	$e^{-j\omega\left(\frac{N-1}{2}\right)}\left[\sum_{n=1}^{\frac{N}{2}} b(n)\cos\left(n-\tfrac{1}{2}\right)\omega\right]$ $b(n)=2h\left(\frac{N}{2}-n\right)$	$\left\lvert\sum_{n=1}^{N/2} b(n)\cos\left(n-\tfrac{1}{2}\right)\omega\right\rvert$	$-\alpha\omega+\theta$. where $\theta=0$ for $\bar{H}(e^{j\omega})>0$ $\theta=\pi$ for $\bar{H}(e^{j\omega})<0$	Low pass, Band pass



Summary of Characteristics of Linear phase FIR filters.

Type	Frequency response $H(e^{j\omega})$	Magnitude response $\lvert H(e^{j\omega})\rvert$	Phase response $\angle H(e^{j\omega})$	Applications.
Anti Symmetrical Impulse response N=odd.	$e^{j\pi/2} \cdot e^{-j\omega(\frac{N-1}{2})} \cdot \sum_{n=1}^{\frac{N-1}{2}} c(n)\sin\omega n$ $c(n) = 2h\left(\frac{N-1}{2} - n\right)$	$\left\lvert \sum_{n=1}^{\frac{N-1}{2}} c(n)\sin\omega n \right\rvert$	$-\alpha\omega + \pi/2 + \theta$ where $\theta = 0$ for $H(e^{j\omega}) > 0$ $\theta = \pi$ for $H(e^{j\omega}) < 0$	Differentiator, Hilbert Transformer
Anti Symmetrical Impulse response N=even	$e^{j\pi/2} \cdot e^{-j\omega(\frac{N-1}{2})} \sum_{n=1}^{N/2} d(n)\sin\omega(n-1/2)$ $d(n) = 2h\left(\frac{N}{2} - n\right)$	$\left\lvert \sum_{n=1}^{N/2} d(n)\sin\omega(n-1/2) \right\rvert$	$-\alpha\omega + \frac{\pi}{2} + \theta$ where $\theta = 0$ for $H(e^{j\omega}) > 0$ $\theta = \pi$ for $H(e^{j\omega}) < 0$	Hilbert Transformer, Differentiator

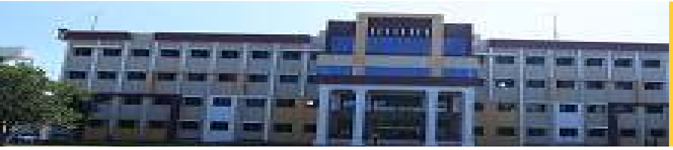


Type	Frequency response $H(e^{j\omega})$	Magnitude response $|H(e^{j\omega})|$	Phase response $\angle H(e^{j\omega})$	Applications.
Symmetric Impulse response N=odd Centre of symmetry at n=0.	$H(e^{j\omega}) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2h(n)\cos\omega n$.	$|H(e^{j\omega})| = \left|h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2h(n)\cos\omega n\right|$	$\theta(\omega) = 0$.	LPF, HPF, BPF, BSF/BRF
Antisymmetric Impulse response N=odd with centre of antisymmetry at n=0.	$e^{-j\pi/2} \sum_{n=1}^{\frac{N-1}{2}} 2h(n)\sin\omega n$.	$\left|\sum_{n=1}^{\frac{N-1}{2}} 2h(n)\sin\omega n\right|$	$\theta(\omega) = -\pi/2$	Hilbert transformers and differentiators

Fourier series Method

- The desired frequency response $H_d(e^{j\omega})$ of a system is periodic in 2π. From the Fourier series analysis we know that any periodic function can be expressed as a linear combination of complex exponentials.
- Therefore the desired frequency response of an FIR filter can be represented by the Fourier series

$$H_d(e^{j\omega}) = \sum_{n=0}^{N-1} h_d[n] e^{-j\omega n} \rightarrow (1)$$



Fourier series Method

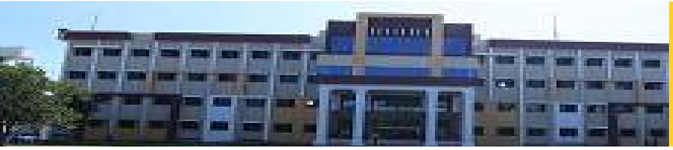
- Where the frequency coefficients $h_d(n)$ are the desired impulse response sequence of the filter.

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}\,d\omega \rightarrow (2)$$

- The Z-transform of the sequence is given by

$$H(z) = \sum_{n=-\infty}^{\infty} h_d[n]z^{-n} \rightarrow (3)$$

- Equ.3 represents a non-causal digital filter of infinite duration.



Fourier series Method

- To get an FIR filter transfer function the series can be truncated by assigning

$$h[n] = \begin{cases} h_d[n] & for \quad |n| \le \frac{N-1}{2} \\ 0 & otherwise \end{cases} \rightarrow (4)$$

- Then

$$H(z) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h[n]z^{-n} = h\left[\frac{N-1}{2}\right]z^{-\left(\frac{N-1}{2}\right)} + \ldots + h[2]z^{-2} + h[1]z^{-1}$$

$$+ h[0] + h[-1]z + h[-2]z^{2} + \ldots + h\left[-\left(\frac{N-1}{2}\right)\right]z^{\left(\frac{N-1}{2}\right)}$$

$$H(z) = h[0] + \sum_{n=1}^{\left(\frac{N-1}{2}\right)} \left[h[n]z^{-n} + h[-n]z^{n}\right] \rightarrow (5)$$

Fourier series Method

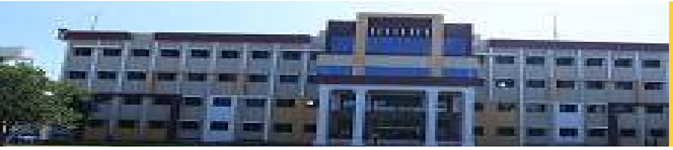
- For a symmetrical impulse response having symmetry at n=0

$$h[-n] = h[n]$$

- Therefore the equ.5 can be written as n

$$H(z) = h[0] + \sum_{n=1}^{\left(\frac{N-1}{2}\right)} h[n]\left[z^{n} + z^{-n}\right] \rightarrow (7)$$

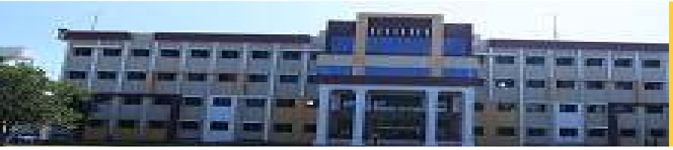
- The above T.F is not physically realizable.
- The realizability can be brought by multiplying the equ.7 with $z^{-\left(\frac{N-1}{2}\right)}$
- Where (N-1)/2 is delay in samples.



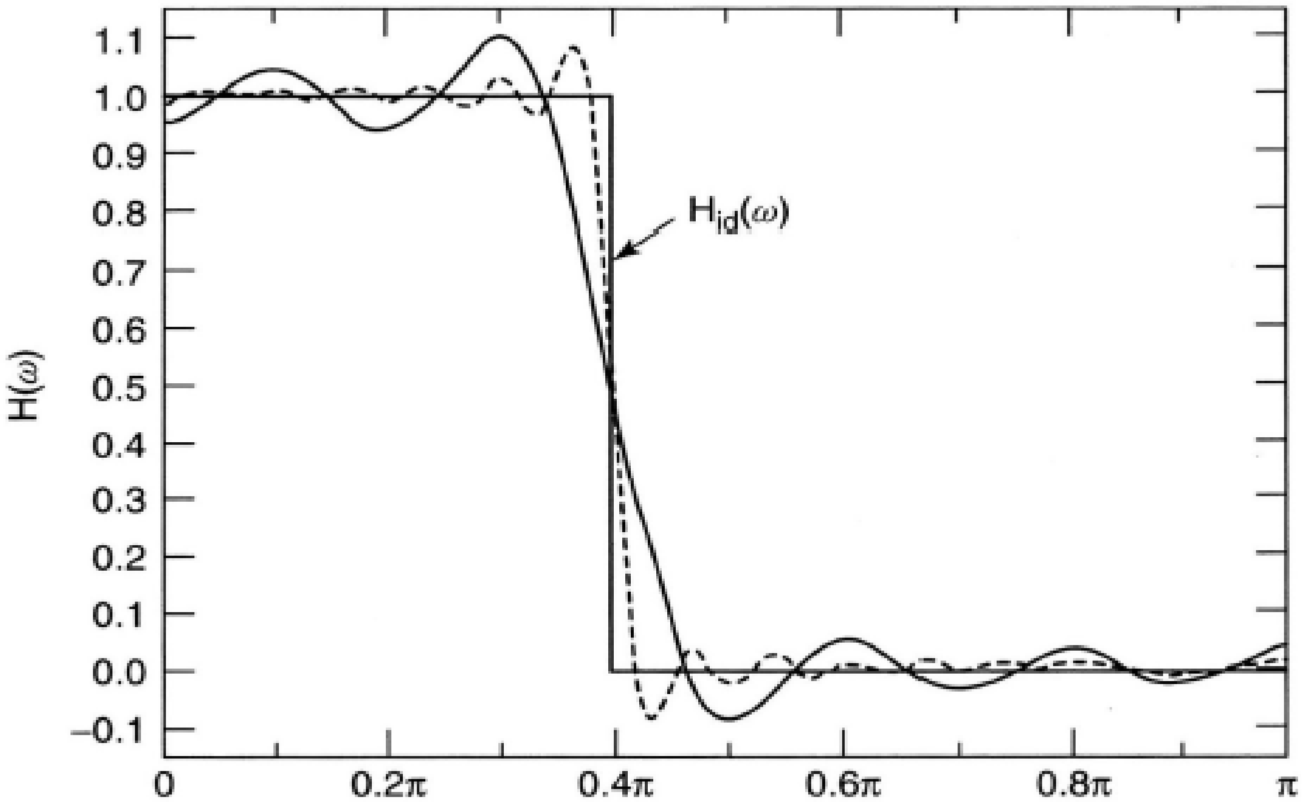
Fourier series Method

$$H^{'}(z) = z^{-\left(\frac{N-1}{2}\right)} H(z) = z^{-\left(\frac{N-1}{2}\right)} \left[h[0] + \sum_{n=1}^{\left(\frac{N-1}{2}\right)} h[n]\left(z^{n} + z^{-n}\right) \right] \rightarrow (8)$$

- From the equ.8 the causality was brought by multiplying the T.F with the delay factor.
- This modification does not effect the amplitude response of the filter.
- However the abrupt truncation of Fourier series results in oscillations in the pass band and stop band.



Fourier series Method

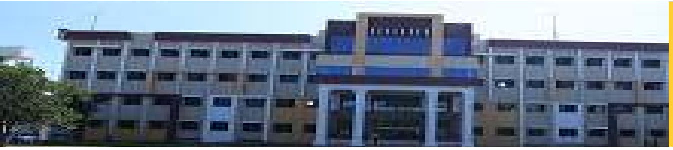


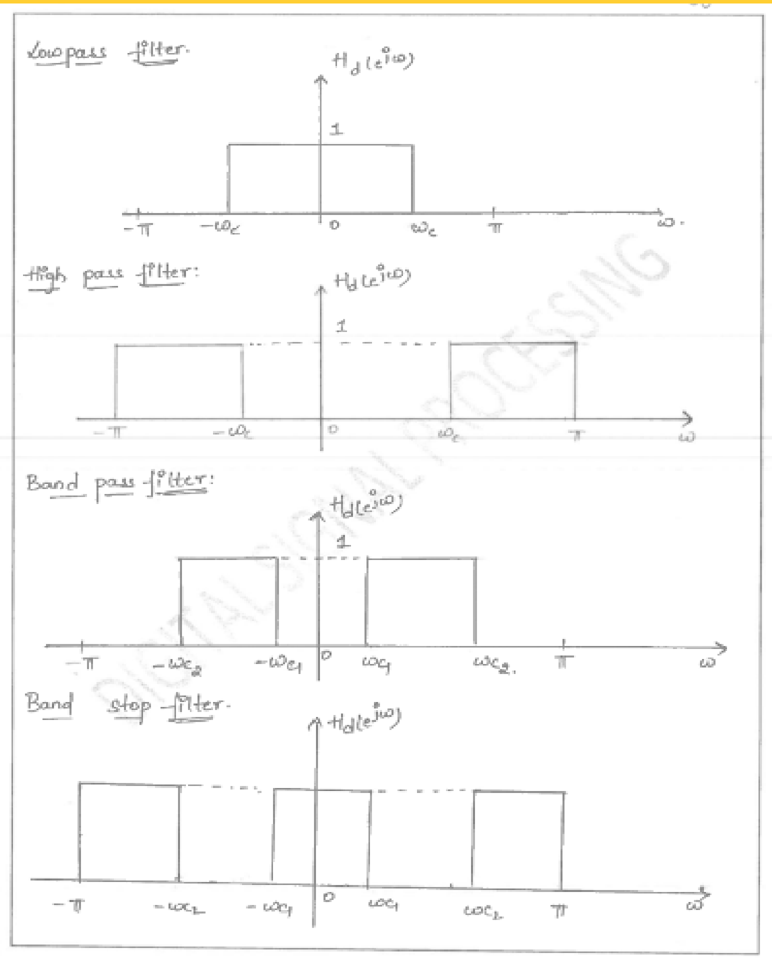
Fourier series Method – GIBBS phenomenon

This oscillation are due to slow conversion of the Fourier series particularly near the points of discontinuity. This effect is known as "Gibb's phenomenon".

- Gibbs phenomenon: As M increases, the maximum deviation from the ideal value decreases except near the point of discontinuity, where the error remains the same, however large the value M we choose. (i.e., as M increases, the maximum amplitude of the oscillation does not approach zero)

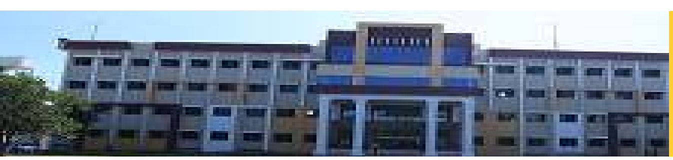
To reduce these oscillations, the Fourier coefficients of the filter are modified by multiplying the infinite impulse response with a finite weighting sequence w[n] called as window and that technique called as windowing method.




Low pass filter.
High pass filter:
Band pass filter:
Band stop filter.

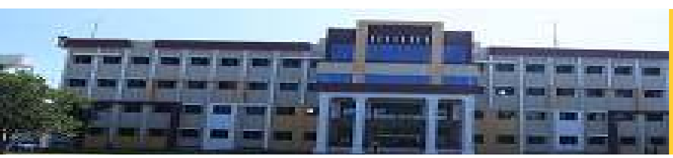
Specification and desired Impulse response for FIR filter design by Fourier Series Method.

Type of filter	Specifications	Impulse response.
1. Low pass	$H_d(e^{j\omega}) = \begin{cases} 1; & -\omega_c \le \omega \le +\omega_c \\ 0; & -\pi \le \omega \le -\omega_c \\ 0; & \omega_c \le \omega \le \pi \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{+\omega_c} e^{j\omega n} d\omega$ [$\because H_d(e^{j\omega}) = 0$ in range $-\pi \le \omega < -\omega_c$ and $+\omega_c < \omega \le \pi$]
2. High pass	$H_d(e^{j\omega}) = \begin{cases} 1; & -\pi \le \omega \le -\omega_c \\ 1; & \omega_c \le \omega \le \pi \\ 0; & -\omega_c < \omega < +\omega_c \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{j\omega n} d\omega.$ [$\because H_d(e^{j\omega}) = 0$ in range $-\omega_c < \omega < +\omega_c$.
3. Band pass	$H_d(e^{j\omega}) = \begin{cases} 1; & -\omega_{c2} \le \omega \le -\omega_{c1} \\ 1; & \omega_{c1} \le \omega \le \omega_{c2} \\ 0; & -\pi \le \omega < -\omega_{c2} \\ 0; & -\omega_{c1} < \omega < +\omega_{c1} \\ 0; & \omega_{c2} < \omega \le \pi \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\omega_{c2}}^{-\omega_{c1}} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\omega_{c1}}^{\omega_{c2}} e^{j\omega n} d\omega$ [$\because H_d(e^{j\omega}) = 0$ in the range $-\pi \le \omega < -\omega_{c2}$; $-\omega_{c1} < \omega < +\omega_{c1}$ and $+\omega_{c2} < \omega \le \pi$
4. Band stop.	$H_d(e^{j\omega}) = \begin{cases} 1; & -\pi \le \omega \le -\omega_{c2} \\ 1; & -\omega_{c1} \le \omega \le +\omega_{c1} \\ 1; & \omega_{c2} \le \omega \le \pi \\ 0; & -\omega_{c2} < \omega < -\omega_{c1} \\ 0; & \omega_{c1} < \omega < \omega_{c2}. \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_{c2}} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{-\omega_{c1}}^{\omega_{c1}} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\omega_{c2}}^{\pi} e^{j\omega n} d\omega.$ [$\because H_d(e^{j\omega}) = 0$ in range $-\omega_{c2} < \omega < -\omega_{c1}$ and $\omega_{c1} < \omega < \omega_{c2}$

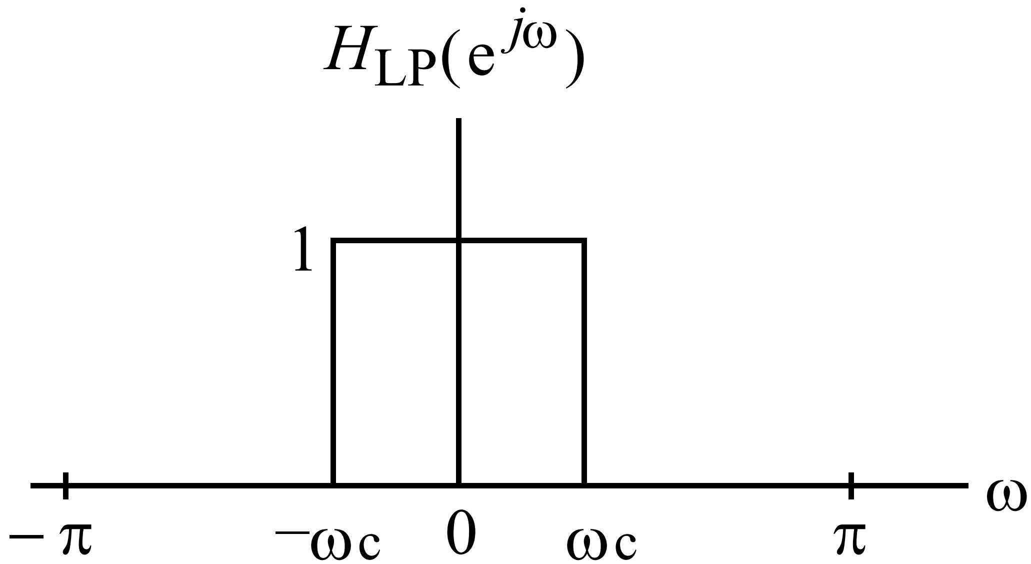


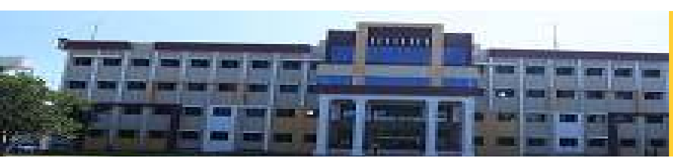
The Normalised Ideal (Desired) Freq. response & Impulse response for FIR filter design using Windows:

Type of filter	Ideal (desired) freq. response	Ideal (Desired) Impulse response.
Low pass	$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha}; & -\omega_c \le \omega \le +\omega_c \\ 0; & -\pi \le \omega < -\omega_c \\ 0; & \omega_c < \omega \le \pi \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\alpha} e^{j\omega n} d\omega$. $[\because H_d(e^{j\omega}) = 0$ in range $-\pi \le \omega < -\omega_c$ and $\omega_c < \omega \le \pi]$
High pass	$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha}; & -\pi \le \omega \le -\omega_c \\ e^{-j\omega\alpha}; & \omega_c \le \omega \le \pi \\ 0; & -\omega_c < \omega < \omega_c \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{-j\omega\alpha} \cdot e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{-j\omega\alpha} e^{j\omega n} d\omega$ $[\because H_d(e^{j\omega}) = 0$ in range $-\omega_c < \omega < +\omega_c]$
Band pass	$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha}; & -\omega_{c2} \le \omega \le -\omega_{c1} \\ e^{-j\omega\alpha}; & \omega_{c1} \le \omega \le \omega_{c2} \\ 0; & -\pi \le \omega < -\omega_{c2} \\ 0; & -\omega_{c1} < \omega < \omega_{c1} \\ 0; & \omega_{c2} < \omega \le \pi \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\omega_{c2}}^{-\omega_{c1}} e^{-j\omega\alpha} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\omega_{c1}}^{\omega_{c2}} e^{-j\omega\alpha} \cdot e^{j\omega n} d\omega$ $[H_d(e^{j\omega}) = 0$ in range $-\pi \le \omega < -\omega_{c2}$; $-\omega_{c1} < \omega < \omega_{c1}$ and $\omega_{c2} < \omega \le \pi]$
Band Stop	$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha}; & -\pi \le \omega \le -\omega_{c2} \\ e^{-j\omega\alpha}; & -\omega_{c1} \le \omega \le \omega_{c1} \\ e^{-j\omega\alpha}; & \omega_{c2} \le \omega \le \pi \\ 0; & -\omega_{c2} < \omega < -\omega_{c1} \\ 0; & \omega_{c1} < \omega < \omega_{c2} \end{cases}$	$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_{c2}} e^{-j\omega\alpha} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{-\omega_{c1}}^{\omega_{c1}} e^{-j\omega\alpha} \cdot e^{j\omega n} d\omega$ $+ \frac{1}{2\pi}\int_{\omega_{c2}}^{\pi} e^{-j\omega\alpha} e^{j\omega n} d\omega$ $[\because H_d(e^{j\omega}) = 0$ in range $-\omega_{c2} < \omega < -\omega_{c1}$ & $\omega_{c1} < \omega < +\omega_{c2}]$



Filter Coefficients of FIR filters
Ideal Low pass filter



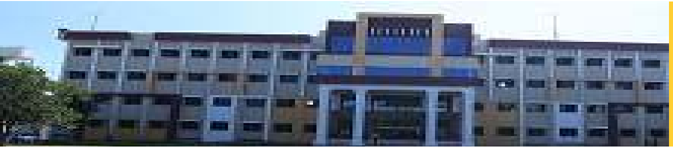


Filter Coefficients of FIR filters
Ideal Low pass filter

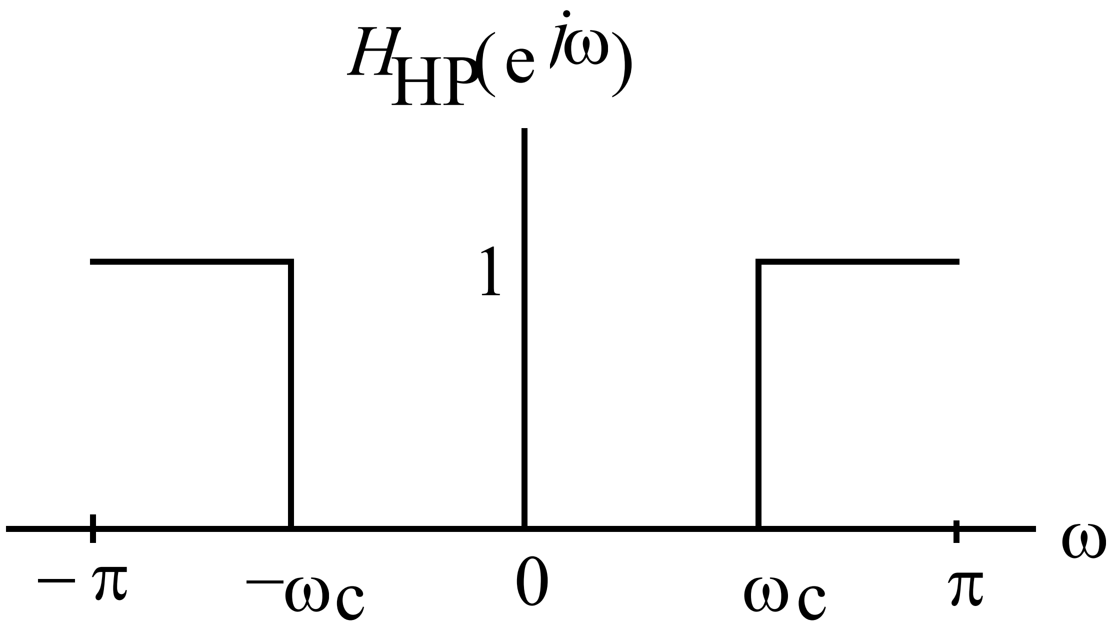
$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega$$

$$h_d[n] = \frac{1}{2\pi}\left.\frac{e^{j\omega n}}{jn}\right|_{-\omega_c}^{\omega_c} = \frac{1}{2\pi}\,\frac{e^{jn\omega_c} - e^{-jn\omega_c}}{jn} = \frac{1}{n\pi}\,\frac{e^{jn\omega_c} - e^{-jn\omega_c}}{2j}$$

$$\boxed{h_d[n] = \frac{\sin(n\omega_c)}{n\pi}}$$



Filter Coefficients of FIR filters
Ideal High pass filter



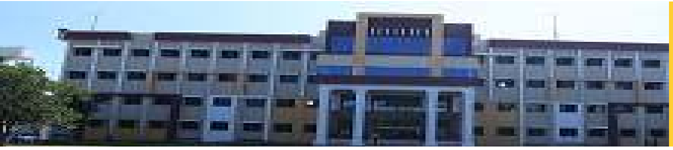
Filter Coefficients of FIR filters
Ideal High pass filter

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\left[\int_{-\pi}^{-\omega_c} e^{j\omega n}\, d\omega + \int_{\omega_c}^{\pi} e^{j\omega n}\, d\omega\right]$$

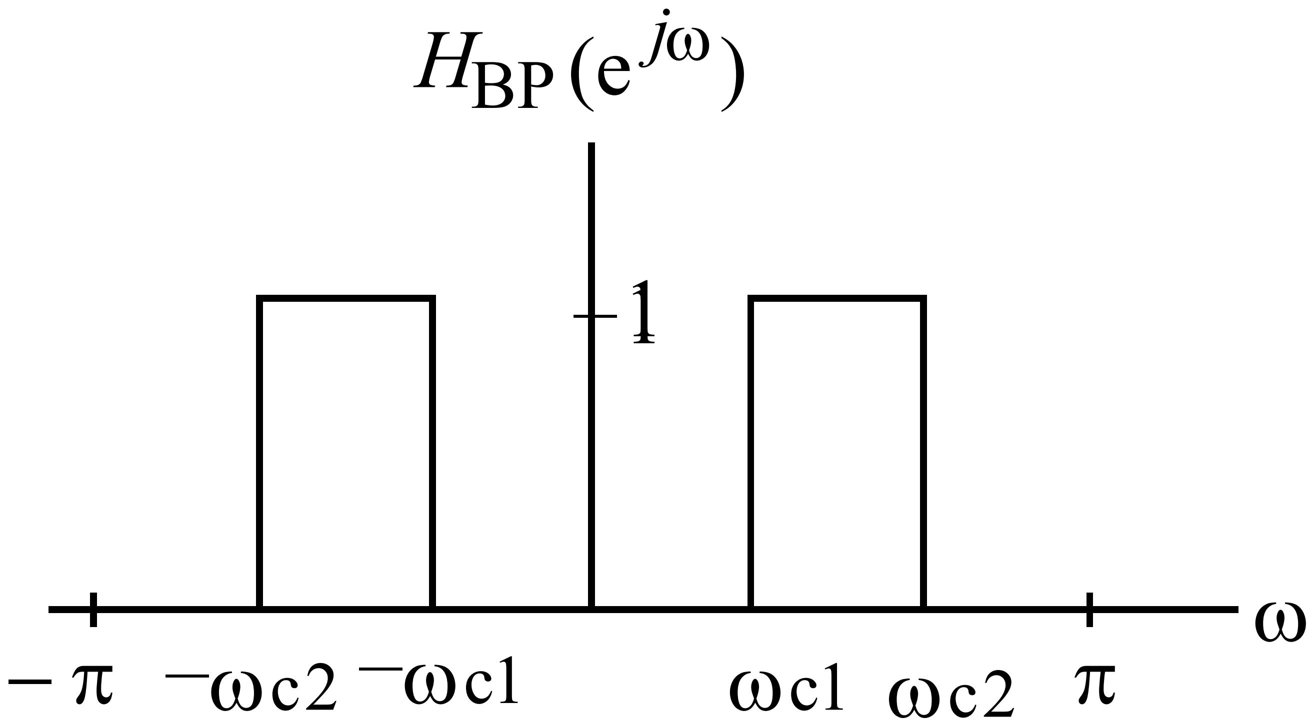
$$h_d[n] = \frac{1}{2\pi}\left[\left.\frac{e^{j\omega n}}{jn}\right|_{-\pi}^{-\omega_c} + \left.\frac{e^{j\omega n}}{jn}\right|_{\omega_c}^{\pi}\right] = \frac{1}{2\pi}\,\frac{e^{-jn\omega_c} - e^{-jn\pi} + e^{jn\pi} - e^{jn\omega_c}}{jn}$$

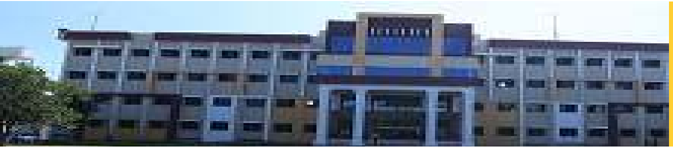
$$h_d[n] = \frac{1}{n\pi}\left[\frac{e^{jn\pi} - e^{-jn\pi}}{2j} - \left(\frac{e^{jn\omega_c} - e^{-jn\omega_c}}{2j}\right)\right]$$

$$\boxed{h_d[n] = \frac{1}{n\pi}\left[\sin(n\pi) - \sin(n\omega_c)\right]}$$



Filter Coefficients of FIR filters
Ideal Band pass filter





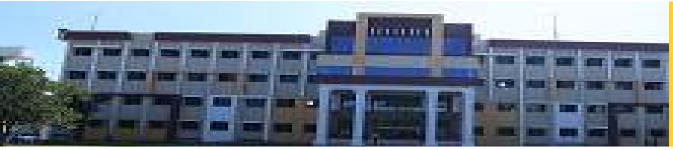
Filter Coefficients of FIR filters
Ideal Band pass filter

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\left[\int_{-\omega_{c2}}^{-\omega_{c1}} e^{j\omega n}\, d\omega + \int_{\omega_{c1}}^{\omega_{c2}} e^{j\omega n}\, d\omega\right]$$

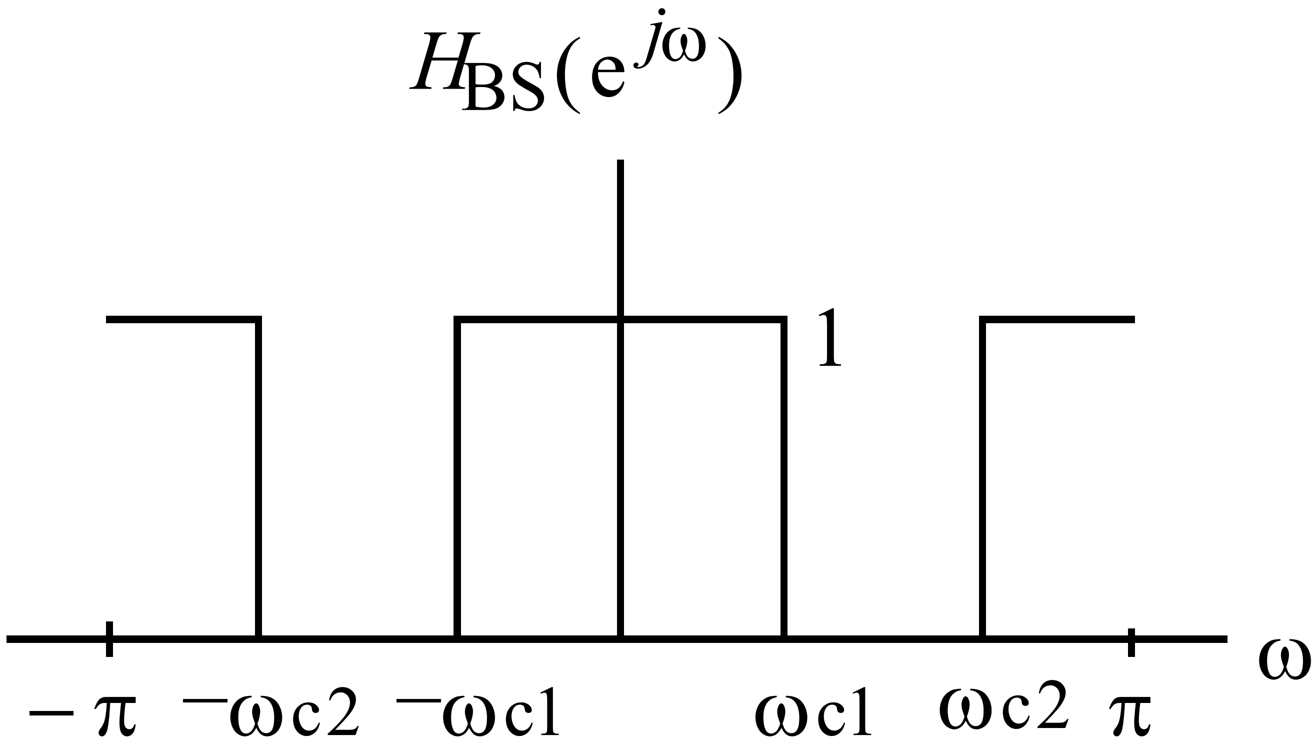
$$h_d[n] = \frac{1}{2\pi}\left[\left.\frac{e^{j\omega n}}{jn}\right|_{-\omega_{c2}}^{-\omega_{c1}} + \left.\frac{e^{j\omega n}}{jn}\right|_{\omega_{c1}}^{\omega_{c2}}\right] = \frac{1}{2\pi}\,\frac{e^{-jn\omega_{c1}} - e^{-jn\omega_{c2}} + e^{jn\omega_{c2}} - e^{jn\omega_{c1}}}{jn}$$

$$h_d[n] = \frac{1}{n\pi}\left[\frac{e^{jn\omega_{c2}} - e^{-jn\omega_{c2}}}{2j} - \left(\frac{e^{jn\omega_{c1}} - e^{-jn\omega_{c1}}}{2j}\right)\right]$$

$$\boxed{h_d[n] = \frac{1}{n\pi}\left[\sin(n\omega_{c2}) - \sin(n\omega_{c1})\right]}$$



Ideal Band stop filter



Filter Coefficients of FIR filters
Ideal Band stop filter

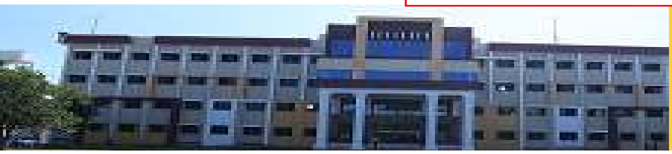
$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\,e^{j\omega n}\,d\omega = \frac{1}{2\pi}\left[\int_{-\pi}^{-\omega_{c2}} e^{j\omega n}\,d\omega + \int_{-\omega_{c1}}^{\omega_{c1}} e^{j\omega n}\,d\omega + \int_{\omega_{c1}}^{\pi} e^{j\omega n}\,d\omega\right]$$

$$h_d[n] = \frac{1}{2\pi}\left[\left.\frac{e^{j\omega n}}{jn}\right|_{-\pi}^{-\omega_{c2}} + \left.\frac{e^{j\omega n}}{jn}\right|_{-\omega_{c1}}^{\omega_{c1}} + \left.\frac{e^{j\omega n}}{jn}\right|_{\omega_{c2}}^{\pi}\right]$$

$$h_d[n] = \frac{1}{2\pi}\,\frac{e^{-jn\omega_{c2}} - e^{-jn\pi} + e^{jn\omega_{c1}} - e^{-jn\omega_{c1}} + e^{jn\pi} - e^{jn\omega_{c2}}}{jn}$$

$$h_d[n] = \frac{1}{n\pi}\left[\frac{e^{jn\pi} - e^{-jn\pi}}{2j} + \frac{e^{jn\omega_{c1}} - e^{-jn\omega_{c1}}}{2j} - \left(\frac{e^{jn\omega_{c2}} - e^{-jn\omega_{c2}}}{2j}\right)\right]$$

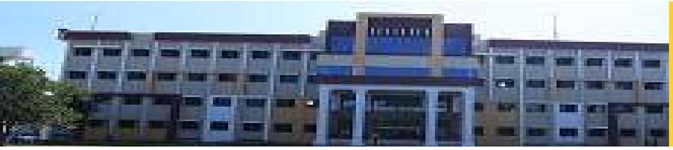
$$\boxed{h_d[n] = \frac{1}{n\pi}\left\{\sin(n\pi - \left[\sin(n\omega_{c2}) - \sin(n\omega_{c1})\right]\right\}}$$



Filter Coefficients of FIR filters

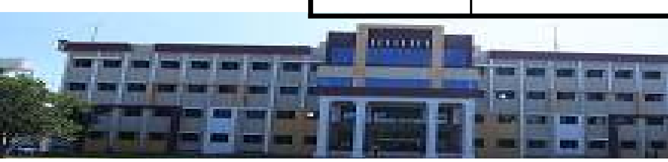
$$Delay\ factor\ \alpha = \frac{N-1}{2}$$

Type	Zero phase $h_d[n]$	Linear phase $h_d[n]$
Low Pass	$h_d[0] = \frac{\omega_c}{\pi}$ $h_d[n] = \frac{\sin(n\omega_c)}{n\pi} \quad for\ \lvert n \rvert > 0$	$h_d[n] = \frac{\omega_c}{\pi} \quad for \quad n = \alpha$ $h_d[n] = \frac{\sin(n-\alpha)\omega_c}{(n-\alpha)\pi} \quad for \quad n \neq \alpha$
High Pass	$h_d[0] = 1 - \frac{\omega_c}{\pi}$ $h_d[n] = \left(\frac{\sin(n\pi) - \sin(n\omega_c)}{n\pi}\right)$ $for \quad \lvert n \rvert > 0$	$h_d[n] = 1 - \frac{\omega_c}{\pi} \quad for \quad n = \alpha$ $h_d[n] = \left(\frac{\sin(n-\alpha)\pi - \sin(n-\alpha)\omega_c}{(n-\alpha)\pi}\right)$ $for \quad n \neq \alpha$



Filter Coefficients of FIR filters

Type	Zero phase $h_d[n]$	Linear phase $h_d[n]$
Band Pass	$h_d[0]=\left(\frac{\omega_{c2}-\omega_{c1}}{\pi}\right)$ $h_d[n]=\left(\frac{\sin(n\omega_{c2})-\sin(n\omega_{c1})}{n\pi}\right)$ $for \quad \lvert n\rvert>0$	$h_d[n]=\left(\frac{\omega_{c2}-\omega_{c1}}{\pi}\right) \quad for \quad n=\alpha$ $h_d[n]=\left(\frac{\sin(n-\alpha)\omega_{c2}-\sin(n-\alpha)\omega_{c1}}{(n-\alpha)\pi}\right)$ $for \quad n\neq\alpha$
Band Stop	$h_d[0]=1-\left(\frac{\omega_{c2}-\omega_{c1}}{\pi}\right)$ $h_d[n]=\frac{\{\sin(n\pi-[\sin(n\omega_{c2})-\sin(n\omega_{c1})]\}}{n\pi}$ $for \quad \lvert n\rvert>0$	$h_d[n]=1-\left(\frac{\omega_{c2}-\omega_{c1}}{\pi}\right) \quad for \quad n=\alpha$ $h_d[n]=\frac{\{\sin(n-\alpha)\pi-[\sin(n-\alpha)\omega_{c2}-\sin(n-\alpha)\omega_{c1}]\}}{(n-\alpha)\pi}$ $for \quad n\neq\alpha$



DESIGN PROCEDURE FOR FIR FILTER USING FOURIER SERIES METHOD

Design procedure for FIR filter by Fourier Series method

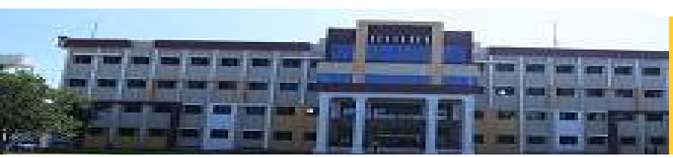
1. Specifications of digital FIR filter are
   i) Desired freq response $H_d(e^{j\omega})$
   ii) Cutoff freq ω_c for Low pass and high pass and ω_{c1} and ω_{c2} for bandpass and bandstop filters.

If analog filter cutoff freq F_c and sampling freq F_s are given, then calculate the cutoff freq of digital filter ω_c using the equation $\omega_c = \frac{2\pi F_c}{F_s}$

   iii) Number of samples of impulse response 'N'.

2. Determine desired impulse response $h_d(n)$ by taking inverse F.T of desired freq response $H_d(e^{j\omega})$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) \cdot e^{j\omega n} \cdot d\omega$$



DESIGN PROCEDURE FOR FIR FILTER USING FOURIER SERIES METHOD

3. Calculate 'N' samples $h_d(n)$ for $n = -\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$ and form the impulse response $h(n)$ of an FIR filter.

$\therefore$ Impulse response $h(n) = h_d(n)$ for $|n| \le \frac{N-1}{2}$

$= 0$ otherwise

As impulse response is symmetric with $n=0$, so $h(-n) = h(n)$. Hence it is sufficient to calculate $h(n)$ for $n = 0$ to $\left(\frac{N-1}{2}\right)$

4. Apply Z-T of impulse response to get non-causal transfer function of FIR filter $H(z)$

$$H(z) = Z\{h(n)\} = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} h(n) \cdot z^{-n}$$

5. Convert non causal transfer function $H(z)$ to causal transfer function $H'(z)$ by multiplying $H(z)$ by $z^{-\left(\frac{N-1}{2}\right)}$

$$H'(z) = z^{-\left(\frac{N-1}{2}\right)} \cdot H(z) = z^{-\left(\frac{N-1}{2}\right)} \cdot \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n) \cdot z^{-n}$$

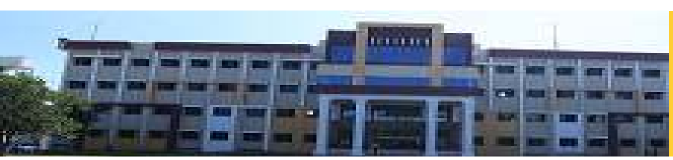
$$H'(z) = z^{-\left(\frac{N-1}{2}\right)} \left[h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) \left(z^{n} + z^{-n}\right) \right]$$

6. Draw a suitable structure for realization of FIR filter designed.

7. Draw freq response $H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$

ie $|H(e^{j\omega})|$ vs ω (0 to π) & $\bar{H}(e^{j\omega})$ vs ω (0 to π)

(Magnitude) (Amplitude)



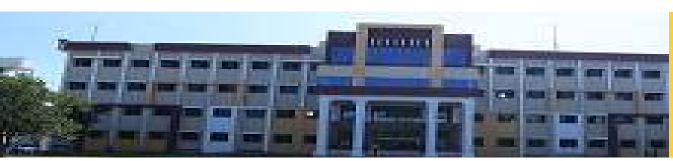
Design of FIR filters: Windows

(i) Start with ideal infinite duration $\{h(n)\}$

(ii) Truncate to finite length. (This produces unwanted ripples increasing in height near discontinuity.)

(iii) Modify to $\tilde{h}(n) = h(n).w(n)$

Weight $w(n)$ is the window



Design of FIR filters: Windows

- Simplest way of designing FIR filters
- Method is all discrete-time no continuous-time involved
- Start with ideal frequency response

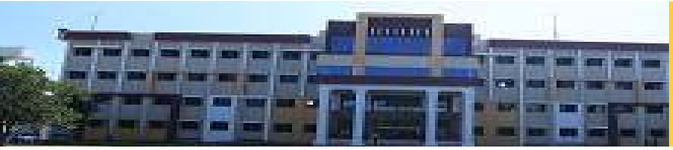
$$H_d\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h_d[n] e^{-j\omega n} \qquad h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d\left(e^{j\omega}\right) e^{j\omega n} d\omega$$

- Choose ideal frequency response as desired response
- Most ideal impulse responses are of infinite length
- The easiest way to obtain a causal FIR filter from ideal is

$$h[n] = \begin{cases} h_d[n] & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

- More generally

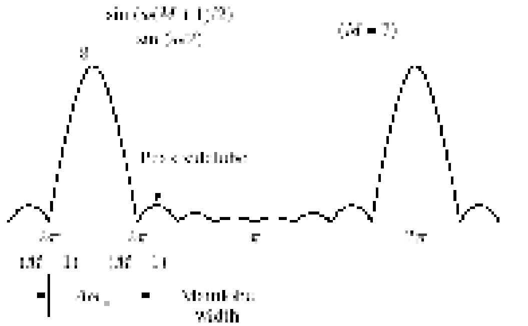
$$h[n] = h_d[n] w[n] \quad \text{where} \quad w[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

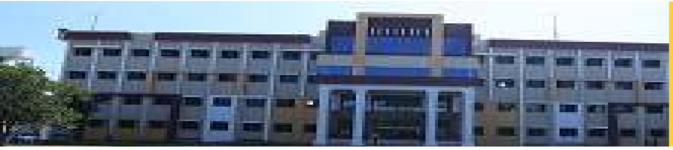


Properties of Windows

- Prefer windows that concentrate around DC in frequency
  - Less smearing, closer approximation
- Prefer window that has minimal span in time
  - Less coefficient in designed filter, computationally efficient
- So we want concentration in time and in frequency
  - Contradictory requirements
- Example: Rectangular window

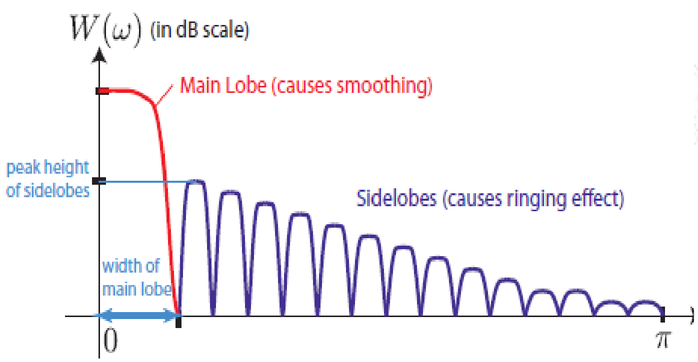
$$W\left(e^{j\omega}\right) = \sum_{n=0}^{M} e^{-j\omega n} = \frac{1-e^{-j\omega(M+1)}}{1-e^{-j\omega}} = e^{-j\omega M/2}\,\frac{\sin[\omega(M+1)/2]}{\sin[\omega/2]}$$

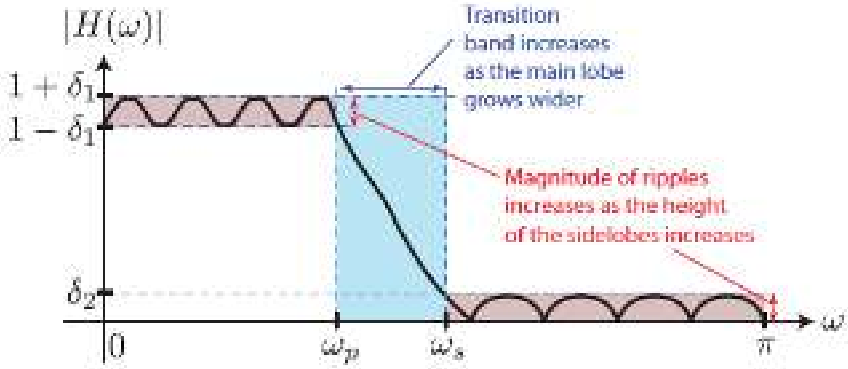


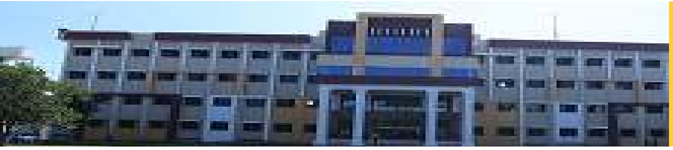


Windowing distortion

- increasing window length generally reduces the width of the main lobe
- peak of sidelobes is generally independent of M







Windows

Commonly used windows

- Rectangular 1
- Bartlett $1 - \frac{2|n|}{N}$
- Hanning $1 + \cos\left(\frac{2\pi n}{N}\right)$
- Hamming $0.54 + 0.46\cos\left(\frac{2\pi n}{N}\right)$
- Blackman $0.42 + 0.5\cos\left(\frac{2\pi n}{N}\right) + 0.08\cos\left(\frac{4\pi n}{N}\right)$

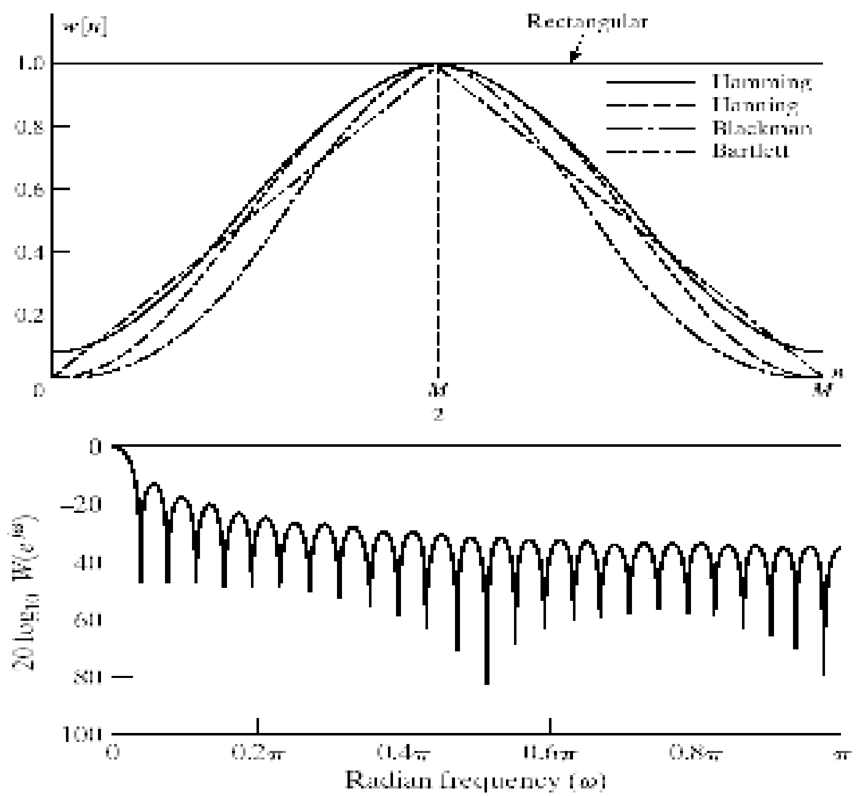
$$|n| < \frac{N-1}{2}$$

- Kaiser $J_0\left[\beta\sqrt{1-\left(\frac{2n}{N-1}\right)^2}\right] \Big/ J_0(\beta)$

Rectangular Window

- Narrowest main lob
  - $4\pi/(M+1)$
  - Sharpest transitions at discontinuities in frequency
- Large side lobs
  - -13 dB
  - Large oscillation around discontinuities
- Simplest window possible

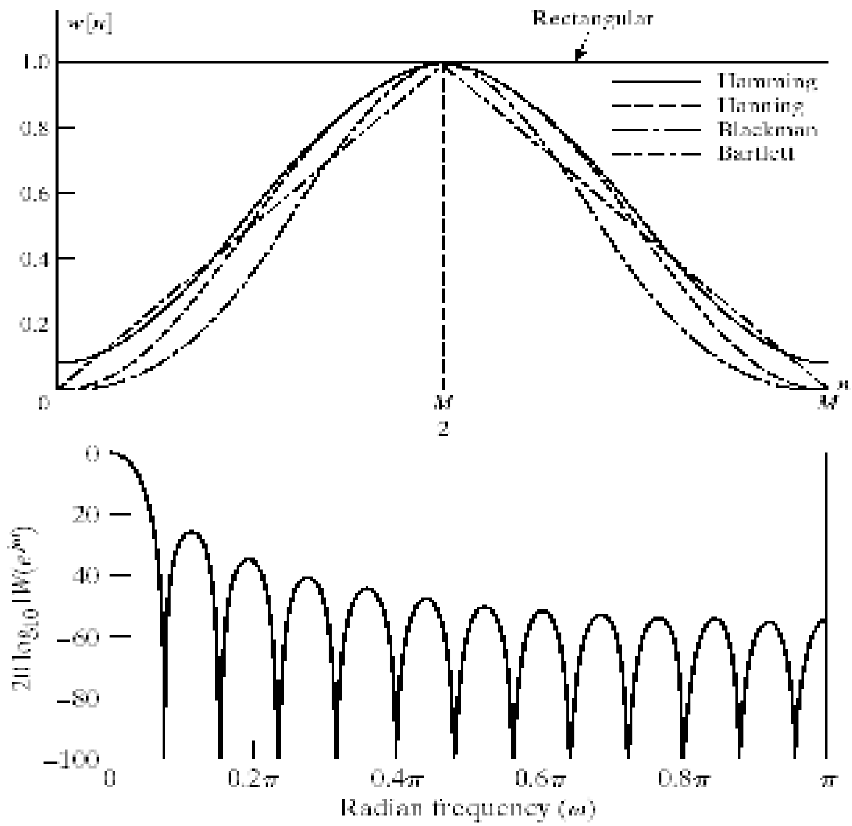
$$w[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

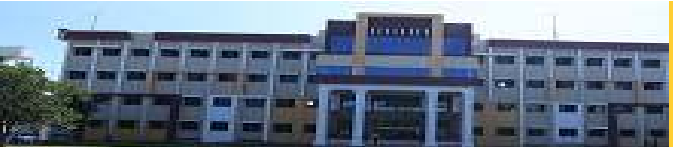


Bartlett (Triangular) Window

- Medium main lob
  - $8\pi/M$
- Side lobs
  - -25 dB
- Hamming window performs better
- Simple equation

$$w[n] = \begin{cases} 2n/M & 0 \le n \le M/2 \\ 2 - 2n/M & M/2 \le n \le M \\ 0 & \text{else} \end{cases}$$

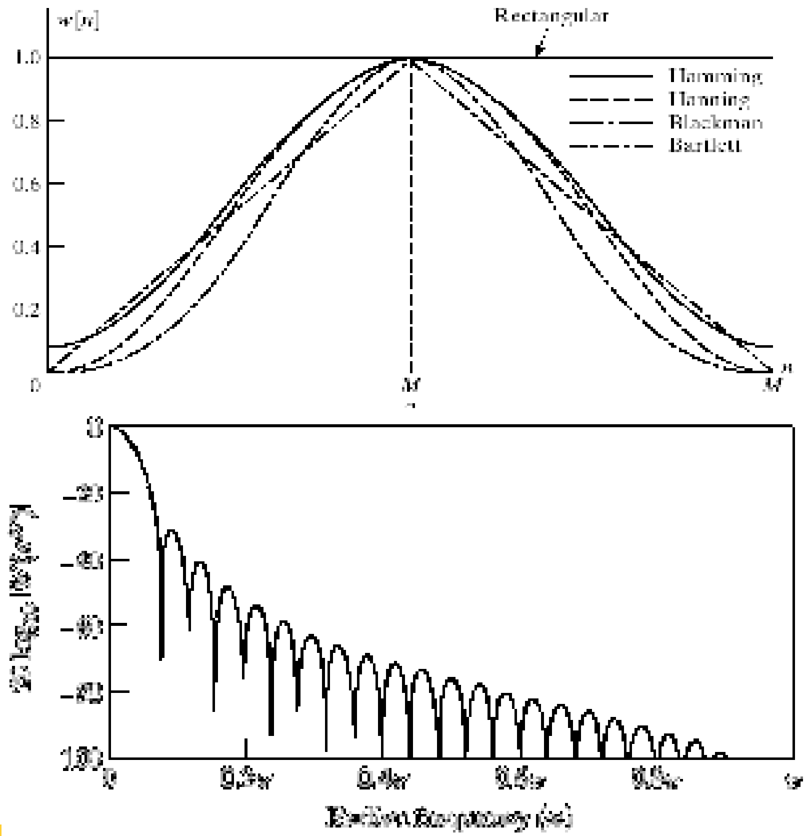


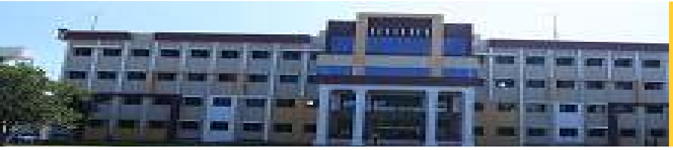


Hanning Window

- Medium main lob
  - $8\pi/M$
- Side lobs
  - -31 dB
- Hamming window performs better
- Same complexity as Hamming

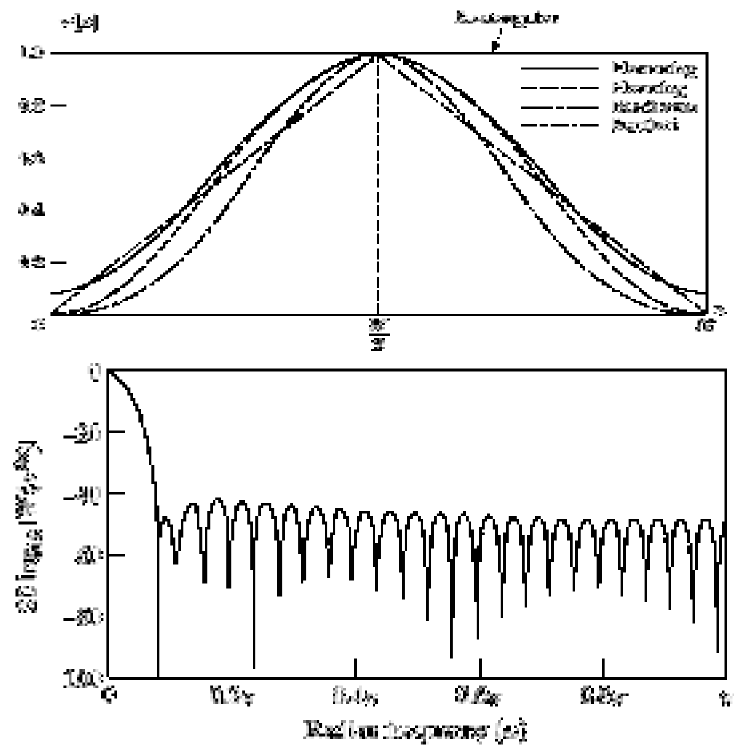
$$w[n] = \begin{cases} \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{M}\right)\right] & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$



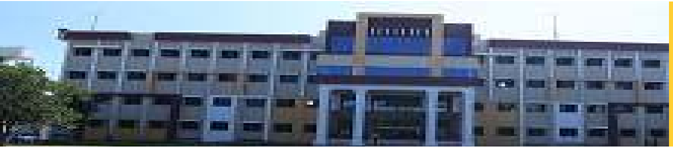


Hamming Window

- Medium main lob
  - $8\pi/M$
- Good side lobs
  - -41 dB
- Simpler than Blackman



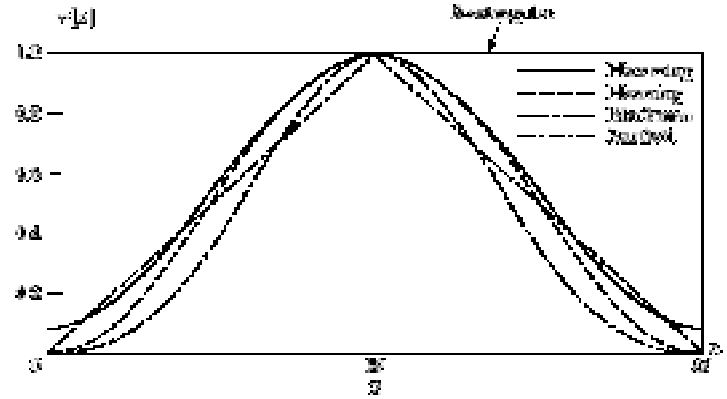
$$w[n] = \begin{cases} 0.54 - 0.46\cos\left(\dfrac{2\pi n}{M}\right) & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

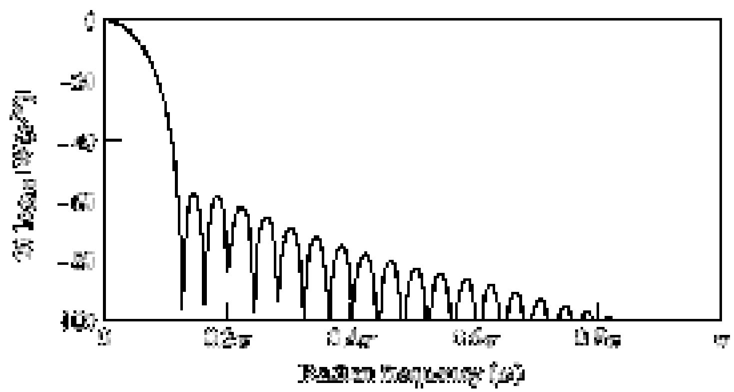


Blackman Window

- Large main lob
  - 12π/M
- Very good side lobs
  - -57 dB
- Complex equation

$$w[n] = \begin{cases} 0.42 - 0.5\cos\left(\dfrac{2\pi n}{M}\right) + 0.08\cos\left(\dfrac{4\pi n}{M}\right) & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$



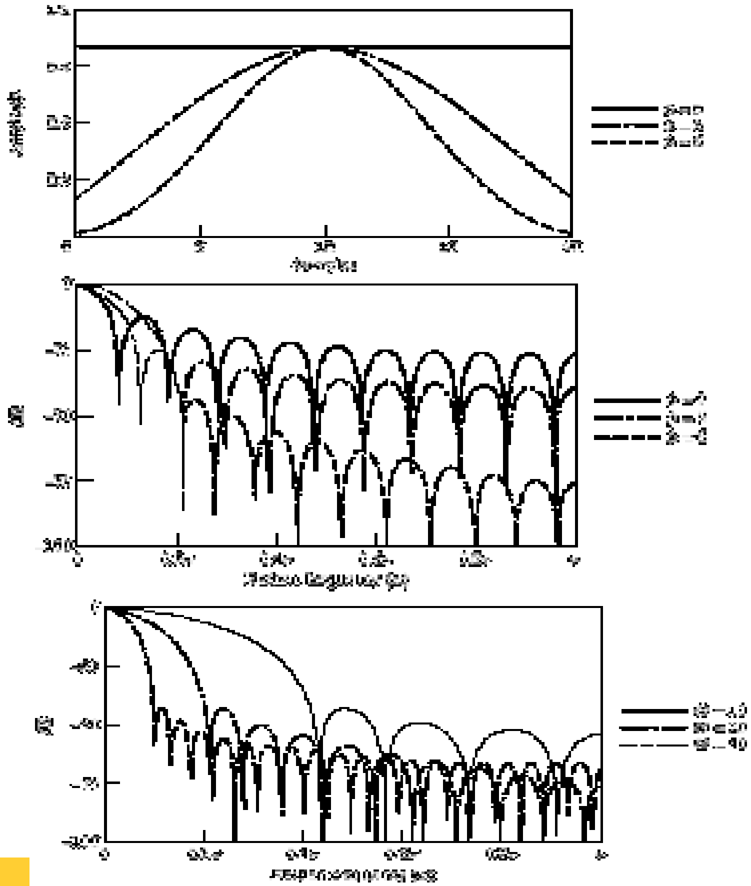


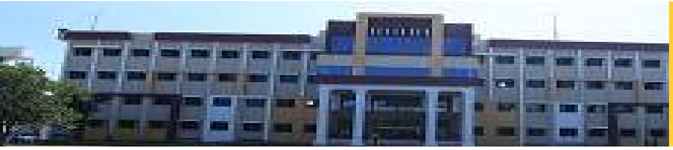
Kaiser Window Filter Design Method

- Parameterized equation forming a set of windows
  - Parameter to change main-lob width and side-lob area trade-off

$$w[n] = \begin{cases} \dfrac{I_0\left[\beta\sqrt{1-\left(\dfrac{n-M/2}{M/2}\right)^2}\,\right]}{I_0(\beta)} & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

  - $I_0(.)$ represents zeroth-order modified Bessel function of 1st kind

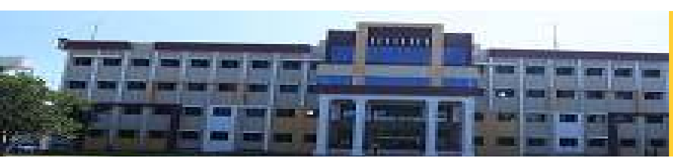




Comparison of windows

COMPARISON OF COMMONLY USED WINDOWS

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20\log_{10}\delta$ (dB)	Equivalent Kaiser Window, β	Transition Width of Equivalent Kaiser Window
Rectangular	−13	$4\pi/(M+1)$	−21	0	$1.81\pi/M$
Bartlett	−25	$8\pi/M$	−25	1.33	$2.37\pi/M$
Hanning	−31	$8\pi/M$	−44	3.86	$5.01\pi/M$
Hamming	−41	$8\pi/M$	−53	4.86	$6.27\pi/M$
Blackman	−57	$12\pi/M$	−74	7.04	$9.19\pi/M$



PROCEDURE FOR FIR FILTER DESIGN USING WINDOW METHOD - 1

FIR filter Design Using Windows :-

Method-1 : Symmetry Condition $h(N-1-n) = h(n)$

1. Specifications of digital FIR filter are

* Desired freq response $H_d(e^{j\omega}) = C e^{-j\alpha\omega}$

C - const ; $\alpha = \frac{N-1}{2}$

* Cutoff freq ω_c for LPF and HPF

ω_{c_1} & ω_{c_2} for BP and BR

[If analog freq's are given then digital freq ω_c is calculated by $\omega_c = \frac{2\pi f}{f_s}$ f - cutoff freq ; f_s - sampling freq]

* No of samples of impulse response 'N'

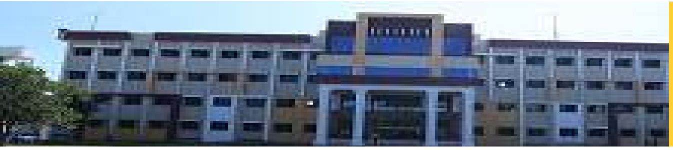
2. Determine desired impulse response $h_d(n)$ by taking IFT of $H_d(e^{j\omega})$ ⇒ $H_d(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) \cdot e^{j\omega n} \cdot d\omega$

3. Choose the window sequence $\omega(n)$ defined for $n = 0$ to $N-1$. Multiply $h_d(n)$ with $\omega(n)$ to get $h(n)$ of filter. Calculate N- samples of impulse response for $n = 0$ to $N-1$.

Impulse response $h(n) = h_d(n) \times \omega(n)$; $n = 0$ to $N-1$

Impulse response is symmetric with Centre of symmetry at $\left(\frac{N-1}{2}\right)$ and so $h(N-1-n) = h(n)$. Hence it is sufficient if we calculate $h(n)$ for $n = 0$ to $\left(\frac{N-1}{2}\right)$

4. Take Z-Transform of $h(n)$ to get transfer fn $H(z)$



PROCEDURE FOR FIR FILTER DESIGN USING WINDOW METHOD -1

of the filter.

$$H(z) = Z\{h(n)\} = \sum_{n=0}^{N-1} h(n) \cdot z^{-n}$$

5. Draw a suitable structure for realization of FIR filter

6. Determine freq response $H(e^{j\omega})$
   * Choose a linear phase magnitude f^n $|H(e^{j\omega})|$. Using $h(n)$, obtain an equation for $|H(e^{j\omega})|$

   (or)

   * Freq response $|H(e^{j\omega})|$ can be obtained by replacing z by $e^{j\omega}$ in transfer f^n $H(z)$

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$$

Calculate freq response values for ω b/n 0 & π and sketch the magnitude response.

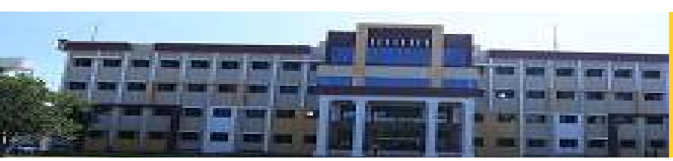
PROCEDURE FOR FIR FILTER DESIGN USING WINDOW METHOD -2

Method-2 : Symmetry condition $h(-n) = h(n)$

Step 1 and 2 are same as method-1

3. Choose desired window sequence $\omega(n)$ defined for $n = -\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$. Multiply $h_d(n)$ with $\omega(n)$ to get impulse response $h(n)$ of filter. Calculate N-samples of $h(n)$ for $n = -\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$.

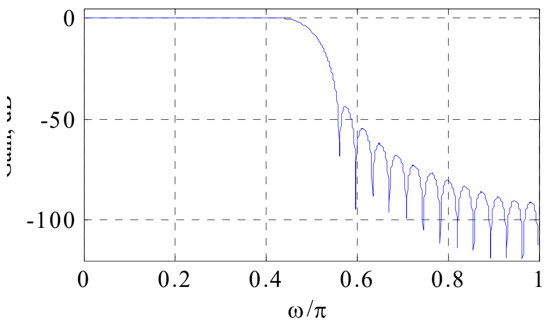
Impulse response $h(n) = h_d(n) \times \omega(n)$; $n = -\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$



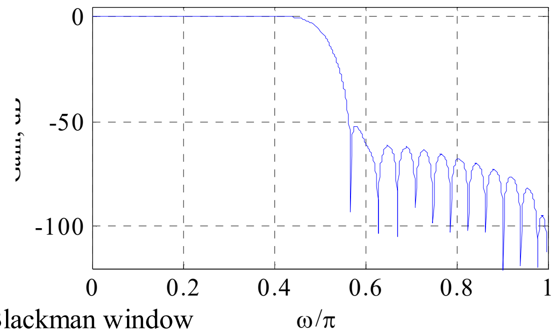
Example

- Lowpass filter of length 51 and $\omega_c = \pi / 2$

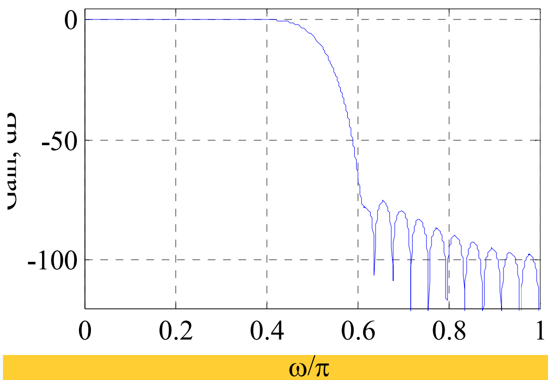
Lowpass Filter Designed Using Hann window

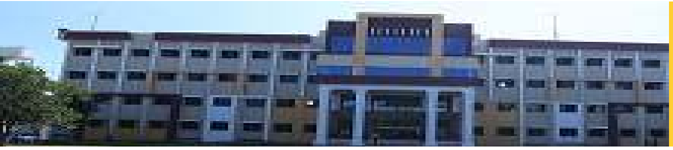


Lowpass Filter Designed Using Hamming window



Lowpass Filter Designed Using Blackman window





Example.1

Design an ideal high pass filter with a frequency response

$$H_d(e^{j\omega}) = \begin{cases} e^{-j5\omega} & for\ \frac{\pi}{4} \le |\omega| \le \pi \\ 0 & for\ |\omega| < \frac{\pi}{4} \end{cases}$$

Find the values of h[n] for N=11, using

a) Hanning window b) Hamming window

Solution: The freq Res is having a term $e^{j\omega(N-1)/2}$ which gives h[n] symmetry about (N-1)/2=5 .i.e. we get a causal sequence. N=11

a) Hanning window

$$w_{Hn}[n] = \begin{cases} 0.5 + 0.5\cos\frac{2\pi n}{N-1} & for \quad |n| \le \left(\frac{N-1}{2}\right) \\ 0 \quad otherwise \end{cases}$$

Example.1 Cont..d

With N=11 $$w_{Hn}[n] = \begin{cases} 0.5 + 0.5\cos\frac{\pi n}{5} & for \quad |n| \le 5 \\ 0 \quad otherwise \end{cases}$$

$$w_{Hn}[0] = 0.5 + 0.5 = 1$$

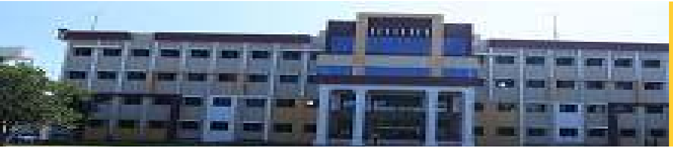
$$w_{Hn}[1] = w_{Hn}[-1] = 0.5 + 0.5\cos\frac{\pi}{5} = 0.9045$$

$$w_{Hn}[2] = w_{Hn}[-2] = 0.5 + 0.5\cos\frac{2\pi}{5} = 0.655$$

$$w_{Hn}[3] = w_{Hn}[-3] = 0.5 + 0.5\cos\frac{3\pi}{5} = 0.345$$

$$w_{Hn}[4] = w_{Hn}[-4] = 0.5 + 0.5\cos\frac{4\pi}{5} = 0.0945$$

$$w_{Hn}[5] = w_{Hn}[-5] = 0.5 + 0.5\cos\frac{5\pi}{5} = 0$$



Example.1 Cont..d

The filter coefficient equation is

$$h_d[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}H_d(e^{j\omega})e^{j\omega n}d\omega=\frac{1}{2\pi}\left[\int_{-\pi}^{-\frac{\pi}{4}}e^{j\omega n}d\omega+\int_{\frac{\pi}{4}}^{\pi}e^{j\omega n}d\omega\right]$$

$$h_d[n]=\frac{1}{2\pi}\left[\left.\frac{e^{j\omega n}}{jn}\right|_{-\pi}^{-\frac{\pi}{4}}+\left.\frac{e^{j\omega n}}{jn}\right|_{\frac{\pi}{4}}^{\pi}\right]=\frac{1}{2\pi}\frac{e^{-jn\frac{\pi}{4}}-e^{-jn\pi}+e^{jn\pi}-e^{jn\frac{\pi}{4}}}{jn}$$

$$h_d[n]=\frac{1}{n\pi}\left[\frac{e^{jn\pi}-e^{-jn\pi}}{2j}-\left(\frac{e^{jn\frac{\pi}{4}}-e^{-jn\frac{\pi}{4}}}{2j}\right)\right]$$

$$\boxed{h_d[n]=\frac{1}{n\pi}\left[\sin(n\pi)-\sin(n\tfrac{\pi}{4})\right]}\qquad\boxed{h_d[0]=1-\frac{\frac{\pi}{4}}{\pi}}$$

Example.1 Cont..d

The desired filter coefficients are

$$h_d[0] = 1 - \frac{\frac{\pi}{4}}{\pi} = 1 - \tfrac{1}{4} = 0.75$$

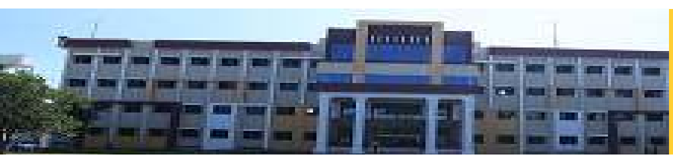
$$h_d[1] = h_d[-1] = \frac{1}{\pi}\left[\sin(\pi) - \sin(\tfrac{\pi}{4})\right] = -0.225$$

$$h_d[2] = h_d[-2] = \frac{1}{2\pi}\left[\sin(2\pi) - \sin(\tfrac{2\pi}{4})\right] = -0.159$$

$$h_d[3] = h_d[-3] = \frac{1}{3\pi}\left[\sin(3\pi) - \sin(\tfrac{3\pi}{4})\right] = -0.075$$

$$h_d[4] = h_d[-4] = \frac{1}{4\pi}\left[\sin(4\pi) - \sin(\tfrac{4\pi}{4})\right] = 0$$

$$h_d[5] = h_d[-5] = \frac{1}{5\pi}\left[\sin(5\pi) - \sin(\tfrac{5\pi}{4})\right] = 0.045$$



Example.1 Cont..d

The filter coefficients using Hanning window are

$$h[n] = h_d[n]\, w_{Hn}[n]$$

$$h[0] = h_d[0]\, w_{Hn}[0] = (0.75)(1) = 0.75$$

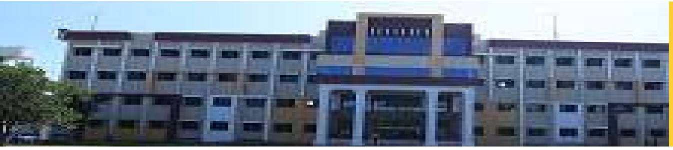
$$h[1] = h[-1] = h_d[1]\, w_{Hn}[1] = (-0.225)(0.9045) = -0.204$$

$$h[2] = h[-2] = h_d[2]\, w_{Hn}[2] = (-0.159)(0.655) = -0.104$$

$$h[3] = h[-3] = h_d[3]\, w_{Hn}[3] = (-0.015)(0.345) = -0.026$$

$$h[4] = h[-4] = h_d[4]\, w_{Hn}[4] = (0)(0.0945) = 0$$

$$h[5] = h[-5] = h_d[5]\, w_{Hn}[5] = (0.045)(0) = 0$$



Example.1 Cont..d

The transfer function of the filter is given by

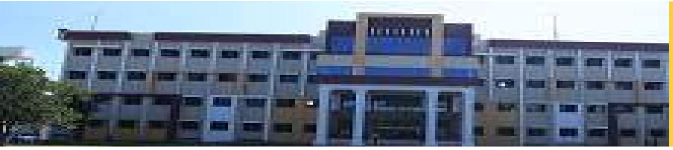
$$H(z) = 0.75 + \sum_{n=1}^{5} h[n]\left[z^{-n} + z^{n}\right]$$

$$H(z) = 0.75 - 0.204\left(z^{-1} + z^{1}\right) - 0.104\left(z^{-2} + z^{2}\right) - 0.026\left(z^{-3} + z^{3}\right)$$

The transfer function of the realizable filter is given by

$$H^{'}(z) = z^{-5}H(z) = -0.026z^{-2} - 0.104z^{-3} - 0.204z^{-4}$$

$$+ 0.75z^{-5} - 0.204z^{-6} - 0.104z^{-7} - 0.026z^{-8}$$



Example.1 Cont..d

The causal filter coefficients using Hanning window are

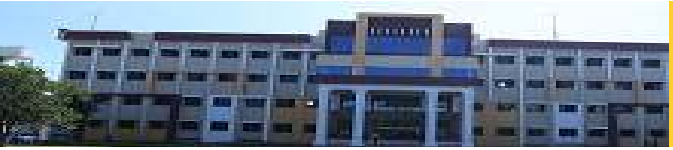
$$h[0] = h[1] = h[9] = h[10] = 0$$

$$h[2] = h[8] = -0.026$$

$$h[3] = h[7] = -0.104$$

$$h[4] = h[6] = -0.204$$

$$h[5] = 0.75$$



Example.1 Cont..d

b) Hamming window

$$w_H[n] = 0.54 + 0.46\cos\tfrac{2\pi n}{N-1} \quad for \quad |n| \le \left(\tfrac{N-1}{2}\right)$$

$$w_H[0] = 0.54 + 0.46 = 1$$

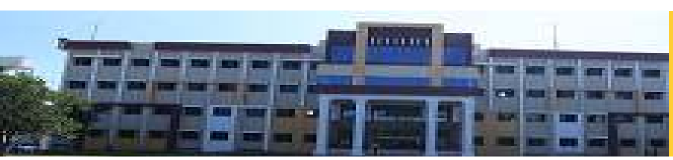
$$w_H[1] = w_H[-1] = 0.54 + 0.46\cos\tfrac{\pi}{5} = 0.912$$

$$w_H[2] = w_H[-2] = 0.54 + 0.46\cos\tfrac{2\pi}{5} = 0.682$$

$$w_H[3] = w_H[-3] = 0.54 + 0.46\cos\tfrac{3\pi}{5} = 0.398$$

$$w_H[4] = w_H[-4] = 0.54 + 0.46\cos\tfrac{4\pi}{5} = 0.1678$$

$$w_H[5] = w_H[-5] = 0.54 + 0.46\cos\tfrac{5\pi}{5} = 0.08$$



Example.1 Cont..d

The filter coefficients using Hamming window are

$$h[n] = h_d[n]\, w_H[n]$$

$$h[0] = h_d[0]\, w_H[0] = (0.75)(1) = 0.75$$

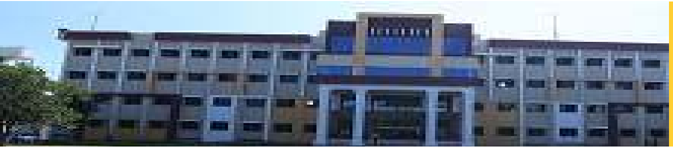
$$h[1] = h[-1] = h_d[1]\, w_H[1] = (-0.225)(0.912) = -0.2052$$

$$h[2] = h[-2] = h_d[2]\, w_H[2] = (-0.159)(0.682) = -0.1084$$

$$h[3] = h[-3] = h_d[3]\, w_H[3] = (-0.015)(0.398) = -0.03$$

$$h[4] = h[-4] = h_d[4]\, w_H[4] = (0)(0.1678) = 0$$

$$h[5] = h[-5] = h_d[5]\, w_H[5] = (0.045)(0.08) = 0.0036$$



Example.1 Cont..d

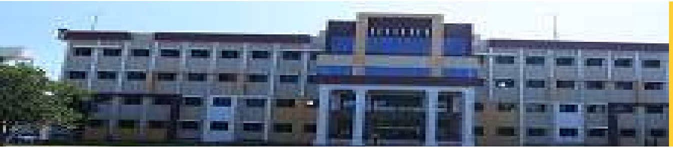
The transfer function of the filter is given by

$$H(z) = 0.75 + \sum_{n=1}^{5} h[n]\left[z^{-n} + z^{n}\right]$$

$$H(z) = 0.75 - 0.2052\left(z^{-1} + z^{1}\right) - 0.1084\left(z^{-2} + z^{2}\right)$$
$$- 0.03\left(z^{-3} + z^{3}\right) + 0.0036\left(z^{-5} + z^{5}\right)$$

The transfer function of the realizable filter is given by

$$H^{'}(z) = z^{-5}H(z) = 0.0036 - 0.03z^{-2} - 0.1084z^{-3} - 0.2052z^{-4}$$
$$+ 0.75z^{-5} - 0.2052z^{-6} - 0.1084z^{-7} - 0.03z^{-8} + 0.0036z^{-10}$$



Example.1 Cont..d

The causal filter coefficients using Hamming window are

$$h[0] = h[10] = 0.0036$$

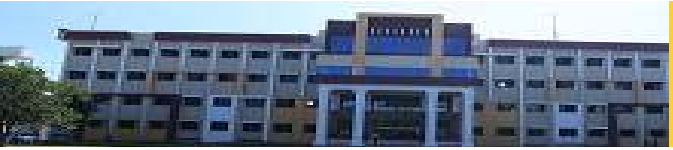
$$h[1] = h[9] = 0$$

$$h[2] = h[8] = -0.03$$

$$h[3] = h[7] = -0.1084$$

$$h[4] = h[6] = -0.2052$$

$$h[5] = 0.75$$

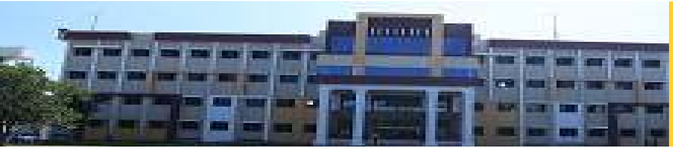


Frequency Sampling Method

- In this approach we are given $H(k)$ and need to find $H(z)$
- This is an interpolation problem and the solution is given in the DFT part of the course

$$H(z) = \frac{1}{N}\sum_{k=0}^{N-1} H(k).\frac{1-z^{-N}}{1-e^{j\frac{2\pi}{N}k}.z^{-1}}$$

- It has similar problems to the windowing approach



PROCEDURE FOR Frequency Sampling Method

Procedure for Type-1 design :-

1. Choose the ideal (desired) freq response $H_d(e^{j\omega})$
2. Sample $H_d(e^{j\omega})$ at N-points by taking $\omega = \omega_k = \frac{2\pi k}{N}$ $k = 0$ to $N-1$ to generate the sequence $H(k)$

$$H(k) = H_d(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}} \quad ; \; k = 0 \text{ to } N-1$$

3. Compute N- samples of impulse response $h(n)$

* When N is odd

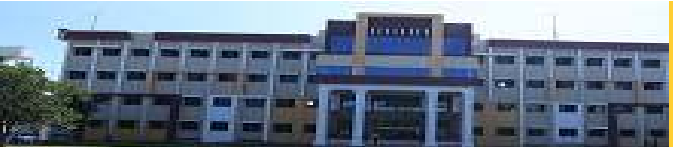
impulse response $h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} \mathrm{Real}\left[H(k)\cdot e^{\frac{j2\pi k n}{N}}\right]\right]$

* When N is even

impulse response $h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N}{2}-1} \mathrm{Real}\left[H(k)\cdot e^{\frac{j2\pi k n}{N}}\right]\right]$

$\left(H\left(\frac{N}{2}\right) = 0\right)$

4. Apply z-transform of $h(n)$ to obtain filter transfer function $H(z) = \sum_{n=0}^{N-1} h(n)\cdot z^{-n}$



PROCEDURE FOR Frequency Sampling Method

Procedure for Type - 2 design :-

1. Choose the ideal (desired) freq response $H_d(e^{j\omega})$
2. Sample $H_d(e^{j\omega})$ at N-points by taking $\omega = \omega_k = \frac{2\pi(2k+1)}{2N}$ $k = 0$ to $N-1$ to generate $H(k)$

$$H(k) = H_d(e^{j\omega})\Big|_{\omega = \frac{2\pi(2k+1)}{2N}} \quad \text{for } k = 0 \text{ to } N-1$$

3. Compute N samples of impulse response $h(n)$ using

* When N is odd

impulse response $h(n) = \frac{2}{N} \sum_{k=0}^{\frac{N-3}{2}} Re\left[H(k) \cdot e^{\frac{j n\pi(2k+1)}{N}}\right]$

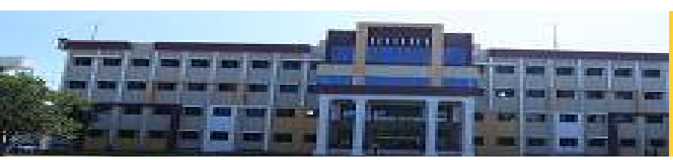
$\left(H\left(\frac{N-1}{2}\right) = 0 \right)$

* When N is even

impulse response $h(n) = \frac{2}{N} \times 2 \sum_{k=0}^{\frac{N}{2}-1} Re\left[H(k) \cdot e^{\frac{j n\pi(2k+1)}{N}}\right]$

4. Apply Z-Transform of $h(n)$ to obtain filter transfer function $H(z)$

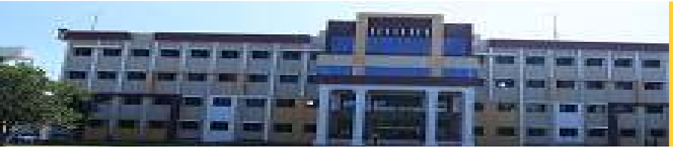
$$\therefore H(z) = \sum_{n=0}^{N-1} h(n) \cdot z^{-n}$$



Kaiser window

- Kaiser window

β	Transition width (Hz)	Min. stop attn dB
2.12	1.5/N	30
4.54	2.9/N	50
6.76	4.3/N	70
8.96	5.7/N	90



FIR Digital Filter Order Estimation

Kaiser's Formula:

$$N \cong \frac{-20\log_{10}(\sqrt{\delta_p \delta_s}) - 13}{14.6(\omega_s - \omega_p)/2\pi} + 1$$

- ie N is inversely proportional to transition band width and not on transition band location